Phonon approach to an array of traps containing Bose-Einstein condensates

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We investigate theoretically an array of traps containing Bose-Einstein condensates, taking into account tunneling of atoms between adjacent traps and atom-atom interactions within each trap. After an expansion valid in the limit of a large number of atoms, we end up studying a variation of the problem of phonons in a one-dimensional lattice. Analysis of these fictitous phonons and their vacua shows that the fluctuations in atom numbers and phases of the condensates behave qualitatively as in the much-studied case of two traps. [S1050-2947(99)06512-9]

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I. INTRODUCTION

The possibility of peculiar interference phenomena, such as an analog of the Josephson effect [1], continues to spur interest in the Bose-Einstein condensate in a double-well potential. Our focus is on fully quantized approaches that lend themselves to analyses of not only average quantities such as atom numbers in each potential well, but also for studies of atom number fluctuations and dynamics of condensate phases. The traditional way is to treat the atom number and the condensate phase as canonical conjugate variables [2–4]. However, with the entry of quantum-optics-oriented authors into the field, nowadays one more often sees the two-mode approximation [5–12]. In the case of a trap split into two, this amounts to taking into account two lowest-energy one-particle states of the double well. We have recently commented on the relations between the two main approaches [12].

On the other hand, a system of multiple wells has recently been found to present intriguing features in the experiments [13]. Briefly, a one-dimensional optical lattice is erected in a more or less homogeneous condensate, so that trapping is in many wells rather than two. In these experiments the optical lattice is oriented vertically and suspends the atoms against gravity. However, atoms slowly leak out of the traps and fall down. Atoms leaking from the traps are seen to combine into a train of pulses. The interpretation is that the relative phases of the condensates in the traps govern the interference underlying the atom pulses.

We assume here and in the rest of this paper that gravity and leakage of atoms from the traps are weak perturbations, and that the effects of gravity on the dynamics of the condensates may be ignored. Even with these restrictions, the experiments of Ref. [13] still present obvious conceptual questions. First, what is the initial state of the system after the optical lattice has been set up? For instance, what are the initial atom number fluctuations and phase fluctuations between the wells? Second, what is the evolution of atom numbers and phases? For example, if phase correlations between the wells deteriorate in time, then so do the distinct pulsations in the flux of atoms. These questions make the subject of the present paper.

The process of dynamically dividing a trap and the ensuing state have recently elicited some controversy [14,15]. We have sought to clarify the issue by studying the splitting of a trap into two by erecting a potential wall in the middle [12]. We make the two-mode approximation, and an expansion valid in the limit of large atom number N. The problem of atom numbers and phases is thereby reverted to studies of a simple harmonic oscillator.

In the present paper we carry out counterparts of the two-mode approximation and large-N expansion for the case of multiple traps. This is in contrast to the recent applications of the Hubbard model to cold atoms in an optical lattice [16], in which small filling factors are considered. Instead of one harmonic oscillator, a system of coupled harmonic oscillators emerges. We end up studying a straightforward variation of the standard textbook exercise [17] of phonons in a one-dimensional lattice. The result is that, for the most part, one may correctly surmise the behavior of atom number and phase fluctuations and their time evolution from the known results for two traps. The main difference, a rather minor one, seems to be that the kind of revivals of fluctuations that one would see in a double trap are absent.

The expansion pertinent to the limit of a large atom number is described in Sec. II. The analogy with phonons is the subject of Sec. III. We discuss a second-quantized representation of the phonons, as well as the ground state and the time-dependent properties of this peculiar one-dimensional lattice in which atom number and condensate phase take the places of lattice displacement and momentum of a lattice atom. The remarks in Sec. IV conclude the paper.

II. FEW-MODE SYSTEM FOR MANY ATOMS

We consider a one-dimensional (1D) array of equivalent traps containing Bose-Einstein condensates. The total number of atoms is denoted by N and the total number of traps by K, both for convenience taken to be even integers. The average number of atoms in each well, $n=N/K$, is regarded as an integer, too, inasmuch as such an assumption helps in the argument. We assume that the gas is weakly interacting, so that the contribution to the chemical potential from atom-atom interactions, for brevity simply called chemical potential, is much smaller than the frequency characterizing the confinement of the atoms in each trap. Here and throughout our verbal descriptions we use the terms energy and frequency interchangeably, but in the equations we always

write the constant $\hbar$ explicitly.

As before, we may add a polynomial of the conserved total number of atoms to the Hamiltonian without any change in the ensuing dynamics. The two-mode ($K=2$) Hamiltonian of our previous paper [12] may therefore also be written as

$$\frac{H}{\hbar}=-\frac{\delta}{2}(a_1^\dagger a_2+a_2^\dagger a_1)+2\kappa[(a_1^\dagger a_1)^2+(a_2^\dagger a_2)^2]. \quad (1)$$

Here we use the indices 1 and 2 for the traps, instead of l and r of Ref. [12]; the boson operator a_1 annihilates one atom from trap 1, and so forth. Given one-particle wave functions $\psi_{1,2}(\mathbf{r})$ representing atoms localized in traps 1 and 2, trapping potential $V(\mathbf{r})$, atom mass m, and s-wave scattering length a, the parameters have the expressions

$$\delta=\frac{2}{\hbar}\,\mathrm{Re}\left\{\int d^3r\,\psi_2^*(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})\right]\psi_1(\mathbf{r})\right\}, \quad (2a)$$

$$\kappa=\frac{2\pi\hbar a}{m}\int d^3r|\psi_1(\mathbf{r})|^4=\frac{2\pi\hbar a}{m}\int d^3r|\psi_2(\mathbf{r})|^4. \quad (2b)$$

In practice, though, these coefficients might often have to be inferred from phenomenological considerations. For instance, the tunneling rate δ could be defined in such a way that in the absence of atom-atom interactions, $\hbar\delta$ gives the energy of the lowest excited state of the system. Similarly, in the absence of tunneling, $\mu=4n\kappa$ would be the chemical potential (in units of frequency), given $n=N/2$ atoms in each well.

Next suppose that there are $K\geqslant 2$ equivalent wells, and take into account tunneling between adjacent wells only. Generalizing from Eq. (1), we write the Hamiltonian as

$$\frac{H}{\hbar}=\sum_{i=1}^{K}\left[-\frac{\delta}{4}(a_{i-1}^\dagger a_i+a_i^\dagger a_{i-1}+a_{i+1}^\dagger a_i+a_i^\dagger a_{i+1})+2\kappa(a_i^\dagger a_i)^2\right]. \quad (3)$$

We have cast the Hamiltonian in a somewhat uneconomical form, displaying Hermiticity explicitly. This will greatly simplify our subsequent calculations. With the interpretation that the operators with the indices 0 and $K+1$ are nonexistent and that the corresponding terms are ignored, Eqs. (1) and (3) become the same in the case $K=2$. However, our present topic is the limit with K at least moderately large. What happens at the $i=1$ and $i=K$ ends of the Hamiltonian is then hopefully a small perturbation. For mathematical convenience we from now on resort to the analog of periodic boundary conditions; for the indices of boson operators appearing in Eq. (3), $0\equiv K$ and $K+1\equiv 1$.

In both Hamiltonians (1) and (3), only intrawell nonlinearities are present. In the two-mode case the justification was that, for a weakly interacting gas, atom-atom interactions are comparable to the effects of tunneling only when tunneling is slow, i.e., the overlap between the wave functions localized in the two traps is small. Hence, quartic interwell terms depending on the overlap of the squares of the wave functions are smaller still [12]. Here we assume that interwell terms may be ignored on similar grounds. The limit $\delta/\kappa\to\infty$ should describe the situation when the atoms tunnel freely, so that in effect there are no traps. In the opposite limit $\delta/\kappa\to 0$ the wells are so deep that the atoms cannot tunnel between them at all.

We write the state vector of the boson system in the usual way as a superposition of number states for the wells,

$$|\psi\rangle=\sum_{N_1,\ldots,N_K}C(N_1,\ldots,N_K)|N_1,\ldots,N_K\rangle. \quad (4)$$

The time-independent Schrödinger equation is equivalent to a set of eigenvalue equations for the coefficients C,

$$\begin{aligned}-\frac{\delta}{4}\sum_i\ &[\sqrt{N_{i-1}(N_i+1)}C(\ldots,N_{i-1}-1,N_i+1,N_{i+1},\ldots)\\ &+\sqrt{N_i(N_{i-1}+1)}C(\ldots,N_{i-1}+1,N_i-1,N_{i+1},\ldots)\\ &+\sqrt{N_{i+1}(N_i+1)}C(\ldots,N_{i-1},N_i+1,N_{i+1}-1,\ldots)\\ &+\sqrt{N_i(N_{i+1}+1)}C(\ldots,N_{i-1},N_i-1,N_{i+1}+1,\ldots)\\ &+2\kappa N_i^2C(\ldots,N_{i-1},N_i,N_{i+1},\ldots)]\\ &=\frac{E}{\hbar}C(N_1,\ldots,N_K) \end{aligned} \quad (5)$$

In accordance with the periodic boundary conditions, the indices again wrap around; e.g., $N_{K+1}\equiv N_1$.

To develop the left-hand side, we take a few algebraic steps. First, we develop the coefficients C into MacLaurin expansions [12,18], as in

$$\begin{aligned}&C(\ldots,N_{i-1}-1,N_i+1,N_{i+1},\ldots)\\ &\simeq C(\ldots,N_{i-1},N_i,N_{i+1},\ldots)\\ &\quad+\left(\frac{\partial}{\partial N_i}-\frac{\partial}{\partial N_{i-1}}\right)C(\ldots,N_{i-1},N_i,N_{i+1},\ldots)\\ &\quad+\frac{1}{2}\left(\frac{\partial^2}{\partial N_{i-1}^2}-2\frac{\partial^2}{\partial N_{i-1}\,\partial N_i}+\frac{\partial^2}{\partial N_i^2}\right)\\ &\quad\times C(\ldots,N_{i-1},N_i,N_{i+1},\ldots). \end{aligned} \quad (6)$$

Second, we write the atom numbers as

$$N_i=N/K+n_i\equiv n+n_i, \quad (7)$$

and expand the square root factors into Laurent series in the presumably large average number of atoms in a well, n. For instance,

$$\sqrt{N_{i-1}(N_i+1)}\simeq n+\frac{1+n_{i-1}+n_i}{2}-\frac{(1+n_{i-1}-n_i)^2}{8n}. \quad (8)$$

The change of variables $N_i\to n+n_i$ is understood in the coefficients C as well. Third, after inserting these expansions into Eq. (5), in each order of the derivatives and separately

for the terms proportional to δ and to κ, we examine the contributions to Eq. (5) order by order in $1/n$.

The contribution that is zeroth order in the derivatives and proportional to κ is simply

$$K(0,\kappa)=2\kappa\left[\sum_i (n^2+2nn_i+n_i^2)\right]C. \tag{9}$$

But the physics following from the Hamiltonian (3) remains unchanged, except for a trivial shift of the zero of energy, if we add a constant of the motion to H. For instance, we might use H' defined by

$$\frac{H'}{\hbar}=\frac{H}{\hbar}-2\kappa\sum_i (n^2+2na_i^\dagger a_i) \tag{10}$$

instead of the Hamiltonian H. However, for the Hamiltonian H' the term corresponding to Eq. (9) would simply read

$$K(0,\kappa)=2\kappa\sum_i n_i^2 C. \tag{11}$$

We ignore the attendant shift in the zero of energy, and use this form in lieu of Eq. (9) in what follows.

On the other hand, the three leading terms in n proportional to δ in the zeroth-order derivative are

$$\begin{aligned}K(0,\delta)&=\delta\sum_i \left\{-n-\frac{1}{4}[(2+n_{i-1}+2n_i+n_{i+1})]\right.\\&\left.+\frac{1}{16n}[2+(n_i-n_{i-1})^2+(n_i-n_{i+1})^2]\right\}C\\&=\delta\sum_i\left[-n-\frac{1}{2}(1+2n_i)+\frac{1}{8n}\right]C\\&+\frac{\delta}{8n}\sum_i (n_i-n_{i-1})^2 C.\end{aligned} \tag{12}$$

In the second equality we have used the periodic boundary conditions. The first sum can then be removed with the same kind of tricks we already used in conjunction with the term $K(0,\kappa)$. We finally have

$$K(0,\delta)=\frac{\delta}{8n}\sum_i (n_i-n_{i-1})^2 C. \tag{13}$$

As far as we can tell, this is the only instance during our analysis in which the periodic boundary conditions are essential. Without them and for the case $K>2$, we do not know how to eliminate all contributions that are zeroth order in n from the term $K(0,\delta)$.

The leading order in n in the terms proportional to δ (there is no contribution proportional to κ) that contain first derivatives is n^{-1}. With the aid of periodic boundary conditions we write

$$K(1,\delta)=\frac{\delta}{4n}\sum_i (-n_{i-1}+2n_i-n_{i+1})\frac{\partial}{\partial n_i}C. \tag{14}$$

On the other hand, the leading power of n in the terms that contain second derivatives may be written, once more using periodic boundary conditions, as

$$K(2,\delta)=-\frac{n\delta}{2}\sum_i\left(\frac{\partial}{\partial n_i}-\frac{\partial}{\partial n_{i-1}}\right)^2 C. \tag{15}$$

Suppose now that the characteristic order of magnitude of the number n_i is $n_i\sim\Delta n$ and correspondingly the order of magnitude of the n_i derivative is $\partial/\partial n_i\sim 1/\Delta n$. The relative size of the first- and second-order derivative terms is then estimated as

$$\left|\frac{K(1,\delta)}{K(2,\delta)}\right|\sim\left(\frac{\Delta n}{n}\right)^2. \tag{16}$$

It will transpire from the solutions that $\Delta n/n\lesssim 1/\sqrt{n}$, so in the limit of large atom number the first derivatives should be negligible in comparison with the second derivatives.

All told, we have found a partial differential equation for the coefficients $C(n_1,\ldots,n_K)$, which in the case of the time independent Schrödinger equation reads

$$\begin{aligned}\sum_{i=1}^K\left[-\frac{n\delta}{2}\left(\frac{\partial}{\partial n_i}-\frac{\partial}{\partial n_{i-1}}\right)^2+\frac{\delta}{8n}(n_i-n_{i-1})^2+2\kappa n_i^2\right]C\\=\frac{E}{\hbar}C.\end{aligned} \tag{17}$$

For the time-dependent Schrödinger equation, the right-hand side is modified in the obvious way. Periodic boundary conditions are assumed, so that $n_{K+1}\equiv n_1$ and $n_0\equiv n_K$. Besides, it should be noted that the conservation of atom number is used in the derivation. Solutions to Eq. (17) should only be considered in the subspace with $\Sigma_i n_i=0$. As a matter of fact, it may be seen easily that the quantity $\Sigma_i n_i$ commutes with the differential operator on the left-hand side of Eq. (17), and indeed is a conserved quantity. Finally, as we have already mentioned, periodic boundary conditions are not necessary for two wells, $K=2$. In that case, using the condition that $n_1+n_2=0$, it may be verified that Eq. (17) coincides with Eq. (17) of Ref. [12].

III. ANALOGY WITH 1D PHONONS

A. Phonon formalism

The Schrödinger equation (17) is derived from the Hamiltonian

$$\frac{\mathcal{H}}{\hbar}=\sum_{k=1}^K\left[\frac{n\delta}{2}(\phi_k-\phi_{k-1})^2+\frac{\delta}{8n}(n_k-n_{k-1})^2+2\kappa n_k^2\right], \tag{18}$$

where $\phi_k\equiv -i\partial/\partial n_k$ satisfy

$$[n_k,\phi_l]=i\delta_{kl}. \tag{19}$$

Occasionally we also think of n_k and ϕ_l as classical canonical variables with the Poisson brackets $\{n_k,\phi_l\}=\delta_{kl}$ among them. As the Hamiltonian is quadratic, the classical and the quantum theories are much alike. The system is akin to the

one encountered in connection with the linear (one-dimensional) string of masses connected by linear springs, except that the coupling between the neighboring sites is more complicated than in the usual textbook examples [17]. Our immediate goal is to formulate a version of the theory of ''lattice vibrations'' for the present system.

We begin by noting the classical equations of motion for the positions and momenta, or equally well, the Heisenberg equations of motion for the corresponding quantum operators:

$$\dot{n}_k = n\delta[-\phi_{k+1}+2\phi_k-\phi_{k-1}], \tag{20a}$$

$$\dot{\phi}_k = \frac{\delta}{4n}[n_{k+1}-2n_k+n_{k-1}]-4\kappa n_k. \tag{20b}$$

Combination of these two gives

$$\ddot{\phi}_k = \frac{\delta^2}{4}[-\phi_{k+2}+4\phi_{k+1}-6\phi_k+4\phi_{k-1}-\phi_{k-2}] - 4n\kappa\delta[-\phi_{k+1}+2\phi_k-\phi_{k-1}]. \tag{21}$$

The equation of motion for each ϕ_k (and n_k) is translationally invariant, given the periodic boundary conditions. We thus try the standard ansatz

$$\phi_k = \beta_q e^{i(kq-\omega_q t)}, \tag{22}$$

where q, β_q, and ω_q are constants. This ansatz satisfies the periodic boundary conditions and defines a complete, linearly independent set of solutions, provided the values of the analog of the phonon wave number q are properly restricted. Here we choose the set of wave numbers

$$q = \frac{2\pi}{K}m, \quad m = -\frac{K}{2}, \ldots, \frac{K}{2}-1. \tag{23}$$

It may be seen immediately that the ansatz (22) succeeds if the dispersion relation of the phonons reads

$$\omega_q^2 = 4\delta^2\sin^4\frac{q}{2}+16n\kappa\delta\sin^2\frac{q}{2}. \tag{24}$$

As always in analysis involving phonons, it is crucial to keep in mind that, for the set of wave numbers (23), the vectors

$$x_q \equiv \left(\frac{e^{iq}}{\sqrt{K}}, \frac{e^{2iq}}{\sqrt{K}}, \ldots, \frac{e^{Kiq}}{\sqrt{K}}\right)^T$$

make an orthonormal basis for K-dimensional complex vectors with respect to the usual vector norm. The $q=0$ mode is problematic in many respects. In what follows, we keep track of it explicitly.

The equations of motion (20) imply that if Eq. (22) is a solution for ϕ_k, then

$$n_k = \frac{4n\delta\sin^2\frac{q}{2}}{-i\omega_q}\beta_q e^{i(kq-\omega_q t)} \tag{25}$$

is a solution for n_k. The limit $q\to 0$ leads to $n_k\to 0$, so that within the present framework there is no representation for the state of the affairs that all n_k are equal [19]. However, it follows from the definition (7) that the case with $n_k \equiv \overline{n} \neq 0$ corresponds to a situation in which the total atom number has changed by $K\overline{n}$. This cannot happen under the Hamiltonian (3). The vanishing of the displacements n_k for the phonon with $q=0$ is as it should be. On the other hand, constants $n_k \equiv 0$ and $\phi_k \equiv \overline{\phi}$ make a solution to the equations of motion, no matter what the value $\overline{\phi}$. There is nothing wrong with a nonzero canonical momentum $\overline{\phi}$ for the phonon $q=0$.

It remains to find the representation of the displacements and momenta n_k and ϕ_k in terms of phonon annihilation and creation operators α_q and $\alpha_q^\dagger$. An argument just as those always made while quantizing phonons [17] shows that

$$n_k = i\sum_q \sqrt{\frac{2n\delta\sin^2\frac{q}{2}}{K\omega_q}}(\alpha_q e^{ikq}-\alpha_q^\dagger e^{-ikq}), \tag{26a}$$

$$\phi_k = \sum_q \sqrt{\frac{\omega_q}{8Kn\delta\sin^2\frac{q}{2}}}(\alpha_q e^{ikq}+\alpha_q^\dagger e^{-ikq}) \tag{26b}$$

is a possible expansion for the operators n_k and ϕ_k. The $q=0$ terms in these expressions admittedly are ill-defined, and so is even the $q\to 0$ limit of the $q=0$ term in ϕ_k. We handle such problems by taking the $q\to 0$ limit for the singular $q=0$ term only at the end of our calculations. So far in our analysis, this procedure has never produced an obviously wrong result.

Conversely, it is easy to verify that if α_q have the boson commutation relations, then n_k and ϕ_k from Eqs. (26) have the proper canonical commutators (19). Moreover, using Eqs. (26), the Hamiltonian (18) may be cast into the ordinary independent-phonon form,

$$\frac{H}{\hbar} = \sum_q \omega_q\left(\alpha_q^\dagger\alpha_q+\frac{1}{2}\right). \tag{27}$$

B. Ground state

We next study the ground state of the system of the condensates in the K wells. In phonon language, we investigate the vacuum of the phonons satisfying $\alpha_k|0\rangle = 0$. Nonetheless, zero-point fluctuations are still present.

First consider atom number fluctuations in a well. We trivially have

$$(\Delta n)^2 \equiv \langle 0|n_k^2|0\rangle = \sum_q \frac{2n\delta\sin^2\frac{q}{2}}{K\omega_q}$$

$$\simeq \frac{n\delta}{\pi}\int_{-\pi}^{\pi} dq\,\frac{\sin^2\frac{q}{2}}{\omega_q} = \frac{2n}{\pi}\arctan\sqrt{\frac{\delta}{4n\kappa}}. \tag{28}$$

In the approximate equality we have replaced the sum over q by an integral, a permissible step for $K \gg 1$.

The key parameter for atom number fluctuations obviously is

$$\xi = \frac{\delta}{4n\kappa}, \tag{29}$$

the ratio of the tunneling rate between adjacent wells and the chemical potential in each well. For $\xi \gg 1$ the atoms tunnel easily. One might then expect that they are shuffled among the wells at random. In the limit $K \gg 1$ this should give a Poissonian atom number in each well. For the average of n atoms in each well, we in fact find the number fluctuations $\Delta n \simeq \sqrt{n}$. In the opposite limit $\xi \ll 1$ it costs a lot of atom-atom interaction energy to move an atom from one well to another, and so tunneling is inhibited. Correspondingly, the number fluctuations are sub-Poissonian, and read

$$\Delta n \simeq \frac{1}{\sqrt{\pi}} \left(\frac{n\delta}{\kappa} \right)^{1/4}. \tag{30}$$

Both of these limits are similar to those found in Ref. [12], the main difference being that our present expressions are cast in terms of the number of atoms per trap n rather than the total number of atoms N. Expression (30) is questionable if $\Delta n \lesssim 1$, as then the expansion (6) fails. Perturbation theory similar to the development in Ref. [12] could again handle the case $\Delta n \lesssim 1$, but we do not go into this.

We now come to the fluctuations of the canonical momenta ϕ_k, conjugates of atom numbers n_k. The notation ϕ_k is no accident; though such an interpretation has certain technical weaknesses [12], we pragmatically regard ϕ_k as the phase operator for the BEC in the well k. There is nothing to fix the global phase for the condensates in an array of wells, and the fluctuations of any given ϕ_k are correspondingly formally infinite. However, the global phase is not a physical observable anyway. Experiments, including the experiments in Ref. [13], rather depend on the relative phases between the wells. We study phase fluctuations between the wells m positions apart,

$$\begin{aligned}
[\Delta\phi_m]^2 &\equiv \langle 0 | (\phi_{k+m} - \phi_k)^2 | 0 \rangle \\
&= \sum_q \frac{\omega_q \sin^2 \frac{mq}{2}}{2Kn\delta \sin^2 \frac{q}{2}} \\
&= \frac{1}{Kn\sqrt{\xi}} \sum_{p=-K/2}^{K/2-1} \frac{\sqrt{1+\xi \sin\left(\frac{p\pi}{K}\right)^2} \sin\left(\frac{mp\pi}{K}\right)^2}{\left|\sin\left(\frac{p\pi}{K}\right)\right|}
\end{aligned} \tag{31a}$$

$$\simeq \frac{1}{\pi n\sqrt{\xi}} \int_0^\pi dq \frac{\sqrt{1+\xi \sin\left(\frac{q}{2}\right)^2} \sin\left(\frac{mq}{2}\right)^2}{\sin\left(\frac{q}{2}\right)}. \tag{31b}$$

It is straightforward to compute the sum in Eq. (31a) exactly, numerically, for a wide range of the parameters ξ, m and K. In this way we have seen that phase fluctuations do not dramatically depend on the distance between the condensates m. We thus use the $m=1$ case as the generic estimate for phase fluctuations between different condensates.

In this vein, we discuss the spread of the relative phase between two neighboring condensates. The integral (31b) gives

$$[\Delta\phi_1]^2 \simeq \frac{\sqrt{\xi} + (1+\xi)\arctan\sqrt{\xi}}{\pi n \xi}, \tag{32}$$

which for $\xi \to 0$ and $\xi \to \infty$ becomes $2/(\pi n\sqrt{\xi})$ and $1/2n$, respectively. It is easy to see that in the latter limiting case the phonon vacuum is a minimum uncertainty state for the operators $n_{k+1} - n_k$ and $\phi_{k+1} - \phi_k$. By comparing with the exact sum form, we have found that the integral approximation is increasingly better, the larger is the number of the wells K. For instance, Eq. (32) is accurate to better than 10% for all ξ once $K \geqslant 44$.

C. Time dependence

In the experiments of Ref. [13] an essentially free condensate is captured in an optical lattice. This is an inherently time-dependent process. We model it by letting the tunneling rate δ decrease on a characteristic time scale τ from a ''very large'' initial value to whatever the final value is. The specific time dependence, if needed, could be $\delta \propto e^{-t/\tau}$. Even if the process started from a steady state, say, ground state of the free condensate, the capture of the condensates into the wells need not be adiabatic, and may leave behind a nonstationary state. We focus on the concomitant time dependence of the relative phases between the wells. Experimentally, such evolution results in changes with time in the shapes of the atom pulses falling from the lattice.

The first issue is adiabaticity itself. Here we without further ado adopt a simple argument that worked exceedingly well in the case of two wells [12]. We assume that a given phonon mode q evolves adiabatically until the tunneling frequency δ has becomes so small that the ensuing phonon frequency ω_q satisfies

$$\omega_q \simeq \frac{1}{2\pi\tau}. \tag{33}$$

The corresponding value of δ is

$$\delta_q \simeq \frac{1}{4\pi\tau \sin^2 \frac{q}{2} [8\pi n\kappa\tau + \sqrt{1 + (8\pi n\kappa\tau)^2}]}. \tag{34}$$

This argument suggests two conclusions. First, it is increasingly hard to stay adiabatic, the lower one goes in phonon

frequencies. This is as expected. Lower frequencies mean longer wavelengths, so the system must adjust itself over an increasing number of neighboring wells. This takes the longer, the larger the length scale. Second, if the time scale for variation of the tunneling rate is shorter than the inverse of the chemical potential, $\tau \lesssim (4n\kappa\tau)^{-1}$, adiabaticity breaks down for all modes even before the system starts sampling atom-atom interactions at all; $\delta_q \gtrsim 4n\kappa\tau$ for all phonon modes q.

The scenario of the breakdown of adiabaticity we adopt here is that, once the tunneling rate reaches the critical value δ_q, the mode q freezes to the vacuum state that prevails at the time of $\delta(t)=\delta_q$. While the present method could easily be adapted to more general situations, from now on we only discuss ''instantaneous'' turning on of the optical lattice, $\tau \lesssim (4n\kappa\tau)^{-1}$. This means that $\delta_q \gtrsim 4n\kappa\tau$ applies to all modes, and the condensate remains in its original unconfined state throughout the turning on of the optical lattice.

As preparation for the ensuing argument, we note that although the operators n_k and ϕ_k are the same no matter what the parameters, the creation and annihilation operators and their associated properties such as the vacuum depend on the tunneling rate δ, chemical potential $\mu=4n\kappa$, and mode frequency ω_q. Suppose we have the annihilation operators α_q corresponding to the parameters δ, κ, ω_q, and another set of annihilation operators $\overline{\alpha}_q$ for the parameters $\overline{\delta}$, $\overline{\kappa}$, $\overline{\omega}_q$. A simple manipulation based on the observation that the matrix $M_{kq}=e^{ikq}/\sqrt{K}$ is unitary gives the connection

$$\overline{\alpha}_q = \cosh\theta_q \alpha_q + \sinh\theta_q \alpha^{\dagger}_{-q}, \tag{35}$$

$$\cosh\theta_q = \frac{1}{2}\left(\sqrt{\frac{\overline{\delta}\overline{\omega}_q}{\delta\omega_q}} + \sqrt{\frac{\delta\omega_q}{\overline{\delta}\overline{\omega}_q}}\right),$$
$$\sinh\overline{\theta}_q = \frac{1}{2}\left(\sqrt{\frac{\overline{\delta}\overline{\omega}_q}{\delta\omega_q}} - \sqrt{\frac{\delta\omega_q}{\overline{\delta}\overline{\omega}_q}}\right). \tag{36}$$

Modes q and $-q$ are pairwise coupled, and have to be discussed together. The vacua of the modes $\overline{\alpha}_q$ and $\overline{\alpha}_{-q}$, the state satisfying $\overline{\alpha}_q|0,0\rangle^{-}_{\alpha,q}=\overline{\alpha}_{-q}|0,0\rangle^{-}_{\alpha,q}=0$ may be expressed in terms of the number states $|n_q,n_{-q}\rangle_{\alpha,q}$ corresponding to the modes α_q, α_{-q} as

$$|0,0\rangle^{-}_{\alpha,q} = \frac{1}{\cosh\theta_q}\sum_{n=0}^{\infty}(-\tanh\theta_q)^n|n,n\rangle_{\alpha,q}. \tag{37}$$

The vacuum $|0,0\rangle^{-}_{\alpha,q}$ is a two-mode squeezed vacuum for the modes α_q and α_{-q} [20]. Quantum optics lingo aside, expression (37) may be verified easily by letting $\overline{\alpha}_q$ (and $\overline{\alpha}_{-q}$) from Eq. (35) act on it, which gives zero.

Let us finally address phase diffusion, or more precisely, phase dispersion. Mathematically, the parameter δ suddenly switches from a value indicating ready tunneling to a final value that might signal strong confinement of the atoms to their wells. The state does not switch suddenly, though, but the trapped lattice of BEC's finds itself in the wrong vacuum corresponding to the initial δ. We use the bar to denote the parameters and phonon operators at the prelattice stage; then the wrong vacuum $|0\rangle_W$ for each doublet of phonon modes q, $-q$ is given by Eq. (37). Moreover, after the capture, we let the system evolve in time, and ask about phase fluctuations between the wells as a function of time. By virtue of the Hamiltonian (27), in the Heisenberg picture each operator α_q simply picks up the time-dependent phase factor $e^{-i\omega_q t}$. A small amount of algebra gives

$$[\Delta\phi_m(t)]^2 \equiv {}_W\langle 0|[\phi_{k+m}(t)-\phi_k(t)]^2|0\rangle_W$$
$$= \sum_q \frac{\omega_q \sin^2\frac{mq}{2}}{4Kn\delta\sin^2\frac{q}{2}}\left[\left(\frac{\delta\overline{\omega}_q}{\overline{\delta}\omega_q}+\frac{\overline{\delta}\omega_q}{\delta\overline{\omega}_q}\right) + \cos\omega_q t\left(\frac{\delta\overline{\omega}_q}{\overline{\delta}\omega_q}-\frac{\overline{\delta}\omega_q}{\delta\overline{\omega}_q}\right)\right]. \tag{38}$$

We first assume that the lattice is switched from the regime in which tunneling completely dominates to the regime when tunneling is negligible. Correspondingly, we expand the expression inside the sum in Eq. (38) into the leading orders in $1/\overline{\delta}$ and δ, and convert the sum into an integral in the standard way. We obtain

$$[\Delta\phi_m(t)]^2 \simeq \frac{1}{2n} + 8n\kappa^2 t^2. \tag{39}$$

The structure of the result is the same as was found earlier for a double well [21,7]. By the time $\Delta\phi_m(t)\sim 1$, the phases between the wells have drifted sufficiently far apart that in an experiments such as those in Ref. [13] neat periodic pulses of atoms are replaced by a randomly fluctuating atomic flux. This takes about the time $t\sim\sqrt{2n}\,(4n\kappa)^{-1}$, the square root of the number of atoms in each well times the inverse of the chemical potential.

Quantitative comparisons with the experiments in Ref. [13] are hampered by the fact that the number of atoms captured in the traps was not uniform; there were more atoms where the original condensate was denser. Besides, some of the parameters needed in the theory are not stated explicitly. This kind of uncertainty notwithstanding, we pause for an estimate of the dephasing time scale in the experiments [13]. We assume that the traps were turned on instantaneously. It seems reasonable to take $n\sim 200$ as the number of atoms in each trap, and use the value for the density $\rho\sim 10^{13}/\mathrm{cm}^3$, which gives the chemical potential $\hbar\mu/k_B\sim 4$ nK. The resulting dephasing time is $\sqrt{2n}\,(4n\kappa)^{-1}\sim 40$ ms. This is comparable to the experimental time scales. It need not be a coincidence that degradation of the interference was observed in Ref. [13] at higher atom densities.

Additional insights are gained if in the second step of the expansion we do not outright take the limit $\delta\to 0$, but just assume $\delta \ll 4n\kappa$. We thus expand as in

$$\cos\omega_q t = \cos\left[4\sqrt{n\kappa\delta}\left|\sin\frac{q}{2}\right|\sqrt{1+\frac{\delta}{4n\kappa}\sin^2\frac{q}{2}}\,t\right] \tag{40}$$
$$\simeq \cos\left[4\sqrt{n\kappa\delta}\left|\sin\frac{q}{2}\right|t\right]. \tag{41}$$

Instead of Eq. (39), we have

$$[\Delta\phi_1(t)]^2=\frac{1}{2n}+\frac{2\kappa}{\delta}\,[1-J_0(4\sqrt{n\kappa\delta}\,t)], \tag{42}$$

where J_0 is the usual Bessel function. For $\delta\neq 0$, the spread of the phase difference does not grow without a bound.

In fact, since $-1\leqslant\cos\omega_q t\leqslant 1$, by virtue of Eq. (38) the phase spread should always remain bounded between the (lower and upper) values

$$[\Delta\phi_m]_l^2=\sum_q \frac{\overline{\omega}_q\sin^2\dfrac{mq}{2}}{2Kn\overline{\delta}\sin^2\dfrac{q}{2}} \tag{43}$$

and

$$[\Delta\phi_m]_u^2\simeq\sum_q \left(\frac{\overline{\delta}\omega_q}{\delta\overline{\omega}_q}\right)\frac{\omega_q\sin^2\dfrac{mq}{2}}{2Kn\delta\,\sin^2\dfrac{q}{2}}$$

$$\simeq\sqrt{\frac{4n\kappa}{\delta}}\sum_q \frac{\omega_q\sin^2\dfrac{mq}{2}}{2Kn\delta\sin^2\dfrac{q}{2}}. \tag{44}$$

In the case of the upper limit, we have taken $\overline{\delta}\gg 4n\overline{\kappa}$ and $\delta\ll 4n\kappa$. Equation (43) simply gives the spread of the phase in the initial state, prior to the optical lattice, whereas Eq. (44) displays the phase spread as appropriate to the vacuum for the final trap parameters δ, κ, *multiplied* by the factor $\sqrt{4n\kappa/\delta}$.

At $t=0$, $\cos\omega_q t=1$ for all q, all phonons are in phase, and the phase spread that prevails before trapping is realized. For the two-mode case there is only one nontrivial q mode in the problem, and for this mode at certain times $\cos\omega_q t=-1$ applies. Therefore the phase fluctuations as in Eq. (44) are also reached at certain times. The phase spread at those times is of the order $\sim\sqrt{n\kappa/\delta}$ larger than the spread appropriate for the ground state of the double well. Equation (32) gives the latter as $\sim\sqrt{\kappa/n\delta}$. The order of magnitude for the maximum phase spread is therefore $\sim\kappa/\delta$.

In the two-mode case one may easily see why there is the extra factor $\sqrt{4n\kappa/\delta}$ in the upper limit for phase spread in Eq. (44). Given the trap parameters, the $t=0$ state is a squeezed vacuum for the (one and only) nontrivial phonon mode, with excess fluctuations in atom number. As is usual for squeezed states [20], with time excess atom number fluctuations will evolve into excess phase fluctuations. After some more time, phase fluctuations again become smaller than the phase fluctuations in the true vacuum, and with $\cos\omega_q t=1$ the initial state should revive.

By virtue of Eq. (42), after a time of the order $\sim(n\kappa\delta)^{-1/2}$ a phase spread of the order $\sim\kappa/\delta$ is reached even in the many-trap system. This is somewhat surprising, in that different phonons fall out of phase with time and $\cos\omega_q t=-1$ cannot hold for all of them at the same time t. The analog of the revival of a two-trap system is incomplete, though. As the oscillations of J_0 in Eq. (42) decay away, with time the repeated revivals are washed out.

IV. CONCLUDING REMARKS

We have studied atom number and phase fluctuations for Bose-Einstein condensates in an array of traps, taking into account tunneling of atoms between adjacent traps and atom-atom interactions within each trap. The conclusions are generally as one would surmise on the basis of the much-studied case of a double-well trap. The main difference, a rather esoteric one, is that certain revivals of two traps are incomplete in the many-well system.

Some comments about the nature of the model are due. First consider the easy case of (deep) traps actually being present. Ignoring atom-atom interactions, the one-particle states then come in (tight) bands. One could think of a_k as boson operators corresponding to the localized Wannier states [17] constructed from the states of the lowest band. It is harder to imagine what a_k should be if the potential wells are shallow or nonexistent. We do not undertake any explicit constructions here, but vaguely think of a_k as boson operators that describe atoms localized approximately in the region where the kth trap would be.

When tunneling dominates ($\delta\gg n\kappa$), the system exhibits two features that suggest that our model is useful even in the limit with no traps at all. First, if the atoms were distributed among the would-be traps independently of one another, in the limit of a large number of traps one would expect a Poissonian atom number statistics for each trap site. As discussed under Eq. (29), this is consistent with the results from our model. Second, from Eq. (24) it may be seen that the spectrum of low-lying excited states for the lattice of traps is

$$\epsilon_l\simeq\frac{8\pi^2\delta l^2}{K^2},\quad l=\pm 1,\pm 2,\ldots. \tag{45}$$

This is as in a one-dimensional box with periodic boundary conditions. Higher excited states ($l\gtrsim K/2$) admittedly develop a peculiar spectrum with a negative effective mass. Inasmuch as only the lowest excitations need be considered, though, our model makes sense even in the limit when there are no traps at all. In that case the parameter δ may be seen to be essentially the zero-point energy if the atoms were confined to a region with a size comparable to the separation between the as of yet nonexistent traps.

Another tricky question, which we have already broached in Sec. II, is atom-atom interactions. We offer here an argument to estimate when our approach should be quantitatively useful. The key idea is that our method is valid if atom-atom interactions have to be taken into account only in the limit when the traps substantially restrict the hopping of the atoms.

The first question is, when is a trap deep enough to confine. Now, from the excitation spectrum it is obvious that in our model the width of the tight-binding band of states is of the order δ. We regard the atoms as confined as soon as the tunneling rate δ is smaller than the zero-point energy in a trap. Take L as the total length of the array of traps, then the size of one trap is $\sim L/K$ and the zero-point energy is E_0

$\sim\hbar^2K^2/mL^2$, where m again is the mass of an atom. We have confining traps if $\hbar\delta \lesssim E_0$.

On the other hand, given the s-wave scattering length a and density of atoms ρ, the atom-atom interaction energy per atom is well known to be $\sim\hbar^2 a\rho/m$. In our Hamiltonian the corresponding frequency $\propto n\kappa$ is proportional to atom number not density, so to estimate κ we take $\kappa\sim\hbar a/mV$, where V is the total volume of the atomic sample. For the sake of the argument we assume that prior to erecting the optical lattice the sample of atoms is essentially spherical, and take $V\sim L^3$. This makes $\kappa\sim\hbar a/mL^3$.

To conclude the estimate, we note that atom-atom interactions begin to matter when $n\kappa = N\kappa/K \gtrsim \delta$. It is therefore correct to ignore atom-atom interactions until the traps are confining if $n\kappa \lesssim E_0/\hbar$. Combining this with our preceding estimates for κ and E_0, we find that our model is valid in the presence of atom-atom interactions if $Na/L \lesssim K^3$.

One could think of variations of the reasoning that reduce the power coming with the number of traps K. Nevertheless, the standard condition for a weakly interacting gas, such that atom-atom interactions have little effect on the atom cloud, reads $Na/L \lesssim 1$. If the latter inequality applies, we believe that our modeling of the array of traps is quantitatively accurate all the way from a free gas to condensates trapped in the optical lattice.

Let us look at the issues of trapping and interactions from one more angle. In the limit $K\to\infty$ our formal theory has only two parameters, n and $\xi = \delta/4n\kappa$. We have no device in our technical development to make a difference between free (not trapped) atoms and noninteracting atoms, and so we occasionally and maybe confusingly speak of these situations as if they were the same. For a dilute enough gas, this is justified to some extent; one has to consider interactions only if the atoms *are* trapped.

The question about the effects of gravity remains. Specifically, should we expect gravity to change qualitatively the dynamics of the system? To this end we note that, to have an array of traps in the first place, the tunneling rate $\hbar\delta$ must be less than the zero-point energy E_0. But since trapping is due to an optical lattice, the trap length parameter should be $L/K\sim\lambda$, the wavelength of driving light. This makes the zero-point energy comparable to recoil energy of laser cooling, $E_0\sim\hbar^2/m\lambda^2$. Suppose we are right at the edge of trapping, so that $\hbar\delta\sim E_0$. Gravitation should be negligible if the difference in gravitational energy between successive traps is smaller than the tunneling rate; $mg\lambda \lesssim E_0$, or the acceleration of gravity g satisfies $g \lesssim \hbar^2/m^2\lambda^3$. Inserting typical parameters, this condition is seen to hold by a margin of several orders of magnitude. Again, one could imagine variations of the argument that put powers of K in the result, but such changes do not necessarily eat up the margin either.

A careful analysis of the effects of gravitation on the dynamics of a trap array is certainly warranted. Besides, it poses an intriguing theoretical problem. Gravitation removes the translational invariance of the trap array, which calls for a modification of the phonon picture we have exploited in the present paper. Nonetheless, as we believe that gravity does not invariably render the development of the present paper moot, we postpone such an analysis to a future occasion.

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[1] J. Javanainen, Phys. Rev. Lett. **57**, 3164 (1986); S. Grossman and M. Holthaus, Z. Phys. A **50**, 323 (1995); F. Dalfovo, L. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, 4213 (1996).
[2] A.J. Leggett and F. Sols, Found. Phys. **21**, 353 (1991).
[3] F. Sols, Physica B **194-196**, 1389 (1994).
[4] I. Zapata, F. Sols, and A.J. Leggett, Phys. Rev. A **57**, R28 (1998).
[5] M.W. Jack, M.J. Collett, and D.F. Walls, Phys. Rev. A **54**, R4625 (1996).
[6] G.J. Milburn, J. Corney, E.M. Wright, and D.F. Walls, Phys. Rev. A **55**, 4318 (1997).
[7] J. Javanainen and M. Wilkens, Phys. Rev. Lett. **78**, 4675 (1997).
[8] K. Mølmer, Phys. Rev. A **58**, 566 (1998).
[9] J. Ruostekoski and D.J. Walls, Phys. Rev. A **58**, R50 (1998).
[10] R.W. Spekkens and J.E. Sipe, Phys. Rev. A **59**, 3868 (1999).
[11] J.F. Corney, G.J. Milburn, and W. Zhang, Phys. Rev. A **59**, 4630 (1999).
[12] J. Javanainen and M. Yu. Ivanov, Phys. Rev. A **60**, 2351 (1999).
[13] B.P. Anderson and M.A. Kasevich, Science **282**, 1686 (1998).
[14] A.J. Leggett and F. Sols, Phys. Rev. Lett. **81**, 1344 (1998).
[15] J. Javanainen and M. Wilkens, Phys. Rev. Lett. **81**, 1345 (1998).
[16] D. Jacksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[17] Consult any advanced textbook of solid-state physics; e.g., C. Kittel, *Quantum Theory of Solids* (Wiley, New York, 1987).
[18] A. Imamoglu, M. Lewenstein, and L. You, Phys. Rev. Lett. **78**, 2511 (1997).
[19] To avoid special cases of mathematical interest only, we assume that $\delta\neq 0$ and $\kappa\neq 0$.
[20] P. Meystre and M. Sargent III, *Elements of Quantum Optics*, 2nd ed. (Springer, Berlin,1991); D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1995); M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
[21] M. Lewenstein and L. You, Phys. Rev. Lett. **77**, 3489 (1996).