Nonadiabatic control of Bose-Einstein condensation in optical traps

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(Received 28 December 1998)

We show that the nonequilibrium atomic phase-space density (degeneracy) of a Bose gas confined in the off-resonant optical trap can be manipulated by means of the breathing modes of the atomic oscillations in the trap. This new phenomenon opens a possibility to control the onset of the Bose-Einstein condensation in the atomic cloud loaded in an optical trap. The effect arises from the instability of squeezing oscillations of atomic phase-space distribution, initiated by the nonadiabatic change of the optical potential. The manipulation of the degree of degeneracy of the atomic distribution is achieved by means of dynamical control of the squeezing parameters. The physical consequences of the proposed mechanism are discussed. [S1050-2947(99)05212-9]

PACS number(s): 03.75.Fi, 61.43.Fs, 77.22.Ch, 75.50.Lk

I. INTRODUCTION

The coherent manipulation of the atomic center-of-mass motion in optical lattices by means of the nonstationary dipole potentials has provided new experimental capabilities to study the dynamical systems with time-dependent potentials [1–4]. The parametric nonadiabatic excitations of the optical lattice give rise to oscillations of the atomic momentum and coordinate distribution dispersions (breathing modes) [2–4] of the lattice and may be used for the manipulation of the coordinate or momentum dispersions of the atomic distribution by means of squeezing in phase space [3,5].

The breathing modes of atomic oscillations in the optical lattices have been observed experimentally with the use of the Bragg scattering techniques [2,4]. The observed decay of the oscillations may be due to both the dissipative and the dephasing effects. The dephasing, caused by nonlinearity effects, can lead to partial revivals (echoes) under certain circumstances. The echo effect for the breathing modes of atomic oscillations in the optical lattices incorporating the features of both spin (photon) [6,7] and the phonon [8] echo mechanisms has been predicted to exist in [9].

In a recent experiment [10], the optical trapping of a Bose-Einstein condensate has been reported. The condensate was obtained by means of evaporative cooling in a magnetic trap and then transferred into an optical trap with large detuning. As the authors point out, the optical confinement of the condensate provides many advantages in comparison to the magnetic traps. One of the important experimental observations in [10] was that the condensates were obtained in the optical trap even when it was loaded with noncondensed magnetically trapped atoms. The authors suggest that this is related to the effect of increasing of the phase-space density of the atomic system with the adiabatically changing shape of the confining potential [11]. As we show below, a similar effect of phase compression should also take place for the parametric changing of the anharmonic optical potential. This happens due to the fact that for the anharmonic potential, the effective role of anharmonicity depends on the temperature, and therefore the average potential changes nonparametrically in case of parametric change of the intensity of the optical lattice. As the authors of Ref. [11] point out, one needs to take into account some relaxation mechanism in order to escape from the conservation of phase volume imposed by the Liouville theorem. In the above cited work, it was assumed that this relaxation mechanism was provided by atomic collisions.

The present paper explores the effect of increasing the phase-space density (degeneracy degree) for the atomic system in the non-adiabatic regime. In this regime, a specific mechanism of relaxation has to be considered, since the dynamical time scale may not be much longer than the relaxation time. We show, that in the nonadiabatic case, the phase-space density can be increased by the nonlinear effects of the atomic breathing oscillations due to the instability of such oscillations with respect to the initial conditions. This relaxation mechanism is known as ''phase mixing'' and does not require atomic collisions. This effect opens a possibility of a fast nonadiabatic control of the onset of the Bose-Einstein condensation (BEC) in the atomic cloud loaded into an optical lattice.

II. THE MODEL AND BASIC EQUATIONS

An atom subjected to a off-resonant laser field experiences an energy shift of the ground state proportional to the intensity of the field. If the field is formed by a standing wave with large detuning, the effective potential for the atoms in the ground state is given by [12]

$$U(x,t)=V(t)[1-\cos(2qx)], \tag{1}$$

where x denotes the atomic center-of-mass coordinate, $V(t)$ is the amplitude of the dipole potential (proportional to the intensity of the laser field) which is assumed to be time-dependent, and q is the wave vector of the laser field.

Following the experimental situation described in Ref. [10], we assume that after the initial cooling to temperature T, the atoms are transferred into the optical trap with large detuning and located near the minimum of the optical potential. The temperature T is assumed to be higher than the BEC transition temperature T_c. Since the detuning is large, the dissipation effects are small and will be disregarded. Note that the dissipation effects may be sufficiently small even when the detuning is not large [4]. In this case, the approach presented below should still be applicable. In this paper we

will consider the case of temperatures T much larger than the energy $\hbar\omega_0$ of the atomic oscillations at the bottom of the potential. Consequently, no quantum dynamical effects will be taken into account. On the other hand, the temperature should be low enough to ensure that the atoms do not escape from the potential well. The periodicity of optical trapping potential is also disregarded. Therefore, the effective potential can be approximated as an oscillator with weak anharmonicity and time-dependent frequency given by

$$U(x,t)\approx\frac{m\omega(t)^2x^2}{2}+\frac{m\eta(t)x^4}{4}, \tag{2}$$

with the harmonic frequency $\omega(t)$ and anharmonicity parameter $\eta(t)$ corresponding to the potential (1) and given by

$$\omega(t)=2q\sqrt{\frac{V(t)}{m}},\quad \eta(t)=-\frac{8}{3}q^4\frac{V(t)}{m}, \tag{3}$$

where m is the atomic mass.

Following [13] and [5], we employ the Wigner function formalism [14] in order to study the nonequilibrium evolution of the atomic phase-space distribution in the optical trap above the BEC transition. As we show in the Appendix, in the range of parameters discussed above, the equation of motion for the Wigner function can be reduced to the Liouville equation. The quantum corrections can in principle be taken into account in the framework of the formalism presented below, but will be disregarded in the present paper. For Gaussian initial conditions and weak anharmonicity, the atomic Wigner function can be approximately represented in the form

$$\rho(p,x,t)=C(t)\exp[-\Phi(p,x,t)],$$

$$\Phi(p,x,t)=\alpha p^2+\beta px+\gamma x^2+\delta x^4+\epsilon px^3, \tag{4}$$

where the distribution parameters satisfy a closed set of equations given by

$$\frac{d}{dt}\alpha(t)=-\frac{1}{m}\beta(t),$$

$$\frac{d}{dt}\beta(t)=-\frac{2}{m}\gamma(t)+2m\omega^2\alpha(t),$$

$$\frac{d}{dt}\gamma(t)=m\omega^2\beta(t),$$

$$\frac{d}{dt}\delta(t)=m\omega^2\epsilon(t)+m\eta\beta(t),$$

$$\frac{d}{dt}\epsilon(t)=-\frac{4}{m}\delta(t)+2m\eta(t)\alpha(t). \tag{5}$$

In this approximation the coefficient $C(t)$ may be time-dependent, since the Wigner function should be normalized. The details are discussed in the Appendix.

The quantity $J=\alpha(t)\gamma(t)-\frac{1}{4}\beta^2(t)$ is related to the information entropy of the system $S_{inf}=-\langle\ln[\rho]\rangle$ where $\langle\ \rangle$ stands for averaging with respect to the distribution (4). In the coherent regime with no anharmonicity, $J=\text{const}$ [5]. From Eq. (5), it follows that J is also conserved in the present case.

The solution given by Eq. (4) can be considered as an extension of the results [15,5], to the case of weak anharmonicity of the trap and sufficiently long evolution times. Namely, as we discuss below, the system has to be in a phase-mixing regime. In this limit, the solution (4) follows from the exact result given by Eqs. (A16) and (A15) from the Appendix.

III. THE PHASE-SPACE VOLUME AND CRITERION OF BEC

The criterion of Bose-Einstein condensation (BEC) for the atoms trapped in the harmonic potential can be obtained analogously to the case of the ideal Bose gas [16]. Expressing the total number of particles N in the system in terms of the chemical potential μ and temperature T with the subsequent setting $\mu=0$ leads to the following criterion of BEC in a finite system [17]

$$N_c=\sum_{k=1}^{\infty}\frac{W_k}{1-W_k}, \tag{6}$$

where N_c is a critical total number of particles and $W_k=\exp(-E_k/T)$. Introducing the density of states $\rho(E)$, we obtain from Eq. (6) [17],

$$N_c=\sum_{k=1}^{\infty}\int_0^{\infty}dE\rho(E)\exp(-kE/T).$$

In the quasiclassical limit, the density of states is expressed in terms of the phase-space volume and the BEC criterion is reduced to

$$N_c=\sum_{k=1}^{\infty}\int\frac{d\Gamma}{(2\pi\hbar)^D}\exp[-kH(p,q)/T], \tag{7}$$

with $d\Gamma=\Pi_{n=1}^{D}dp_ndq_n$ being an element of the D-dimensional phase space. For the harmonic potential and $D>1$, the BEC criterion given by Eq. (7) yields

$$N_c=\left(\frac{T}{\hbar\omega}\right)^D\zeta_D(1), \tag{8}$$

where ω is the frequency of harmonic oscillations in the well and $\zeta_D(z)=z+z^2/2^D+z^3/3^D+\ldots$ is a Riemann ζ function. In one dimension, one has to exclude the ground-state population in order to avoid problems with singularities of the Riemann ζ function. For the harmonic potential, the BEC criterion is given by [17]

$$N_c=\frac{T}{\hbar\omega}\ln\left(\frac{2T}{\hbar\omega}\right), \tag{9}$$

where ω is the frequency of harmonic oscillations in the well. One should note, that the quantity T/ω corresponds to the average phase volume $\langle E\rangle/\omega$ of the atomic system, which is an adiabatic invariant for the slow variation of the parameters of the harmonic potential [18]. Following the ex-

perimental situation of Ref. [10], we will concentrate on the one-dimensional case in the rest of this section. The generalization of the results presented below to $D>1$ is straightforward.

In order to analyze the role of the anharmonic corrections to the confining potential in the quasiclassical approximation, we will relate the BEC criterion to the average phase volume of the atomic system. From Eq. (7), it follows that in this case the BEC criterion is formulated in terms of the functions $Z_k(T)$ defined by

$$Z_k(T)=\int d\Gamma \exp[-kH(p,q)/T], \tag{10}$$

where $Z_1(T)=Z(T)$ is the average phase volume of the particle with the Hamiltonian H. If the energy E of the classical Hamiltonian system is fixed, the phase volume is an adiabatic invariant of the system [18]. For the one-dimensional (1D) Hamiltonian $H=p^2/2m+U(x)$, the phase volume is given by

$$I(E)=\oint dx\sqrt{2m[E-U(x)]}. \tag{11}$$

Making use of the identity $\partial I(E)/\partial E=t(E)$ [18], the element of phase volume is expressed as $d\Gamma=dE\ t(E)$, where $t(E)$ is a period of the finite motion of the system. Taking this into account, we finally obtain

$$Z_k(T)=\frac{1}{2\pi\hbar}\int_0^\infty dE t(E)\exp(-kE/T). \tag{12}$$

We shall estimate the effect of an adiabatic change of the phase volume for the atomic cloud confined in the optical trap in the situation close to the experiment [10]. In this case, the potential is given by Eq. (1). For sufficiently low temperatures, the anharmonicity effects are small and we need to only consider the quartic anharmonicity terms in the potential. Due to the anharmonicity, the period of the finite oscillations of the atoms acquires the energy-dependent corrections [19]

$$t(E)=\frac{2\pi}{\omega}\left(1+\frac{E}{8V}\right). \tag{13}$$

Substituting the dependence given by Eq. (13) into Eq. (12), we obtain

$$Z_k(T)=\frac{I}{k}\left(1+\frac{1}{k}\frac{T}{8V}\right), \tag{14}$$

where $I=(2\pi/\omega)T$ is the average phase volume for the harmonic potential. Making use of the Eq. (7) and taking into account the known result for the harmonic case, we obtain the BEC criterion in the form

$$N_c=\frac{T}{\hbar\omega}\left[\ln\left(\frac{2T}{\hbar\omega}\right)+\frac{\pi^2}{6}\frac{T}{8V}\right]. \tag{15}$$

Since for the adiabatic modulation of the optical potential $T\sim\omega\sim V^{1/2}$, the anharmonic corrections given by Eq. (15) lead to the parametric dependence of the BEC point on the amplitude of the potential. Typically, $T/V\sim 0.1$ [10] and therefore the magnitude of the phase volume compression can be significant for small occupation numbers $\bar{n}=T/(\hbar\omega)\sim 1$.

As we will see below, the average phase-space volume enters the BEC criterion in the modified form if the nonadiabatic squeezing atomic oscillations are present and the energy distribution is nonequilibrium.

IV. THE BEC IN PRESENCE OF SQUEEZING OSCILLATIONS

Let us analyze the nonequilibrium energy distribution in the case when the fast squeezing oscillations are present. As it was shown in the previous section, the average phase-apace volume is not conserved in case of the adiabatic modulation of optical potential. This effect is due to a relaxation mechanism which does not have to be specified in this regime. In the nonadiabatic regime, a similar effect is expected, but in this case one has to consider a specific relaxation mechanism. We consider a collisionless gas of atoms in the anharmonic optical trap with fast parametric modulation of optical potential and show that in such system there exists a relaxation mechanism for the average quantities due to the instabilities of atomic trajectories with respect to the initial conditions. Such a mechanism is related to the effect known in the literature as ''phase mixing'' [21].

In order to clarify the mechanism of the proposed effect, we will make use of the analytical expression for the Wigner function given by Eq. (4). As we show in the Appendix, this solution is valid for arbitrary nonadiabatic evolution of the parameters of the dipole potential provided that the phase mixing [21] is present and the initial condition for the Wigner function has the form of Eq. (4). If the atomic system is excited by means of a quick parametric change of the effective potential, the phase-mixing regime is achieved for sufficiently long times $t\geqslant t_c$, where t_c is a phase correlation time. Extending the method described in [21] to the case of a nonperiodic change of the potential, we obtain an estimate for the correlation time in the form $t_c\cong(1/K\omega)$ with $K=\mu(\eta T/m\omega^4)$ and

$$\mu\simeq\frac{1}{\omega^2}\left|\frac{d\omega}{dt}\right|.$$

For the fast nonadiabatic change of the potential, the correlation time is sufficiently short and the time evolution of the atomic distribution takes place in the phase-mixing regime. One should note that in the classical regime (sufficiently large occupation numbers) considered in a present paper, the phase mixing can also be treated as a classical process. In this case, the solution for the distribution function is given by Eqs. (4) and (5).

The distribution in Eq. (4) is nonequilibrium and the evolution of the parameters is given by (5). In order to obtain the BEC criterion when the squeezing oscillations are present, we will go back to the condition in the form given by Eq. (6) and then substitute the probabilities for different occupation numbers derived for the nonequilibrium distribution Eq. (4). In quasiclassical approximation, this will give the result in the form

$$N_c=\frac{1}{(2\pi\hbar)^D}\sum_{k=1}^{\infty}\tilde{Z}_k(t), \tag{16}$$

analogous to Eq. (7). From Eq. (16), it follows that in the non-equilibrium case the BEC criterion is formulated in terms of the time-dependent functions $\tilde{Z}_k(t)$ defined by

$$\tilde{Z}_k(t)=\int d\Gamma \exp[-k\Phi(p,x,t)]. \tag{17}$$

In this case, $\tilde{Z}(t)$ is the effective phase volume of the particle with the distribution function given by Eq. (4). Making use of the Eqs. (4) and (17), we obtain

$$\tilde{Z}_k(t)=\frac{\tilde{I}}{k}\left(1-\frac{1}{k}\frac{3}{8}\frac{\eta_0}{\omega_0^2}\frac{\alpha(t)}{J}\right), \tag{18}$$

where $\tilde{I}$ is the average phase volume for the harmonic potential. Note that according to Eq. (3),

$$\frac{\eta_0}{m\omega_0^4}=-\frac{1}{6}\frac{1}{V_0}.$$

In the adiabatic limit, $\alpha_{ad}(t)/J\approx(4T/m\omega^2)$ [5] and Eq. (18) reduces to the adiabatic result given by Eq. (14). Making use of the Eq. (16) and taking into account the known result for the harmonic case and that $J=1/4(\hbar\omega_0/T_0)^2$ where $\omega_0=\omega(0)$, $\eta_0=\eta(0)$ and $T_0=T(0)$, we obtain the BEC criterion in the form

$$N_c=\frac{T_0}{\hbar\omega_0}\left[\ln\left(\frac{2T_0}{\hbar\omega_0}\right)-\frac{\pi^2}{6}\frac{3}{2}\left(\frac{T_0}{\omega_0}\right)^2\frac{\eta_0}{\omega_0^2}\alpha(t)\right]. \tag{19}$$

For the quantitative analysis of the nonequilibrium phase-space compression effect described above, a full numerical simulation of the evolution of the atomic distribution with the potential given by Eq. (1), has been performed. The simulation was done by means of direct integration of the atomic equations of motion with subsequent calculation of the phase-space volume. The initial conditions were taken in the form of the classical squeezed state (A12), corresponding to an instantaneous increase of the confining potential at $t=0$ with subsequent excitation of squeezing oscillations of the atomic distribution.

In Fig. 1, we present the results of simulations for the average phase volume $\tilde{Z}(t)$ as a function of time with fast nonadiabatic modulation of the optical potential $\omega(t)$. One can see, that $\tilde{Z}(t)$ exhibits a fast decrease during the time interval less than the period of undisturbed oscillations. Note, that the nonconservation of $\tilde{Z}(t)$ is related to a transition into the phase mixing regime. Making use of the estimate for the correlation time t_c given above and the parameters of the modulation, we obtain

$$\omega(0)t_c\simeq\frac{10}{\mu}\frac{V_0}{T}\approx 2,$$

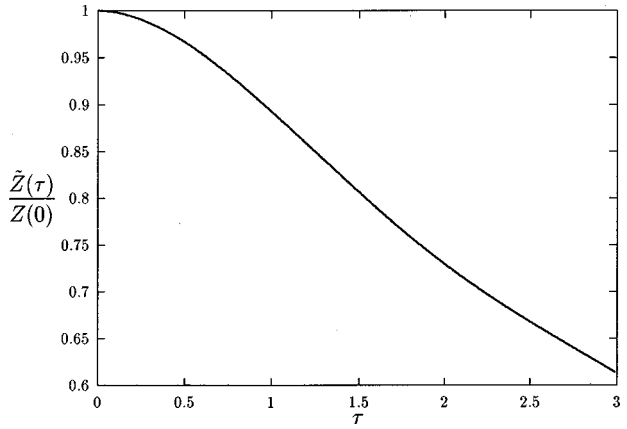

FIG. 1. The average phase-space volume $\tilde{Z}(\tau)$ as a function of a dimensionless time $\tau=\omega_0 t$ for the fast time modulation of the optical potential $\omega(t)=\omega_0(1+\mu\tau)$ with $(\eta_0 T_0/m\omega_0^4)=0.1$ and $\mu=5.0$. The initial conditions correspond to the equilibrium thermal state for $t<0$ excited by means of the instant increase of the effective potential at $t=0$ so that $[\omega_0/\omega(t<0)]=0.25$.

in qualitative agreement with the results presented in Fig. 1. This estimate also agrees with direct numerical analysis of the average phase correlator of the system described by Eq. (A6).

For the time $t>t_c$, the atomic system is in a phase-mixing regime and the evolution of the distribution function is defined by Eq. (5) for the parameters of the classical squeezed state in the form of Eq. (4) with time-dependent frequency $\omega(t)$. In Fig. 2, we present the results of simulations for the average phase volume $\tilde{Z}(t)$ in the phase-mixing regime as a function of time, with the nonadiabatic modulation of the optical potential. One can see, that the average phase volume $\tilde{Z}(t)$ decreases over the time interval of the order of the modulation time and exhibits oscillations.

We assume that the correlation time t_c can be made shorter than the mean-free time for the atomic collisions, so that the evolution of the atomic distribution during the modulation and phase mixing can be described within the collisionless model presented above.

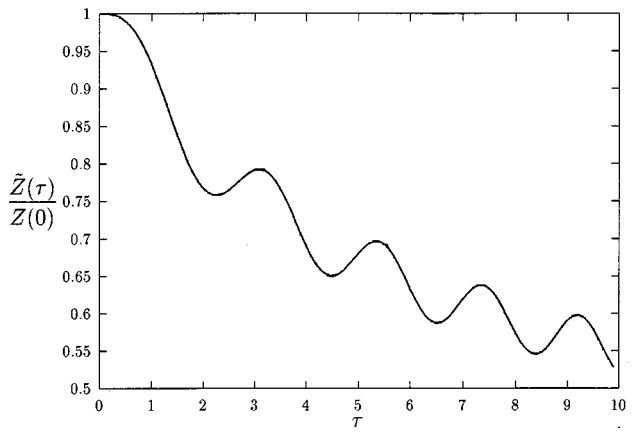

FIG. 2. The average phase-space volume $\tilde{Z}(\tau)$ as a function of a dimensionless time $\tau=\omega_0 t$ in the phase-mixing regime $t>t_c$ for the time modulation of the optical potential $\omega(\tau)=1+\epsilon\tanh(\lambda\tau)$ with $\epsilon=5.0$ and $\lambda=0.05$.

When the onset point of the BEC is achieved, the atomic phase space distribution forms an initial state for the nonequilibrium kinetic process of Bose condensation. In order to achieve a specific target state of the atomic distribution at a given time, an optimal control analysis [22] analogous to [5] should be done. This will be a subject of future work. The potential design may be realized in the laboratory by closed loop learning control techniques [23,24].

The nonadiabatic modulation of the optical potential may not only change the initial conditions, but also affect the BEC and therefore provide a tool to control the kinetics of the BEC process. This may happen when the modulation time in the phase-mixing regime becomes comparable to the typical time of the Bose condensation [20]. This problem is much more complex, since the global behavior of the phase space volume as a function of time becomes important and therefore introduction of the full tools of the optimal control theory is desirable. The full control of the Bose condensate opens up the prospect of manipulating an atom laser cloud released from the trap. The modest time scale of the dynamics should provide a testing ground for real time feedback control of the overall condensate.

V. CONCLUSION

This paper investigated the effect of phase-space compression making use of the breathing oscillations in optical lattices. The compression originates from the fact that the entropy and the phase volume do not coincide in the case when the energy of the system is distributed and the system is in the nonequilibrium state. An analytical model demonstrates the possibility and degree of such phase-space compression. This effect opens a possibility to control the onset and subsequent dynamics of the Bose-Einstein condensation of the atomic cloud loaded in an optical lattice.

APPENDIX

In this appendix, we show that in case of weak anharmonicity, the atomic Wigner function can be presented in the form of Eq. (4) from the text. For the potential (2), the equation of motion for the Wigner function is given by (Ref. [14])

$$\frac{\partial\rho}{\partial t}+\frac{p}{m}\frac{\partial\rho}{\partial x}-(m\omega^2 x+m\eta x^3)\frac{\partial\rho}{\partial p}-\frac{1}{4}m\eta x\frac{\partial^3\rho}{\partial p^3}=0. \tag{A1}$$

One can expect that at the BEC transition, the fourth term on the lhs of Eq. (A1) presenting the ''quantum anharmonic'' contribution should be of the same order of magnitude as the third one, which is the ''classical'' anharmonic term. However, in the regime above the BEC transition discussed in the text and due to the numerical coefficients, the quantum anharmonic term appears to be one order of magnitude smaller than the classical one. The ratio ξ of the fourth and the third terms on the lhs in Eq. (A1) can be approximately estimated as

$$\xi\sim\frac{\hbar^2}{4\langle x^2\rangle\langle\Delta p^2\rangle}\sim\frac{1}{4n^2}, \tag{A2}$$

where $\langle x^2\rangle$ and $\langle\Delta p^2\rangle$ are the average coordinate and momentum dispersions, respectively and $n=(T/\hbar\omega)$ is the average occupation number. Assuming $n\geqslant 2$, we conclude that $\xi\sim 0.1$ and therefore the equation of motion for the Wigner function given by Eq. (A1) reduces to the Liouville equation in the form

$$\frac{\partial\rho}{\partial t}+\frac{p}{m}\frac{\partial\rho}{\partial x}-(m\omega^2 x+m\eta x^3)\frac{\partial\rho}{\partial p}=0. \tag{A3}$$

We introduce the ''polar'' coordinates E,Θ in the phase space by

$$\frac{p^2}{2m}=E\sin^2\Theta,$$

$$\frac{m\omega^2 x^2}{2}+\frac{m\eta x^4}{4}=E\cos^2\Theta, \tag{A4}$$

with the Jacobian

$$\Sigma(p,x)=\frac{\partial(E,\Theta)}{\partial(p,x)}=-\frac{1}{\sqrt{2mU(x)}}\frac{\partial U(x)}{\partial x}, \tag{A5}$$

where

$$U(x)=\frac{m\omega^2 x^2}{2}+\frac{m\eta x^4}{4}$$

is the potential energy. Note, that the variable E has the meaning of the microscopic energy and is conserved along each trajectory in the phase space. The polar coordinates introduced by Eq. (A4), are not identical to the action-angle variables commonly used for the analysis of nonlinear problems [21]. As we will see below, the variables E,Θ are more convenient for the analysis of nonlinear system dynamics in the presence of phase mixing and in the adiabatic regime. In terms of the coordinates given by Eq. (A4), the equations of motion take the form

$$\frac{d\Theta}{dt}=\Sigma(E,\Theta)-\frac{1}{2\omega}\frac{d}{dt}(\omega)\sin(2\Theta),$$

$$\frac{dE}{dt}=\frac{1}{\omega}\frac{d}{dt}(\omega)E[1+\cos(2\Theta)], \tag{A6}$$

with the initial conditions $\Theta(0)=\Theta_0$ and $E(0)=E_0$. The second equation in Eq. (A6) reflects the energy conservation if $\omega\equiv\omega_0=\text{const}$ for $t>0$.

From Eq. (A5), an element of the phase volume $d\Gamma$ is expressed in terms of the polar variables in 1D as

$$d\Gamma=\frac{dpdx}{2\pi\hbar}=\frac{1}{2\pi\hbar}\frac{dEd\Theta}{\Sigma(E,\Theta)}. \tag{A7}$$

Using this result and the equations of motion, one can show that the element of the phase space volume is time-independent. From the first of Eq. (A6), we obtain the relations

$$\frac{\partial \Sigma}{\partial \Theta} = \frac{\partial}{\partial t}\left[\ln\left(\frac{\partial \Theta}{\partial \Theta_0}\right)\right] + \frac{1}{\omega}\frac{d\omega}{dt}\cos(2\Theta), \tag{A8}$$

and

$$\frac{\partial}{\partial t}\left[\ln\left(\frac{\partial E}{\partial E_0}\right)\right] = \frac{1}{\omega}\frac{d\omega}{dt}[1+\cos(2\Theta)]. \tag{A9}$$

From Eqs. (A8), (A9), and (A5), it follows that

$$\frac{\partial}{\partial t}\ln\left[\left(\frac{\partial \Theta}{\partial \Theta_0}\right)\left(\frac{\partial E}{\partial E_0}\right)\frac{\Sigma_0}{\Sigma}\right] = 0$$

and therefore

$$d\Gamma = \frac{1}{2\pi\hbar}\frac{dE\,d\Theta}{\Sigma(E,\Theta)} \equiv \frac{1}{2\pi\hbar}\frac{dE_0 d\Theta_0}{\Sigma(E_0,\Theta_0)}, \tag{A10}$$

implying the conservation of the phase space volume element. One should note, that this conclusion holds for arbitrary potential function $U(x)$ provided that the motion is finite.

In case of weak anharmonicity, the Jacobian $\Sigma(E,\Theta)$ can be approximately obtained from Eq. (A5) as

$$\Sigma(E,\Theta) \approx -\omega\left(1+\frac{3}{4}\frac{\eta E}{m\omega^4}[1+\cos(2\Theta)]\right). \tag{A11}$$

In order to solve the Liouville equation given by Eq. (A3) with a given initial distribution, we will employ the method of characteristics. This method requires expressing the current variables $E(t),\Theta(t)$ in terms of the initial values E_0,Θ_0 with the substitution $\rho(E,\Theta,t)=\rho(E_0,\Theta_0,0)$, where $\rho(E_0,\Theta_0,0)$ is the initial distribution function at $t=0$. To obtain the characteristics $E_0=E_0(E,\Theta,t)$ and $\Theta_0=\Theta_0(E,\Theta,t)$, one has to solve the equation of motion (A6).

We will consider the situation when the atomic system is excited by means of sudden nonadiabatic increase of the confining potential at $t=0$. In this case, the nonequilibrium initial condition for the Wigner function is given by [5]

$$\begin{aligned}\rho(p,x,0) &= C(0)\exp[-\Phi(p,x,0)],\\ \Phi(p,x,0) &= \alpha_0 p^2 + \gamma_0\left(x^2 + \frac{\eta}{2\omega^2}x^4\right),\end{aligned} \tag{A12}$$

and transforming to the (E,Θ) coordinates, i.e., the characteristics, the solution of the Liouville equation is expressed in the form

$$\begin{aligned}\rho(p,x,t) &= C(0)\exp[-\Phi(E,\Theta,t)],\\ \Phi(E,\Theta,t) &= \frac{2}{\omega_0}E_0(E,\Theta,t)\left[\alpha_0(m\omega_0)\sin^2(\Theta+\Psi)\right.\\ &\quad \left. + \frac{\gamma_0}{m\omega_0}\cos^2(\Theta+\Psi)\right],\end{aligned} \tag{A13}$$

where $\Theta_0 \equiv \Theta + \Psi(E,\Theta,t)$ and Ψ should be expressed in terms of the current coordinates (E,Θ,t). Note, that this transformation is much simplified if the system does not utilize the initial phase of the oscillations and $\Psi=\Psi(t)$. This indeed happens in the phase mixing regime which is established for the times greater than a correlation time t_c, estimated in the text.

Transferring back to the (p,x) coordinates, we obtain

$$\begin{aligned}\rho(p,x,t) &= C(0)\exp[-\Phi(p,x,t)],\\ \Phi(p,x,t) &= \tilde{\alpha}p^2 + \tilde{\beta}px\sqrt{1+\frac{\eta x^2}{2\omega^2}} + \tilde{\gamma}x^2\left(1+\frac{\eta x^2}{2\omega^2}\right),\end{aligned} \tag{A14}$$

where

$$\begin{aligned}\tilde{\alpha}(t) &= \frac{1}{2m\omega}\left[\alpha_0(m\omega)+\frac{\gamma_0}{m\omega}\right]\\ &\quad + \frac{1}{2m\omega}\cos(2\Psi)\left[\alpha_0(m\omega)-\frac{\gamma_0}{m\omega}\right],\\ \tilde{\beta}(t) &= \sin(2\Psi)\left[\alpha_0(m\omega)-\frac{\gamma_0}{m\omega}\right],\\ \tilde{\gamma}(t) &= \frac{m\omega}{2}\left[\alpha_0(m\omega)+\frac{\gamma_0}{m\omega}\right]\\ &\quad - \frac{m\omega}{2}\cos(2\Psi)\left[\alpha_0(m\omega)-\frac{\gamma_0}{m\omega}\right].\end{aligned} \tag{A15}$$

Since $\Psi=\Psi(E,\Theta,t)$ in the absence of the phase mixing, the coefficients $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$are the functions of (p,x,t). Making use of Eq. (A15) in the limit of short times $t<t_c\cong(1/K\omega)$ where

$$K = \mu\frac{\eta T}{m\omega^4} \quad \text{and} \quad \mu \simeq \frac{1}{\omega^2}\left|\frac{d\omega}{dt}\right|,$$

one can show that the phase volume is conserved, as expected.

In the phase-mixing regime, $\Psi(E,\Theta,t)=\Psi(t)$ and the average phase volume may not be conserved [21]. In this case, we obtain from Eqs. (A14) and (A15) the solution for the single-particle distribution function in a form of Eq. (4) from the text with

$$\begin{aligned}\delta(t) &= \frac{\eta}{2\omega^2}\gamma(t),\\ \epsilon(t) &= \frac{\eta}{4\omega^2}\beta(t),\end{aligned} \tag{A16}$$

leading to the result for the phase-space volume

$$\tilde{Z}(t) = \tilde{I}\left(1-\frac{3}{8}\frac{\eta_0}{\omega_0^2}\frac{\alpha(t)}{J}\right) \tag{A17}$$

from the text.

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