Traversal time in macroscopic quantum tunneling

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We present an interpretation of traversal time results in Josephson junctions, based on a stochastic model of tunneling processes which include dissipative effects. With the use of semiclassical analysis, the experiments performed up until now have supplied a time duration of the order of 10^2 ps (a remarkably long time, even considering the macroscopic nature of the systems under examination). According to our interpretation, this duration is only the imaginary part of a complex quantity, the real part of which (presumably the true physical duration) lies in the range of a few picoseconds. [S1050-2947(99)03012-7]

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Quantum mechanics at a macroscopic level is indubitably a fascinating topic and, within this context, the Josephson effect is one of the most suitable for observing phenomena such as macroscopic quantum tunneling (MQT) [1], macroscopic quantum coherence (MQC), and energy levels quantization (ELQ) [2].

As for the observation of MQT in a current biased Josephson junction, the single result reported in the literature [3] has been confirmed only recently, by the results of an analogous experiment [4]. The traversal time of the barrier has been deduced from these measurements by using semiclassical analyses. In both cases, a result of the order of 100 ps was obtained. To be more precise, for the junction of Ref. [3], the semiclassical time given by $\pi/\omega = 86$ ps is nearly coincident with the value of 78 ps inferred (not directly obtained) from the measurements at 18 mK. For the junction of Ref. [4], $\pi/\omega = 158$ ps, but from measurements at 50 mK a value of 91 ps was deduced. The shortening which resulted is due to the thermal contribution [5].

However, the traversal time evaluated as π/ω is presumably only the imaginary part of a complex quantity whose real part remains unknown, since it is not accessible for a direct measurement, as in other experimental situations. Although not yet fully shared, it is, however, largely accepted that the tunneling time has to be considered as a complex quantity [6]. The real part, related to the phase of the complex transmission coefficient, is named *phase time*, while the imaginary part, related to the absolute value of the transmission coefficient (the attenuation under the barrier) is sometimes named *loss time*. The latter is essentially equivalent to the semiclassical time in the opaque barrier limit [7]. This work is devoted to the evaluation of the phase time which, in agreement with the above framework, is just the real part of the tunneling time.

Since the pioneering work done by Caldeira and Leggett [8], considerable efforts aimed at understanding this matter, have been made from a theoretical point of view [9]. In quantum-theoretical analysis, the main difficulties lie in the inclusion of dissipative effects (which are always present in macroscopic systems), as well as other thermodynamical variables, such as temperature. In this work, we search for an alternative approach based on a model which considers tunneling as a stochastic process. Our approach represents an alternative method for solving partial differential equations when dissipative effects are present. It has already demonstrated its capability to interpret other experimental results on tunneling [10]. The advantage of this method lies in its being able to discriminate between real and imaginary parts of the traversal time. An evaluation of the real part can be performed once the imaginary one has been determined.

The deterministic equation of motion for a current biased junction can be written as [11]

$$C\left(\frac{\Phi_0}{2\pi}\right)^2 \ddot{\phi} + R^{-1}\left(\frac{\Phi_0}{2\pi}\right)^2 \dot{\phi} + \frac{\partial}{\partial \phi} V(\phi) = 0, \tag{1}$$

where R and C are the shunt resistance and the capacitance of the junction, respectively, ϕ is the phase difference of the Cooper-pair wave function across the gap, and Φ_0 is the flux quantum. The function $V(\phi)$ represents a tilted periodic potential given by $V(\phi) = (-I_0\Phi_0/2\pi)[\cos\phi + (I\phi/I_0)]$, where I_0 is the critical current of the junction. The equation of motion (1) is homologous with the equation of motion

$$m\ddot{x} + \eta\dot{x} + \frac{\partial}{\partial x} V(x) = 0 \tag{2}$$

for a particle of mass m moving in a potential $V(x)$, in the presence of dissipation with a damping coefficient η. The potential is well described by a cubic function $V(x) = \epsilon x^2 - \rho x^3$, where $\epsilon = m\omega^2/2, \omega$ is the angular frequency at the bottom of the well, ρ is a positive constant whose value is taken to fit $V(\phi)$. The equation of motion (2) is easily integrated in the reversed potential $-V$, for $\eta = 0$ (absence of dissipation). Its solution gives the instanton bounce trajectory [12]

$$x(t) = x_b \operatorname{sech}^2\left(\frac{\omega t}{2}\right), \tag{3}$$

where $x_b=\epsilon/\rho$ is the bounce amplitude (which corresponds to the barrier length) and t is the imaginary time. The particle trapped in the well of the potential corresponds to the zero-voltage state of the junction. The superconducting condition becomes lost when the particle escapes from the well (by means of tunneling or thermal overcoming of the barrier) and slides down toward a lower minimum of the potential. Thus, the zero-voltage state is unstable due to quantum and/or thermal fluctuations; the relative importance of the two contributions depends on the temperature. At a very low temperature, quantum tunneling prevails, and its motion can be described approximately by Eq. (3). The presence in Eq. (2) of the dissipative term $\eta\dot{x}$—which represents the interaction of local motion with the phonon modes—strongly increases the complexity of the analysis. What is clearly established is that dissipation increases the action integral, in agreement with the intuitive behavior that viscous forces reduce the probability of tunneling.

The solution of motion (3) holds exactly for $\eta=0$ at zero temperature. At moderate temperatures, below the crossover one (defined as $T_c=\hbar\omega/2\pi k_B$, around 50 mK in our cases), tunneling will occur essentially in proximity with the top of the barrier. Here the potential is well described by the parabolic function $V(x)=V_0-\epsilon(x-x_0)^2$, where V_0 is the barrier height and x_0 its coordinate [4]. In this situation, the equation of motion (2) is reduced to that of a damped harmonic oscillator [13] once—in view of the forbidden character of the process—its analytical continuation into complex plane has been considered. One way to obtain this continuation is to consider the motion as occurring in imaginary time ($t\rightarrow it$) and imaginary frequencies ($\omega\rightarrow i\omega$) [14]. Another way is to replace the damping parameter $a(=\eta/2m)$ with ia [15]. By putting $x-x_0=y$, in both cases, the equation of motion becomes $\ddot{y}+2ia\dot{y}+\omega^2y=0(\omega^2>0)$. Its solution is then

$$y(t)=\exp(-iat)\left(\cos\tilde{\omega}t+i\frac{a}{\tilde{\omega}}\sin\tilde{\omega}t\right)y_0, \qquad (4)$$

where $\tilde{\omega}=\sqrt{\omega^2+a^2}$, and y_0 is the amplitude of the pseudo-oscillations ($2y_0$ is the length of the barrier traversal) which, in imaginary time, are not damped.

We are now in a position to evaluate the action integral relative to the solution of motion (4) and, in particular, the variation of this quantity due to the modification of the trajectory [the second term in Eq. (4)]. By considering the Euclidean Lagrangian $L=T+V=m\dot{y}^2/2-\epsilon y^2$, we can easily verify that the increase of the absolute value of the action is of the order of ηy_0^2 [8,12]. By collecting only terms of the first order in a, and for moderate values of the argument $2at$, its real part, ΔS_r results

$$\Delta S_r\simeq\frac{amy_0^2}{2}[\tilde{\omega}t\sin(2\tilde{\omega}t)+\cos(2\tilde{\omega}t)]. \qquad (5)$$

This quantity produces a dephasing $\Delta\varphi=\Delta S_r/\hbar$, whose derivative, with respect to the frequency, contributes to the real part of the delay by an amount given by ($\tilde{\omega}\simeq\omega$)

$$t_r=\frac{\partial}{\partial\omega}\Delta\varphi=\frac{am}{\hbar}y_0^2\omega t^2\left[\cos(2\omega t)-\frac{\sin(2\omega t)}{2\omega t}\right]\approx at^2, \qquad (6)$$

where we have assumed $m\omega y_0^2\approx\hbar$ and $\omega t\approx\pi$ (half-oscillation). This result is in agreement with the analysis of Ref. [10]. There, by considering tunneling in the presence of dissipation as a stochastic process, we demonstrated that tunneling time is a complex quantity whose average, for moderate values of the semiclassical time L/v, is just $\langle t\rangle\simeq a(L/v)^2+iL/v$, where L/v can be identified with our t.

The analysis of Ref. [10] is based on analyses of Refs. [13–15] which, in turn, derive from the original work by Kac regarding a stochastic model related to the telegrapher equation [16]. Noteworthy is the fact that this model can be applied to solving other (linear) partial differential equations, which include dissipative effects, given a solution of their principal parts [13]. The solution of the damped harmonic oscillator [Eq. (4) is its analytical continuation to imaginary time and frequency] represents an example of such a method. In order to proceed further with our analysis, we integrated the equation of motion (2) in the presence of (moderate) dissipation. This was not a trivial task since, as we shall see, there is no exact analytical solution to it, but only some approximate ones. We consider three kinds of solutions.

(i) By means of a Fourier-integral expansion of the solution (3) in the absence of dissipation, we substitute $\cos(\omega t)$ functions by $y(t)$, as given by Eq. (4), and then, by integrating over ω, we obtain a $x(t)$ solution of motion.

(ii) By adopting the same procedure which allowed us to obtain Eq. (4), we try to use the same distribution $h(t,r)$ [13] for solving Eq. (2), whose solution for $\eta=0$ is given by $x(t)$ of Eq. (3). This represents an improper procedure, since it is exactly applicable only to linear equations, while our Eq. (2) is not linear for $V(x)=\varepsilon x^2-\rho x^3$.

(iii) From a numerical integration of Eq. (2), we are able to obtain an exact solution to the equation of motion. This solution is not suitable for our analysis, but we can use it for testing the worth of analytical solutions (i) and (ii).

Differential equation (2) can be numerically solved, for $V_-(x)=-\varepsilon x^2+\rho x^3$, as follows

$$\ddot{x}(t)=-2a\dot{x}+\omega^2\left[x(t)-\frac{3x^2}{2x_b}\right],$$

$$\dot{x}(t+\Delta t)=\dot{x}(t)+\ddot{x}(t)\Delta t,$$

$$x(t+\Delta t)=x(t)+\dot{x}(t)\Delta t+\frac{1}{2}\ddot{x}(t)(\Delta t)^2, \qquad (7)$$

where $a=\eta/2m$, and the boundary conditions are $\dot{x}(0)=0,x(0)=x_b$. For $t>0$, the resulting trajectories show, as expected, the shape of damped oscillations. Also, for $t\rightarrow\infty$, they asimptotically tend towards the value of the potential-minimum coordinate $x_m=2x_b/3$. With a small increase in $x(0)$ [e.g., $x(0)=1.02537x_b$ for $a=0.25$ and $\omega=10$], the trajectory becomes aperiodic [similar to Eq. (3) which is the trajectory in the absence of dissipation] since it meets the metastable position again at $x=0$ (see Fig. 1).

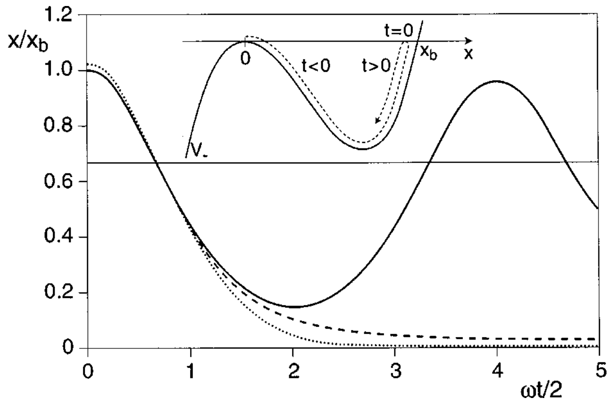

FIG. 1. Trajectory shapes in the positive-time domain (half bounce in the inset which shows the inverted potential V_-) obtained, for $a=0.25$ and $\omega=10$ using different procedures: numerical integration [Eqs. (7)]—continous line; analytical solution [Eqs. (10) or (13)]—dashed line. The quantities a and ω are expressed in the same (arbitrary) units. The dotted line represents the solution obtained by numerical integration with a small variation in the initial condition.

According to procedure (i), we find that the trajectory is given by

$$x(t)=x_b\int_{-\infty}^{\infty}A(\tilde{\omega})y(t)d\tilde{\omega}, \tag{8}$$

where $y(t)$ is given by Eq. (4) and $A(\tilde{\omega})$ is

$$A(\tilde{\omega})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{sech}^2\left(\frac{\omega t}{2}\right)\cos(\tilde{\omega}t)dt=\frac{2\tilde{\omega}}{\omega^2\sinh(\pi\tilde{\omega}/\omega)}. \tag{9}$$

By substituting Eq. (9) in Eq. (8), we obtain [17]

$$x(t)=x_b\left[\text{sech}^2\left(\frac{\omega t}{2}\right)+\frac{2a}{\omega}\tanh\left(\frac{\omega t}{2}\right)\right]\exp(-at) \tag{10}$$

which can be considered an approximate solution to our Eq. (2), at least for moderate values of the $\omega t/2$ argument. A check can be made by comparing Eq. (10) with the numerical results of item (iii): we can see that there is good agreement between the two kinds of solution for argument values up to 1–1.5. However, the extent of the agreement is increased when we consider the aperiodic solution mentioned above. In this case the agreement—depending on the value of the parameter a—is acceptable for greater argument values, nearly over all the significant extent of the bounce trajectory (see Fig. 1).

As for the procedure of item (ii), in principle the solution for which we are searching is given simply by quadrature [13]

$$X(t)=\int_0^{\infty}x(r)h(t,r)dr, \tag{11}$$

where $x(r)=x_b\,\text{sech}^2(\omega r/2)$ is the solution for the classically-allowed motion [Eq. (3)] in the reversed potential $V_-(x)=-\epsilon x^2+\rho x^3$, and $h(t,r)$ is the distribution of the randomized paths (t is the true time, r is the effective time). For our trajectory, Eq. (11) has no analytical solution; however, we can solve the problem approximately by expanding the terms containing a [13] in the Laplace transform $f(s)$ of Eq. (11). To the second order in a, we obtain

$$f(s)=\left(1+\frac{a}{2}-\frac{a^2}{2s^2}\right)F(s)+aF'(s)+\frac{a^2}{2}F''(s), \tag{12}$$

where $F(s)=\int_0^{\infty}\exp(-rs)\,\text{sech}^2(\omega r/2)dr$. Then, by antitransforming, we easily obtain [18]

$$\frac{X(t)}{x_b}\simeq\frac{e^{-at}}{\cosh^2\left(\frac{\omega t}{2}\right)}+\frac{2a}{\omega}\tanh\left(\frac{\omega t}{2}\right)-\frac{2a^2}{\omega^2}\ln\left[\cosh\left(\frac{\omega t}{2}\right)\right]+\cdots. \tag{13}$$

This function, which for $a=0$ rightly gives the unperturbed solution, represents an asymmetric function for $a>0$ (with respect to $t=0$) whose behavior accounts for the damping of the motion. The coincidence of this result with Eq. (10), at the first order in a, is worthy of note. We therefore consider that the inclusion of the second–order term [third term in Eq. (13)] represents a better approximation of the solution. This can be verified by comparison with the numerical solution given by Eqs. (7) and reported in Fig. 1. As before with the harmonic oscillator, we are interested in the analytical continuation of Eq. (13) to imaginary time and frequency. We therefore obtain a solution of the motion of the type

$$\frac{X(t)}{x_b}=\frac{e^{-iat}}{\cosh^2(\omega t/2)}+i\frac{2a}{\omega}\tanh\left(\frac{\omega t}{2}\right)+\frac{2a^2}{\omega^2}\ln\left[\cosh\left(\frac{\omega t}{2}\right)\right]+\cdots. \tag{14}$$

Analogously to what was done for the harmonic oscillator, we consider the Euclidean Lagrangian $L=T+V=m\dot{x}^2/2+\varepsilon x^2-\rho x^3$: that is we look at the motion as it was allowed in the reversed potential. The increase in the action integral due to dissipation is again of the order of ηx_b^2 [12], and its real part—retaining terms of the first order in a—is of type [19]

$$\Delta S_r\simeq 2amx_b^2\left[\left(\frac{\omega t}{2}\right)\frac{\tanh(\omega t/2)}{\cosh^4(\omega t/2)}\right]. \tag{15}$$

Again [see Eq. (6)], by assuming $m\omega x_b^2\approx\hbar$, we evaluate t_r as

$$t_r=\frac{\partial}{\partial\omega}\frac{\Delta S_r}{\hbar}=\frac{at}{\omega}\left[\tanh\left(\frac{\omega t}{2}\right)+\left(\frac{\omega t}{2}\right)\text{sech}^2\left(\frac{\omega t}{2}\right)-4\left(\frac{\omega t}{2}\right)\tanh^2\left(\frac{\omega t}{2}\right)\right]\text{sech}^4\left(\frac{\omega t}{2}\right), \tag{16}$$

of course, if the quantity $m\omega x_b^2$ is a number of $\hbar$, the result for t_r must be multiplied for this number. Equation (16) for $\omega t/2\gg 1$ becomes

$$t_r \simeq 2at^2[4e^{-\omega t}+O(e^{-2\omega t})], \qquad (17)$$

while for $\omega t/2 \ll 1$ becomes $\sim at^2$, which essentially confirms the result of Eq. (6). This result can be interpreted in the following way. When the analytical continuation to imaginary time is considered [10] and for $r \gg t$, the shape of the distribution $h(t,r)$ is of the $\exp(-at^2/2r)$ type. This means that the asymptotic average value of t is rather small: that is, it tends to $1/2a$ [20]. The width of this distribution is given by $\sqrt{2r/a}$, and decreases by increasing the dissipative parameter a [13]. So, for large values of a (or large time values), we have $2r \sim at^2$, which gives an upper limit to the real time duration of the process. On the contrary, within the limit of small values for a, we obtain a distribution cut at $t = r$. In this case, therefore, the upper limit of the real time is given by ar^2, with $t \lesssim r$, which has to be assumed as the semiclassical unperturbed time.

For the junction tested in Ref. [4], we estimated $2a$ $(=\eta/m)$ to be of the order of one $(\mathrm{ns})^{-1}$ [in the equivalent equation of motion of the junction, Eq. (1), the quantity $2a$ is replaced by $(\mathrm{RC})^{-1}$], while the semiclassical time r is 0.091 ns [5]. This means that the real-time delay should be given approximately, according to Eq. (16), by $t_r \approx ar^2 \approx 4$ ps for $(\omega t/2) \ll 1$, or, according to Eq. (17), by $t_r \approx 8at^2e^{-\pi} \approx 1.4$ ps for $\omega t \simeq \pi$.

The situation is slightly different for the junction of Ref. [3] since $RC \approx 0.1$ ns and $r = 0.078$ ns. Therefore, $ar^2 \simeq 30$ ps, while t_r, evaluated by Eq. (17), is about 10 ps. Thus, in the experimental situations considered above [3,4], whenever t_r would be accessible to a direct measurement it should be of the order of a few picoseconds, and not of the order of 10^2 ps, as deduced from semiclassical analyses.

We cannot overlook the fact that, in the model adopted, the real component of the delay originates only from the dissipative parameter, and that it vanishes as $a \to 0$. In the absence of dissipation, is the real component zero in any case? Certainly this cannot be true: any dephasing of the wave function across the boundaries of the barrier can contribute to the real component of delay: the phase time mentioned at the beginning. This contribution is usually ignored in semiclassical treatments, but will, more or less, always be present. An estimate of this contribution can be made according to the transition elements theory [21]. This theory supplies an approximate measurement of the reduction of traversal time (with respect to the semiclassical time) given by $\exp(-S/\hbar)$, where S is the reduced action between classical turning points of motion [12].

For physical systems under examination in Refs. [3] and [4], the action in the zero temperature limit ($S = \pi V_0/\hbar\omega$, V_0 being the barrier height [22]) is about 7–8 $\hbar$; therefore the reduction factor is of the order of 10^{-3}. This means that, for a semiclassical imaginary time of about 10^2 ps, the contribution to the real time duration should be one of subpicoseconds. However, finite temperature effects produce a lowering of the action and, consequently, an increase in the real time component likely up to picosecond scale. We can therefore conclude that the evaluation made here, performed on dissipative systems, supplies the main part of the real time which, however, remains distinctly different from the imaginary one.

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