Discrete formulation of teleportation of continuous variables

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Teleportation of continuous variables can be described in two different ways, one in terms of Wigner functions, the other in terms of discrete basis states. The latter formulation provides the connection between the theory of teleportation of continuous degrees of freedom of a light field and the standard description of teleportation of discrete variables. [S1050-2947(99)06812-2]

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Teleportation is a process by which one party, Alice, can transfer any (unknown) quantum state $|\psi\rangle$ to a distant second party, Bob, by sending him just the classical information containing the outcome x_0 of an appropriate measurement performed by Alice, provided the two parties share a nonlocal entangled pair of particles. Alice's measurement is a joint measurement on two systems, one of which is the particle in the state $|\psi\rangle$, while the other forms half of the entangled state. Bob can create the state $|\psi\rangle$ on his part of the entangled state by applying a unitary operation U_{x_0}, the form of which is determined by the classical outcome x_0. The original protocol of Bennett *et al.* [1] concerned quantum states in a finite-dimensional Hilbert space, so that the measurement outcome x_0 is discrete. Most experimental efforts towards accomplishing teleportation using entangled photons [2,3] follow that discrete path.

A recent experiment [4], however, succeeded in teleporting *continuous* degrees of freedom of a light field, following the theoretical proposal of Ref. [5]. The description of that experiment made use of the Wigner function, so that its connection to the original teleportation proposal [1] may not be entirely clear. Here we describe the experiment in the (discrete) photon number-state basis and thereby provide that connection. Moreover, the present formulation is simpler than the one in Ref. [6] of teleportation of N variables. The inverse route of linking continuous to discrete descriptions by reformulating the protocol of [1] in terms of the Wigner function for discrete variables [7] will not be followed here.

In the experiment of Ref. [4], states of a given single mode [8] of the electromagnetic field were teleported. One way of describing the field is in terms of quadrature amplitudes (see, e.g., Ref. [9]), which are analogous to the position and momentum variables of a harmonic oscillator (in fact, the electromagnetic field variables are quantized by first rewriting the Hamiltonian into the form of an infinite set of harmonic oscillators). An alternative description is in terms of number states. In particular, the entangled state that Alice and Bob share is a two-mode squeezed state, which can be written as [9]

$$|S_r\rangle_{2,3}=\frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n|n\rangle_2|n\rangle_3, \tag{1}$$

where mode 2 is located in Alice's laboratory, and mode 3 in Bob's. The parameter r is a measure for the amount of squeezing. The fluctuations in the squeezed variable are reduced by $\exp(-2r)$, at the cost of increasing the fluctuations in the complimentary variable by $\exp(2r)$. For $r\to\infty$ the two-mode squeezed state is maximally squeezed and fully entangled. It is interesting to tabulate the amount of entanglement [11], $E=-\mathrm{Tr}_2\,\rho\log_2\rho$ with $\rho=\mathrm{Tr}_3|S_r\rangle_{2,3}\langle S_r|$, for a finitely squeezed state,

$$E=\cosh^2 r\,\log_2(\cosh^2 r)-\sinh^2 r\log_2(\sinh^2 r). \tag{2}$$

Figure 1 shows that the amount of entanglement is approximately linear in the amount of squeezing. In particular, for $r=0.69$, for which $\exp(-2r)=0.5$ (the squeezing parameter for the experiment [4]), the amount of entanglement is $E=1.46$. The requirements on the amount of entanglement and corresponding fidelity needed to distinguish quantum from ''classical'' teleportation, are discussed in [10].

Alice is given another field mode 1 which is in the state $|\psi\rangle_1$ to be teleported. This state can be expanded as

$$|\psi\rangle_1=\sum_{n=0}^{\infty}\alpha_n|n\rangle_1. \tag{3}$$

As in the original teleportation protocol, Alice has to perform a joint measurement on modes 1 and 2. In [4] the joint measurement consisted of two measurements of the two

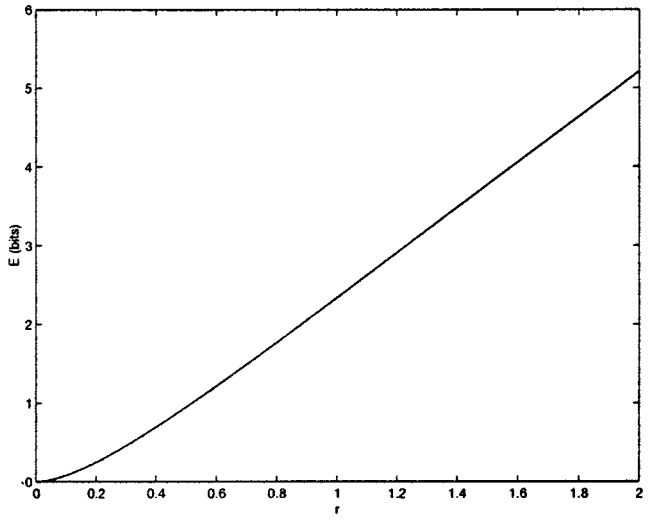

FIG. 1. Entanglement E in units of bits as a function of the squeezing parameter r.

commuting observables $\hat{X}=\hat{x}_2-\hat{x}_1$ and $\hat{P}=(\hat{p}_2+\hat{p}_1)/2$, where $\hat{x}_i$ and $\hat{p}_i$ are proportional to the quadrature amplitudes referred to above,

$$\hat{x}_i=\frac{1}{2}(a_i+a_i^\dagger),$$

$$\hat{p}_i=\frac{1}{2i}(a_i-a_i^\dagger), \tag{4}$$

in terms of the creation and annihilation operators acting on the modes $i=1,2$. The joint eigenstate of the two operators $\hat{X}$ and $\hat{P}$ with eigenvalues X and P can be expanded in the eigenstate basis of $\hat{x}_i$,

$$|\phi(X,P)\rangle_{1,2}=\int dX_1\int dX_2\,\delta(X_2-X_1-X) \times\exp[iP(X_1+X_2)]|X_1\rangle_1|X_2\rangle_2. \tag{5}$$

This state is fully entangled, and is in fact of the same form as the original EPR state [13].

Now in order to discuss the limit of infinite squeezing, in which the state (1) is no longer normalizable, we now truncate the Hilbert space and consider only photon numbers up to and including N, where we may take the limit $N\to\infty$ in the end. In particular, the two-mode squeezed state in the limit of infinite squeezing $r\to\infty$ becomes

$$|S_\infty\rangle_{2,3}=\frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}|n\rangle_2|n\rangle_3. \tag{6}$$

We can rewrite the eigenstate (5) in that truncated space as

$$|\phi(X,P)\rangle_{1,2}=\sum_{m=0}^{N}\sum_{n=0}^{N}\gamma_{mn}(X,P)|m\rangle_1|n\rangle_2, \tag{7}$$

where we do not yet have to specify the precise form of the coefficients $\gamma_{mn}(X,P)$ (but see below). It is easy to verify that the reduced density matrix of either mode 1 or 2 in the eigenstate (5) is proportional to the identity matrix. This implies that after Alice's measurement no information about the identity of the state $|\psi\rangle$ will be left behind in either system 1 or 2, which is a necessary condition for faithful teleportation [1]. The fact that

$$\mathrm{Tr}_2|\phi(X,P)\rangle_{1,2}\langle\phi(X,P)|=\frac{1}{N+1}I_1 \tag{8}$$

with I_1 the $(N+1)\times(N+1)$ identity operator on mode 1, implies that the coefficients γ_{mn} satisfy

$$(N+1)\sum_{l=0}^{N}\gamma_{ml}^*(X,P)\,\gamma_{nl}(X,P)=\delta_{mn}. \tag{9}$$

That is, the matrix $\sqrt{N+1}\,\gamma_{mn}$ is unitary. In order to show explicitly that this is a necessary and sufficient condition for teleportation to be possible, we rewrite the joint initial state of modes 1,2,3, in the case of infinite squeezing, as

$$|\psi\rangle_1|S_\infty\rangle_{2,3}=\frac{1}{N+1}\sum_{X=X_0}^{X_N}\sum_{P=P_0}^{P_N}\sum_{l=0}^{N}\sum_{m=0}^{N} \times\gamma_{lm}(X,P)|l\rangle_1|m\rangle_2\sum_{n=0}^{N}\beta_n(X,P)|n\rangle_3. \tag{10}$$

Here we used that the eigenfunctions of the operators $\hat{X}$ and $\hat{P}$ form a complete set, so that the sum—P and X have become discrete variables now—over all eigenvalues X and P of the operators $|\phi(X,P)\rangle_{1,2}\langle\phi(X,P)|$ gives the identity. The coefficients β_n are given by

$$\beta_n(X,P)=\sqrt{N+1}\sum_{m=0}^{N}\gamma_{mn}^*(X,P)\,\alpha_m. \tag{11}$$

It follows directly from Eq. (10) that after Alice finds two measurement outcomes X_0 and P_0, Bob's state is collapsed onto

$$|\Psi\rangle_3=\sum_{n=0}^{N}\beta_n(X_0,P_0)|n\rangle_3 =\sqrt{N+1}\sum_{m=0}^{N}\sum_{n=0}^{N}\gamma_{mn}^*(X_0,P_0)\,\alpha_m|n\rangle_3. \tag{12}$$

In order for Bob to be able to recover the original state $|\psi\rangle$ from $|\Psi\rangle$, we see now that the matrix $\sqrt{N+1}\,\gamma_{mn}$ indeed must be unitary: Bob has to apply the operation

$$U_{X_0,P_0}:|n\rangle_3\mapsto\sqrt{N+1}\sum_{m=0}^{N}\gamma_{mn}|m\rangle_3 \tag{13}$$

to effect the transformation

$$|\Psi\rangle_3\mapsto|\psi\rangle_3, \tag{14}$$

which completes the teleportation process.

Thus, Bob's unitary operation (13) and Alice's measurement outcome (7) are both described by a single unitary matrix γ_{mn} (just as in the example given in [1]). In the experiment [4] this translates into the fact that Alice's measurement outcomes are classical currents that Bob directly converts into field amplitudes and subsequently mixes with his part of the two-mode squeezed state.

For completeness, let us now calculate the explicit form of the eigenstates $|\phi(X,P)\rangle_{1,2}$ of the operators $\hat{X}$ and $\hat{P}$ with eigenvalues X and P in the number-state basis. First, the (truncated) eigenstate $|\phi(0,0)\rangle_{1,2}$ with zero eigenvalues is easily found by simply solving the eigenvalue equations

$$(a_1-a_2^\dagger)|\phi(0,0)\rangle_{1,2}=0,$$
$$(a_2-a_1^\dagger)|\phi(0,0)\rangle_{1,2}=0, \tag{15}$$

with the result

$$|\phi(0,0)\rangle_{1,2}=\frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}|n\rangle_1|n\rangle_2. \tag{16}$$

Then, introducing the two commuting operators $\hat{Y}=(\hat{x}_1+\hat{x}_2)$ and $\hat{Q}=(\hat{p}_1-\hat{p}_2)/2$, it is easy to verify, using the

commutation relations between $\hat{X}$ and $\hat{Q}$, and between $\hat{P}$ and $\hat{Y}$, that

$$|\phi(X,P)\rangle_{1,2}=\exp(iP\hat{Y})\exp(iX\hat{Q})|\phi(0,0)\rangle_{1,2} \quad (17)$$

is indeed the desired eigenstate with eigenvalues X and P. Using standard identities for exponentials of creation and annihilation operators [12] and the relations (15) this state can be rewritten as

$$|\phi(X,P)\rangle_{1,2}=\exp[-[P^2+(X/2)^2]/4]\exp[(iP-X/2)a_1]\times\exp[(iP+X/2)a_2]|\phi(0,0)\rangle_{1,2}, \quad (18)$$

which can be expanded as

$$|\phi(X,P)\rangle_{1,2}=\exp\{-(P^2+(X/2)^2)/4\}\times\sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{l=\min(m,n)}^{N}\left(\frac{(l!)^2}{m!n!}\right)^{1/2}\times\frac{(iP-X/2)^{l-m}(iP+X/2)^{l-n}}{(l-m)!(l-n)!}|m\rangle_1|n\rangle_2. \quad (19)$$

This then yields in the coefficients γ_{mn}. Because of the still relatively complicated form of the coefficients γ_{mn}, the question of how finite squeezing affects the fidelity of the teleportation process is better discussed in the Wigner-state formalism [5] (see also [14]).

In conclusion, the teleportation experiment of Ref. [4] of continuous degrees of freedom of a light beam can be formulated in the number-state basis, thus providing a connection with the original formulation of the teleportation protocol. The measurements of quadrature amplitudes on Alice's side correspond to entangled measurements that leave no information behind in Alice's field modes about the state to be teleported. This enables Bob to recreate that state in a field mode in his laboratory by applying a particular unitary operation, described by the same unitary matrix γ_{mn} that describes Alice's measurement scheme.

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