Anomalous condensate fluctuations in strongly interacting superfluids

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We show that the condensate occupation of a superfluid Bose liquid quite generally exhibits anomalously large fluctuations at finite temperatures $T\neq 0$. In three dimensions, the variance $\langle \delta\hat{N}_0^2\rangle$ of the number $\hat{N}_0$ of particles in the condensate scales nonlinearly with volume V like $T^2V^{4/3}$ at low T, generalizing the result obtained by Giorgini, Pitaevskii, and Stringari for a weakly interacting Bose gas. In two dimensions there is only a quasicondensate whose fluctuations are of the same order as the mean value. [S1050-2947(99)08612-6]

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The experimental realization of Bose Einstein condensation (BEC) in alkali-metal vapors has stimulated extensive theoretical research on the physics of BEC [1]. It is well known that the standard grand-canonical ensemble used in textbooks gives unphysically large condensate fluctuations, $\langle \delta\hat{N}_0^2\rangle \simeq \langle \hat{N}_0\rangle^2 \propto V^2$, even at $T=0$. Much recent theoretical work has thus been devoted to the investigation of fluctuations in different statistical ensembles [2–5]. With a fixed total particle number, the pathologies of the grand-canonical ensemble are removed, however the condensate fluctuations remain anomalously large, $\langle \delta\hat{N}_0^2\rangle \propto T^2V^{4/3}$, at least for an ideal Bose gas in a box. Exploiting the fact that BEC in a noninteracting gas is described exactly by the spherical model, this was first realized by Fierz [6]. Surprisingly, the same behavior was found to persist in the presence of weak interactions, as recently shown by Giorgini, Pitaevski, and Stringari using the Bogoliubov theory [7].

In this paper, we demonstrate that the anomalous condensate fluctuations are a general feature of any superfluid Bose system, valid for arbitrary interactions. The large fluctuations at finite temperature will be shown to be a direct consequence of Bogoliubov's well known $1/k^2$ theorem for the static susceptibility $\chi_{\varphi\varphi}(\vec{k})$ of superfluids [8]. They are thus related to the existence of gapless modes and correspondingly are absent in artificial models with a finite gap in the spectrum, as introduced recently in this context [9]. We give a very simple derivation similar to the one employed to prove that the thermal depletion of the condensate in dimension $d=3$ is $\propto T^2$ [10]. For a Bose liquid in $d=2$, there is only a quasicondensate whose root-mean-square fluctuations $\langle \delta\hat{N}_0^2\rangle^{1/2}$ are of the same order as the mean value $\langle \hat{N}_0\rangle$, both scaling with volume with a temperature-dependent exponent $1-\eta(T)/2$. Being a rather general property of a Bose condensed system, the anomalous scaling with volume or particle number will also be present for BEC's in harmonic traps, where the condensate fluctuations are crucial for determining the linewidth of an atom laser [11,12] as recently shown by Graham [13].

We restrict ourselves to Bose liquids in a fixed volume $V_d=L^d$ with periodic boundary conditions. As shown by Feynman, the spectrum of excitations as $k\to 0$ is exhausted by phononlike modes with $\omega_k=ck$. For low enough temperatures such that $mc^2\gg k_BT$, only phase fluctuations are relevant [14] and the field operator may be written as [15]

$$\hat{\Psi}(\vec{x})=\sqrt{n_0}e^{i\hat{\varphi}(\vec{x})} \tag{1}$$

with n_0 being the bare condensate density at $T=0$, which would be observed in a small sample, i.e., in the absence of low-energy excitations. The operator of phase fluctuations is

$$\hat{\varphi}(\vec{x})=\sum_{\vec{k}}{}' \left(\frac{mc}{2V_dn\hbar k}\right)^{1/2}(\hat{c}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}+\hat{c}_{\vec{k}}^{\dagger}e^{-i\vec{k}\cdot\vec{x}}), \tag{2}$$

where m is the particle mass, c the actual velocity of sound, and n the mean particle density. The operators $\hat{c}_{\vec{k}}^{\dagger}$ and $\hat{c}_{\vec{k}}$ are phonon creation and annihilation operators, respectively. The prime indicates summation over all $\vec{k}\neq\vec{0}$ with $|\vec{k}|<\Lambda$, Λ being a momentum cutoff (the term $k=0$ is omitted because it corresponds to an irrelevant global phase factor which can always be absorbed in the bare condensate density n_0 and drops out in any gauge-invariant quantity, as calculated below). The actual, renormalized condensate density is obtained from the off-diagonal elements of the one-particle density matrix, which gives

$$\lim_{|\vec{x}|\to\infty}\langle\hat{\Psi}^{\dagger}(\vec{x})\Psi(\vec{0})\rangle=n_0\lim_{|\vec{x}|\to\infty}e^{-\langle[\hat{\varphi}(\vec{x})-\hat{\varphi}(\vec{0})]^2\rangle/2} \tag{3}$$

as long as phonon interactions and higher-energy excitations are neglected. For large systems, $\lim_{|\vec{x}|\to\infty}\langle\hat{\varphi}(\vec{x})\hat{\varphi}(\vec{0})\rangle$ is always small compared to $\langle\hat{\varphi}^2(\vec{x})\rangle$, which is independent of $\vec{x}$. Separating off the quantum fluctuations of $\hat{\varphi}$ at $T=0$, the mean condensate density at temperature T may be written as

$$n_0(T)=n_0(T=0)e^{-[\langle\hat{\varphi}^2(\vec{0})\rangle|_T-\langle\hat{\varphi}^2(\vec{0})\rangle|_{T=0}]}. \tag{4}$$

In a strongly interacting Bose liquid, the condensate density at zero temperature may be much smaller than the total density n, as is the case in superfluid ^{4}He, where $n_0(T=0)/n\simeq 0.1$ [16].

In $d=3$, the exponent in Eq. (4) is independent of the system size and proportional to T^2, thus describing the well known thermal depletion of the condensate [10],

$$\frac{n_0(T)-n_0(0)}{n_0(0)}=-\frac{m(k_BT)^2}{12nc\hbar^3}. \tag{5}$$

In $d=2$, the thermal contribution to phase fluctuations diverges logarithmically with system size [17],

$$\langle\hat{\varphi}^2(\vec{0})\rangle|_T-\langle\hat{\varphi}^2(\vec{0})\rangle|_{T=0}\simeq\frac{mk_BT}{2\pi n\hbar^2}\ln(\Lambda L)\equiv\eta(T)\ln(\Lambda L). \tag{6}$$

Hence, the condensate density at finite temperature depends on the system size via $n_0(T)=n_0(0)(\Lambda L)^{-\eta(T)}$, with $\eta(T)\propto T$. In particular, for any finite T the condensate density vanishes in the thermodynamic limit, reflecting the fact that there is no true condensate in $d=2$ in agreement with the rigorous proof given by Hohenberg [18].

The operator of fluctuations around the mean field is given by

$$\delta\hat{\Psi}(\vec{x})=\hat{\Psi}(\vec{x})-\sqrt{n_0(T)}=\sqrt{n_0}(e^{i\hat{\varphi}(\vec{x})}-e^{-\langle\hat{\varphi}^2(\vec{0})\rangle/2}). \tag{7}$$

In terms of $\delta\hat{\Psi}(\vec{x})$, the number of particles out of the condensate is

$$\hat{N}_{\rm out}=\int d\vec{x}\,\delta\hat{\Psi}^\dagger(\vec{x})\,\delta\hat{\Psi}(\vec{x}). \tag{8}$$

Since the total number of particles is fixed, the fluctuations of the condensate occupation $\hat{N}_0$ may be calculated from $\langle\delta\hat{N}_0^2\rangle=\langle\delta\hat{N}_{\rm out}^2\rangle$, i.e., the condensate and noncondensate particles act as effectively infinite particle reservoirs for each other [2,6]. Using Eq. (7) one obtains

$$\langle\delta\hat{N}_0^2\rangle\simeq2n_0^2e^{-2\langle\hat{\varphi}^2(\vec{0})\rangle}\int d\vec{x}\,d\vec{x}'\langle\hat{\varphi}(\vec{x})\hat{\varphi}(\vec{x}')\rangle^2, \tag{9}$$

since $\langle\hat{\varphi}(\vec{x})\hat{\varphi}(\vec{x}')\rangle$ is small at large distances both in $d=2$ and in $d=3$, provided that the temperature is low. Evaluating the correlation function at $T=0$, we find

$$\langle\delta\hat{N}_0^2\rangle|_{T=0}\simeq2\left(\frac{n_0(T=0)}{n}\right)^2\left(\frac{mc}{2\hbar}\right)^2\times\begin{cases}\dfrac{1}{2\pi^2}\Lambda V_3, & d=3\\[2ex] \dfrac{1}{4\pi}V_2\ln(\Lambda^2V_2), & d=2.\end{cases} \tag{10}$$

The fluctuations are thus normal, scaling essentially linearly with the volume of the system.

At finite temperatures, it is convenient to express the integral in Eq. (9) in terms of the static susceptibility $\chi_{\varphi\varphi}(\vec{k})$. Using the classical version of the fluctuation dissipation theorem, which is always applicable at low k because $\hbar\omega_k\ll k_BT$, we find

$$\int d\vec{x}\,d\vec{x}'\langle\hat{\varphi}(\vec{x})\hat{\varphi}(\vec{x}')\rangle^2=\sum_{\vec{k}}{}'\,[k_BT\chi_{\varphi\varphi}(\vec{k})]^2. \tag{11}$$

Since the superfluid density n_s which usually appears in $\chi_{\varphi\varphi}$ [8] coincides with the full density n at zero temperature for any translation-invariant superfluid, we have $\chi_{\varphi\varphi}(\vec{k})=m/n\hbar^2k^2$ for $k\to0$, independent of the interaction strength. Hence, at low temperatures,

$$\langle\delta\hat{N}_0^2\rangle|_T=2\left(\frac{n_0(0)}{n}\right)^2\left(\frac{mk_BT}{\hbar^2}\right)^2\sum_{\vec{k}}{}'\,\frac{1}{k^4}. \tag{12}$$

For the weakly interacting Bose gas, there is no depletion of the condensate at $T=0$ to lowest order, and thus with $n_0(0)/n$ replaced by one, our result reduces to the one given in [7]. The somewhat surprising fact that the velocity of sound does not appear in Eq. (12) may be traced back to the well known low k divergence [15,19],

$$\lim_{k\to0}n_k=\frac{n_0(0)}{n}\,\frac{k_BT}{2\varepsilon_k}, \tag{13}$$

of the momentum distribution at low but finite temperatures, which, apart from the ratio $n_0(0)/n$, only depends on the bare single-particle energy $\varepsilon_k=(\hbar k)^2/2m$. In order to make this connection more explicit, we note that the large distance behavior of the phase correlation function which appears in Eq. (9) is directly related to the momentum distribution n_k of the noncondensed particles at low k via [15]

$$\lim_{|\vec{x}|\to\infty}\langle\hat{\Psi}^\dagger(\vec{x})\Psi(\vec{0})\rangle=n_0(T)[1+\langle\hat{\varphi}(\vec{x})\hat{\varphi}(0)\rangle]=n_0(T)+\int\frac{d^3k}{(2\pi)^3}n_k\exp-i\vec{k}\vec{x}. \tag{14}$$

This shows that Eq. (9) can be written in the simple form

$$\langle\delta\hat{N}_0^2\rangle=2\sum_{\vec{k}}{}'\,n_k^2, \tag{15}$$

which immediately gives Eq. (12) using the result (13) for the momentum distribution at finite temperature. The anomalous scaling of the condensate fluctuations with volume is therefore a direct consequence of the $1/k^2$ divergence of the momentum distribution or—equivalently—the related slow $1/r$ decay of the one-particle density matrix at large distances to its limiting nonzero value $n_0(T)$. Incidentally, this argument also explains why the result for an *ideal* Bose gas, where $n_0(0)=n$ trivially, is just twice the result (12). Indeed, the momentum distribution for noninteracting bosons at small nonzero k is $n_k^{(0)}\to k_BT/\varepsilon_k$ [the factor 2 difference compared with Eq. (13) is due to the fact that at low k the quasiparticles of the interacting system are just an equal weight superposition of a bare particle with momentum k plus a bare hole with momentum $-k$ [20]]. Using the standard result for the occupation number fluctuations of ideal bosons, we have

$$\langle\delta\hat{N}_0^2\rangle^{(0)}=\sum_{\vec{k}}{}'\,n_k^{(0)}(1+n_k^{(0)})\simeq\sum_{\vec{k}}{}'\,(n_k^{(0)})^2. \tag{16}$$

To leading order, the fluctuations in the interacting and the noninteracting case are thus simply related by a factor $2[n_0(0)/2n]^2$. Contrary to the argument in [7], the similar behavior of the fluctuations in an ideal and interacting Bose gas is therefore not accidental.

As implied by Eq. (12), at $T\neq 0$ the fluctuations of the ground-state occupation are anomalous for any interaction. Expressing the result in terms of the standard thermal wavelength $\lambda_T = h(2\pi m k_B T)^{-1/2}$, we have

$$\langle \delta\hat{N}_0^2\rangle|_T = B\left(\frac{n_0(0)}{n}\right)^2\left(\frac{L}{\lambda_T}\right)^4 \propto V_3^{4/3} \tag{17}$$

with a universal constant $B=\Sigma_{\vec{n}}{}'|\vec{n}|^{-4}/2\pi^2=0.8375$ whose numerical value is readily obtained from the lattice sum (12) (this result is roughly a factor 5 smaller than the corresponding one given in Ref. [7]). Note that contributions $\vec{k}=2\pi\vec{n}/L$ with one or two of the components of $\vec{n}\in\mathbb{Z}^3$ being zero should be included in the sum in the case of periodic boundary conditions as assumed here. For Dirichlet boundary conditions in a box with hard walls, the corresponding constant in an ideal Bose gas is in fact smaller by a factor 0.864 [21].

In $d=2$, the one-particle density matrix decays to zero like $1/r^{\eta(T)}$ at small nonzero temperatures and the associated momentum distribution n_k is thus proportional to $k^{-(2-\eta)}$ as $k\to 0$. The resulting scaling of the condensate fluctuations with system size is temperature dependent,

$$\langle \delta\hat{N}_0^2\rangle|_T \propto \left(\frac{n_0(0)}{n}\right)^2\left(\frac{L}{\lambda_T}\right)^4 (\Lambda L)^{-2\eta(T)} \propto V_2^{2-\eta(T)}, \tag{18}$$

increasing like the square of the average condensate number $\langle\delta\hat{N}_0^2\rangle\propto\langle\hat{N}_0\rangle^2$. The fluctuations in $d=2$ are thus much larger than those in $d=3$. The fact that the rms fluctuations of the condensate are of the same order as the average is yet another indication that the condensate in $d=2$ is not well defined.

In conclusion, we have shown that the anomalous fluctuations of the condensate occupation at finite temperature, found previously for both ideal and weakly interacting Bose gases, are a general property of any superfluid with arbitrary interactions. The anomalous behavior of these fluctuations may be traced back via Bogoliubov's $1/k^2$ theorem to the corresponding low-k singularity in the momentum distribution. They are thus expected to appear quite generally for any order parameter of a phase with a broken continuous symmetry. As has been shown in [7], the condensate fluctuations in harmonic traps are similarly determined by the associated low lying collective excitations. As a result, the anomalous scaling $\langle\delta\hat{N}_0^2\rangle|_T\propto T^2N^{4/3}$ with the particle number N is also present in finite geometries, provided the excitation spectrum is effectively continuous ($k_BT\ll\hbar\omega$ in harmonic traps).

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[1] For a recent theoretical review see, F. Dalfovo, S. Giorgini, L. P. Pitaevski, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[2] H. D. Politzer, Phys. Rev. A **54**, 5048 (1996).
[3] P. Navez *et al.*, Phys. Rev. Lett. **79**, 1789 (1997).
[4] C. Weiss and M. Wilkens, Opt. Express **1**, 272 (1997).
[5] M. Holthaus, E. Kalinowksi, and K. Kirsten, Ann. Phys. (N.Y.) **270**, 198 (1998).
[6] M. Fierz, Helv. Phys. Acta **29**, 47 (1956).
[7] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **80**, 5040 (1998).
[8] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, MA, 1975).
[9] Z. Idziaszek *et al.*, Phys. Rev. Lett. **82**, 4376 (1999).
[10] R. A. Ferrell *et al.*, Ann. Phys. (N.Y.) **47**, 565 (1968).
[11] I. Bloch, T. W. Hänsch, and T. Esslinger, Phys. Rev. Lett. **82**, 3008 (1999).
[12] E. W. Hagley *et al.*, Science **283**, 1706 (1999).
[13] R. Graham, Phys. Rev. Lett. **81**, 5262 (1998).
[14] For a microscopic derivation of the fact that phase fluctuations alone dominate the low-energy physics, see, e.g., V. N. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, Cambridge, 1987), Chap. 7.
[15] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1981), Sec. 27.
[16] D. M. Ceperley, Rev. Mod. Phys. **67**, 279 (1995); note that both phase and amplitude fluctuations are necessary to calculate the renormalized condensate density $n_0(0)$, however for the thermal effects at low temperatures only the phase is relevant.
[17] J. W. Kane and L. P. Kadanoff, Phys. Rev. **155**, 80 (1967).
[18] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).
[19] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).
[20] P. Nozieres and D. Pines, *The Theory of Quantum Liquids, Vol. 2* (Addison-Wesley, Redwood City, CA, 1990).
[21] M. Holthaus (unpublished).