

# Relativistic partial-wave analysis using the velocity basis of the Poincaré group

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The velocity basis of the Poincaré group is used in the direct product space of two irreducible unitary representations of the Poincaré group. The velocity basis with total angular momentum  $j$  will be used for the definition of relativistic Gamow vectors. [S1050-2947(99)04012-3]

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## I. INTRODUCTION

Resonances are obtained in the scattering of two (or more) elementary particles, and quasistationary states decay into a two (or many) particle system with masses  $m_i$  and spins  $s_i$ ,  $i=1, 2, \dots$ . Relativistic resonances and decaying states are therefore described in the direct product space of two irreducible representation spaces of the Poincaré group  $\mathcal{H} = \mathcal{H}_1(m_1, s_1) \otimes \mathcal{H}_2(m_2, s_2)$ . Nonrelativistic resonances and decaying states have been described by Gamow vectors [1]. Gamow vectors are characterized by a value of angular momentum  $j$  in the center-of-mass frame and by a complex energy  $z_R = (E_R - i\Gamma/2)$ , representing resonance energy  $E_R$  and lifetime  $\hbar/\Gamma$ . They are generalized eigenvectors in a Rigged Hilbert Space  $\Phi \subset \mathcal{H} \subset \Phi^\times$  of the self-adjoint Hamiltonian  $H$  with complex eigenvalue  $z_R$  [1]. Relativistic resonances and unstable particles are characterized by their spin (total angular momentum in the center-of-mass frame of the decay products) and the value  $\mathbf{s} = \mathbf{s}_R \equiv (M_R - i\Gamma/2)^2$  of the invariant mass squared  $\mathbf{s} = (p_1 + p_2)^2 = (E^2 - \mathbf{p}^2)$  where  $M_R$  is the resonance mass and  $\hbar/\Gamma_R$  is its lifetime. We want to find relativistic Gamow vectors which are generalized eigenvectors of the total mass operator  $M^2 = P_\mu P^\mu = (P_{1\mu} + P_{2\mu})(P_1^\mu + P_2^\mu)$  with complex eigenvalue  $\mathbf{s}_R$  and with spin  $j$ . These must be obtained from the direct product space  $\mathcal{H}_1(m_1, s_1) \otimes \mathcal{H}_2(m_2, s_2)$ .

Eigenspaces of  $M^2$  with real values of invariant mass  $\mathbf{s}$  and total angular momentum  $j$  are obtained by the relativistic partial wave analysis [2–4] using the Wigner basis, i.e., using momentum eigenvectors  $|\mathbf{p}_i, s_{3i}(m_i, s_i)\rangle$  in the spaces  $\mathcal{H}_i$  and eigenvectors  $|\mathbf{p}, j_3(\mathbf{s}, j)\rangle$  of  $P_\mu = P_{1\mu} + P_{2\mu}$  in the direct product space  $\mathcal{H}$ .

In contrast to the nonrelativistic case, in the relativistic case Lorentz transformations intermingle energy and momenta. If one wants to make an analytic continuation of  $\mathbf{s}$  from the values  $(m_1 + m_2)^2 \leq \mathbf{s} < \infty$  to the complex values  $\mathbf{s}_R$  (of the pole position in the second sheet of the relativistic  $S$ -matrix  $S_j(\mathbf{s})$ ) this will also lead to complex momenta. To restrict the unwieldy set of complex momentum representa-

tions [5] we want to construct complex mass representations of the Poincaré group  $\mathcal{P}$  whose momenta are “minimally complex” in the sense that though  $p_\mu$  and  $m$  are complex, the 4-velocities  $\hat{p}_\mu \equiv p_\mu/m$  remain real. This can be carried out because, as explained in Sec. II, the four-velocity eigenvectors  $|\hat{\mathbf{p}}, j_3(\mathbf{s}, j)\rangle$  provide as valid basis vectors for the representation space of  $\mathcal{P}$  as the usual momentum eigenvectors. Moreover, they are more useful for physical reasoning than the momenta eigenvectors, because the 4-velocities seem to fulfill to rather good approximation “velocity superselection rules” which the momenta do not [6]. Therefore we will use the velocity basis  $|\hat{\mathbf{p}}_i, s_{3i}(m_i, s_i)\rangle$  for the relativistic partial wave analysis and obtain the Clebsch-Gordan coefficients of the Poincaré group for the velocity basis. This is done in Sec. III for  $s_1 = s_2 = 0$ , which applies to the case of  $\pi^+ \pi^-$  in the final state. This gives the velocity eigenvectors  $|\hat{\mathbf{p}}, j_3(\mathbf{s}, j)\rangle$  of the direct product space  $\mathcal{H} = \sum_{j=0}^{\infty} \int_{(m_1+m_2)^2}^{\infty} d\mu(\mathbf{s}) \mathcal{H}(\mathbf{s}, j)$  from which we obtain the four-velocity scattering states  $|\hat{\mathbf{p}}, j_3(\mathbf{s}, j)^\pm\rangle$  using the Lippmann-Schwinger equation as, e.g., done in [7]. The relativistic Gamow vectors  $|\hat{\mathbf{p}}, j_3(\mathbf{s}_R, j)^\pm\rangle$  will be obtained in a subsequent paper from the scattering states by analytic continuation. In the Appendix, we derive the Clebsch-Gordan coefficients for the velocity basis of  $\mathcal{P}$  for the general case.

## II. VELOCITY BASIS OF THE POINCARÉ GROUP

We denote the ten generators of the unitary representation  $\mathcal{U}(a, \Lambda)$  of  $(a, \Lambda) \in \mathcal{P}$ , by

$$P^\mu, J^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3. \quad (2.1)$$

The standard choice of the invariant operators and of a complete set of commuting observables (c.s.c.o.) is

$$M^2 = P_\mu P^\mu, \quad W = -w_\mu w^\mu,$$

$$P_i (i=1, 2, 3), \quad S_3 = M^{-1} \mathcal{U}(L(p)) w_3 \mathcal{U}^{-1}(L(p)), \quad (2.2)$$

here

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$$w_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma}, \quad (2.3)$$

$M^{-1}$  is the inverse square root of the positive definite operator  $P^\mu P_\mu$ , and  $\mathcal{U}(L(p))$  is the representation of the boost that depends upon the parameters  $p_\mu$  ( $\mu=0,1,2,3$ ), which are the eigenvalues of the operators  $P_\mu$ . Only three of these parameters are independent in an irreducible representation, because of the relation  $m^2 = p_\mu p^\mu$ . The standard boost (“rotation free”) matrix  $L_{\nu}^{\mu}(p)$  is given by

$$L_{\nu}^{\mu}(p) = \begin{pmatrix} \nu=0 & \nu=n \\ \mu=0 & \frac{p^0}{m} & -\frac{p_n}{m} \\ & \frac{p^m}{m} & \delta_n^m - \frac{p_n}{m} \\ \mu=m & & 1 + \frac{p^0}{m} \end{pmatrix}. \quad (2.4)$$

Note that  $p_\mu = \eta_{\mu\nu} p^\nu$  and we use the metric<sup>1</sup>

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

It has the property that

$$L^{-1}(p)_{\nu}^{\mu} p^\nu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.5)$$

One feature shown in (2.4) which we want to make use of, is that the boost  $L_{\nu}^{\mu}(p)$  does not depend upon  $p$  but only upon the 4-velocity  $p/m \equiv \hat{p}$ . The complete basis system in the irreducible representation space  $\mathcal{H}(m^2, j)$  which consists of eigenvectors of the c.s.c.o. (2.2) is the Wigner basis usually denoted as

$$|p, j_3(m, j)\rangle. \quad (2.6)$$

It has the transformation property under the translation (a, I) and the Lorentz transformation (0,  $\Lambda$ ):

$$\mathcal{U}(a, I)|p, j_3\rangle = e^{ip^\mu a_\mu} |p, j_3\rangle \quad (2.7a)$$

$$\mathcal{U}(0, \Lambda)|p, \xi\rangle = \sum_{\xi'} |\Lambda p, \xi'\rangle D_{\xi'\xi}(\mathcal{R}(\Lambda, p)), \quad (2.7b)$$

where  $\mathcal{R}$  is the Wigner rotation

$$\mathcal{R}(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p). \quad (2.7c)$$

The Wigner rotation depends upon the ten parameters of  $\Lambda$  and upon the parameters  $\hat{p}^\mu = p^\mu/m$ . In an unitary irreducible representation (UIR) there are three independent  $\hat{p}^\mu$  and:

$$|p, j_3\rangle = \mathcal{U}(L(p))|p=0, j_3\rangle, \quad (2.7d)$$

where we have omitted the fixed values  $m, j$  as we shall often do in an UIR. Every vector (of a dense subspace of physical states) of  $\mathcal{H}(m, j)$  can be written according to Dirac’s basis vector decomposition as

$$\phi = \int d\mu(p) \sum_{\xi} |p, \xi\rangle \langle p, \xi | \phi \rangle, \quad (2.8a)$$

where one has many arbitrary choices for the measure. It is usually chosen to be given by

$$d\mu(p) = \rho(p) d^3p, \quad (2.8b)$$

where one can choose any (measurable) function  $\rho$ , in particular a smooth function. The choice of  $\rho$  is connected to the “normalization” of the Dirac kets through:

$$\langle \xi' | p' | p, \xi \rangle = \frac{1}{\rho(p)} \delta^3(p - p') \delta_{\xi\xi'}. \quad (2.8c)$$

One convention<sup>2</sup> for  $\rho$  is the Lorentz invariant measure:

$$\rho(p) = \frac{1}{2E(p)}, \quad \text{where } E(p) = \sqrt{m^2 + p^2}. \quad (2.8d)$$

The mathematically precise form of the Dirac decomposition is the Nuclear Spectral Theorem for the complete system of commuting (essentially self-adjoint) operators. It is the same as (2.8), however with well defined mathematical quantities. The state vectors  $\phi$  in (2.8a) must be elements of a dense subspace  $\Phi$  of the representation space  $\mathcal{H}$  of an UIR:

$$\phi \in \Phi \subset \mathcal{H}(m, j); \quad (2.9)$$

and the basis vectors  $|p, \xi\rangle \in \Phi^\times$  are elements of the space of antilinear functionals on  $\Phi$  which fulfill the condition:

$$\langle P_i \psi | p, \xi \rangle = p_i \langle \psi | p, \xi \rangle \quad \text{for every } \psi \in \Psi. \quad (2.10a)$$

This condition means the  $|p, \xi\rangle$  are generalized eigenvectors of  $P_i$ , which is also written as

$$P_i^\times |p, \xi\rangle = p_i |p, \xi\rangle, \quad (2.10b)$$

where  $P_i^\times$  is an extension of  $P_i^\dagger (= P_i)$ ; and the “component of  $\phi$  along the basis vector  $|p, \xi\rangle$ ,” the  $\langle p, \xi | \phi \rangle = \langle \phi | p, \xi \rangle^*$ , are antilinear continuous functionals  $F(\phi) = \langle p, \xi | \phi \rangle^*$  on the space  $\Phi$ .

The space  $\Phi$  is a dense nuclear subspace of  $\mathcal{H}$  [9]. [E.g.,  $\Phi$  could be chosen to be the subspace of differentiable vectors of  $\mathcal{H}$  equipped with a nuclear topology defined by the countable number of norms:  $\|\phi\|_p = \sqrt{\langle \phi, (\Delta + 1)^p \phi \rangle}$ , where  $\Delta = \sum_\mu P_\mu^2 + \sum_{\mu\nu} \frac{1}{2} J_{\mu\nu}^2$  is the Nelson operator [10]. But it

<sup>1</sup>Some of the references we use here have different convention, e.g.,  $\eta_{\mu\nu} \rightarrow -\eta_{\mu\nu}$  [7], and  $L^{-1}(p) \rightarrow L(p)$  [3].

<sup>2</sup>This is the convention of [8,2–4], but not of [7]

could also be chosen as another dense nuclear subspace of  $\mathcal{H}$ .] The three spaces form a Gel'fand triplet, or Rigged Hilbert Space

$$\Phi \subset \mathcal{H} \subset \Phi^\times \quad (2.11)$$

and the bra-ket  $\langle | \rangle$  is an extension of the scalar product  $(,)$ . The  $\langle \mathbf{p}, \xi | \phi \rangle = \langle \phi | \mathbf{p}, \xi \rangle^*$  are the Wigner momentum wavefunctions.

The Wigner kets (2.6) are not the only basis system of  $\mathcal{H}(m, j)$  that one can use to expand every vector  $\phi \in \Phi$ . For every different choice of c.s.c.o. in the enveloping algebra  $\mathcal{E}(\mathcal{P})$  (the algebra generated by  $P_\mu, J_{\mu\nu}$ ) one obtains a different system of basis vectors; in this way one can obtain, e.g., Lorentz basis (eigenvectors of the Casimir operators of  $SO(3,1)_{J_{\mu\nu}}$  [3,9]), or the spinor basis (whose Fourier transforms are the relativistic fields [7]) etc. We want to choose still another basis system, which is similar to the Wigner basis except that it is a basis of eigenvectors of the 4-velocity operator  $\hat{P}_\mu \equiv P_\mu M^{-1}$  rather than the momentum operator  $P_\mu$ .

With the 4-velocity operator, one defines the operators

$$\hat{w}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \hat{P}^\nu J^{\rho\sigma} = w_\mu M^{-1}, \quad (2.12)$$

and the spin tensor

$$\Sigma_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \hat{P}^\rho \hat{w}^\sigma.$$

The complete set of commuting observables is then given by

$$\hat{P}_m, \quad S_3, \quad \hat{W} = -\hat{w}_\mu \hat{w}^\mu = \frac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu}, \quad M^2, \quad (2.13)$$

and we denote its generalized eigenvectors by

$$|\hat{\mathbf{p}}, j_3; \mathbf{s} = m^2, j\rangle, \quad (2.14)$$

where  $\hat{p}_\mu = p_\mu/m$  are the eigenvalues of  $\hat{P}_\mu$ .

The basis vector expansion for every  $\phi \in \Phi$  with respect to the basis system (2.14) is given by

$$\phi = \sum_{j_3} \int \frac{d^3 \hat{\mathbf{p}}}{2\hat{p}^0} |\hat{\mathbf{p}}, j_3\rangle \langle j_3, \hat{\mathbf{p}} | \phi \rangle, \quad (2.15a)$$

where we have chosen the invariant measure

$$d\mu(\hat{\mathbf{p}}) = \frac{d^3 \hat{\mathbf{p}}}{2\hat{p}^0} = \frac{1}{m^2} \frac{d^3 p}{2E(\mathbf{p})} \quad (2.15b)$$

$$\hat{p}^0 = \sqrt{1 + \hat{\mathbf{p}}^2}.$$

As a consequence of (2.15b), the  $\delta$ -function normalization of these velocity-basis vectors is

$$\langle \xi, \hat{\mathbf{p}} | \hat{\mathbf{p}}', \xi' \rangle = 2\hat{p}^0 \delta^3(\hat{\mathbf{p}} - \hat{\mathbf{p}}') \delta_{\xi\xi'} = 2p^0 m^2 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\xi\xi'}. \quad (2.15c)$$

Mathematically, every c.s.c.o. is equally valid. But, for a given physical problem one c.s.c.o. may be more useful than

another. For instance a c.s.c.o. that contains physically distinguished observables (e.g., observables whose eigenstates happen to appear predominantly in nature) is more useful for calculations in physics than the c.s.c.o. whose eigenvectors are very different from physical eigenstates. Two different c.s.c.o.'s lead to different basis systems, whose vectors can be expanded with respect to each other. But this expansion is usually very complicated and intractable, for which reason the choice of the physically right c.s.c.o. is very important for each particular physical problem. This is the reason for which the Lorentz basis of the Poincaré group is pretty useless for physics, because the Casimir operators of  $SO(3,1)$  are not important observables as compared to the momentum. However, the two c.s.c.o. (2.2) and (2.13) are not even different in an irreducible representation of  $\mathcal{P}$ , since its operators differ only by a factor of the operator  $M$ , which is an invariant. The basis systems (2.6) and (2.14) are therefore the same, i.e., their values differ by a normalization-phase factor  $N(p, j_3)$

$$|\hat{\mathbf{p}}, j_3(m, j)\rangle = |\mathbf{p}, j_3(m, j)\rangle N(p, j_3). \quad (2.16)$$

The Poincaré transformations (2.7) act on the basis vectors (2.16) in the following way

$$\mathcal{U}(a, \mathbf{I}) |\hat{\mathbf{p}}, j_3\rangle = e^{im\hat{p}^\mu a_\mu} |\hat{\mathbf{p}}, j_3\rangle \quad (2.17a)$$

$$\mathcal{U}(L(\hat{p})) |\hat{\mathbf{p}} = \mathbf{0}, j_3\rangle = |\hat{\mathbf{p}}, j_3\rangle. \quad (2.17b)$$

The distinction between the basis vectors  $|\mathbf{p}, \xi\rangle$  and  $|\hat{\mathbf{p}}, \xi\rangle$  becomes important if one does not have an unitary irreducible representation of  $\mathcal{P}$  but a representation with many different values for  $(m^2, j)$ , e.g.,  $\mathcal{H} = \sum_{m^2, j} \oplus \mathcal{H}(m, j)$ . Then one has besides the observables (2.1), additional observables  $X_\alpha$  (generators of an intrinsic symmetry group or a spectrum generating group) and an additional system of commuting observables:

$$B = B_1, B_2, \dots, B_N \quad (2.18)$$

whose eigenvalues,  $b = (b_1, b_2, \dots, b_N)$ , characterize the elementary particles described by  $\mathcal{H}(m, j) = \mathcal{H}^b(m, j)$ .<sup>3</sup> In order that (2.2) and (2.18) combine into a c.s.c.o., the operators  $B$  have to commute with  $M^2, P_\mu, W$  and  $S_3$ . If also the other observables  $X_\alpha$ , which change the particle species number  $b$ , commute with  $M^2, P_\mu, W$  and  $S_3$ , then the combination of (2.2) and (2.18) gives a useful c.s.c.o. However, if the  $X_\alpha$  do not commute with  $M^2$  (i.e., the particle species number changing operators  $X_\alpha$  transform also from one mass eigenstate to another mass eigenstate changing also the mass  $m_b$  into  $m_{b'}$ ) then the  $X_\alpha$  will also not commute with  $P_\mu$ ,  $[X_\alpha, P_\mu] \neq 0$ . In this case, it may still happen [6] that a "velocity superselection rule" holds:

$$[X_\alpha, \hat{P}_\mu] = 0 \quad (\text{or at least } [X_\alpha, \hat{P}_\mu] \approx 0). \quad (2.19)$$

Then combination of (2.18) with (2.13), i.e., the

<sup>3</sup>The quantum numbers  $b$  are called the particle species numbers in [7].

$$\hat{P}_i, \hat{w}_3, \hat{W}, M^2, B_1, \dots, B_N \quad (2.20)$$

will form a useful c.s.c.o., but the combination of (2.2) with (2.18) will not. The generalized eigenvectors of (2.20),  $|\hat{\mathbf{p}}, \xi, b, m, j\rangle$ , will then be a much more useful basis system for every  $\phi \in \Phi \subset \mathcal{H} = \Sigma \oplus \mathcal{H}^b(m, j)$  than the corresponding momentum eigenvectors. Using the eigenvectors of (2.20), we have the Dirac basis vector expansion:

$$\phi = \sum_{m,b} \sum_{j,\xi} \int \frac{d^3 \hat{\mathbf{p}}}{2\hat{p}^0} |\hat{\mathbf{p}}, \xi, b, m, j\rangle \langle j, m, b, \xi, \hat{\mathbf{p}} | \phi \rangle$$

for every  $\phi \in \Phi$ . (2.21)

The momentum eigenvectors  $|\mathbf{p}, \xi, b \dots\rangle$  may either not exist (if  $[B, P_\mu] \neq 0$ ), or if they do exist, they are not useful because the  $X_\alpha$  change the value of  $p$ , which then becomes a function of  $b$ ,  $p = p_b$ . As a consequence, quantities like form factors depend upon  $b$  through  $p$ . In contrast, using the velocity eigenvectors  $|\hat{\mathbf{p}}, \xi, b, \dots\rangle$  under the assumption (2.19) will lead to form factors with universal (independent of  $b$ ) dependence upon the four-velocity. This was the original motivation for the introduction of the velocity-basis vectors  $|\hat{\mathbf{p}}, \xi, b, \dots\rangle$  [6].

The subject of the present work is the description of relativistic decaying states by representations of the Poincaré group, combining Wigner's idea [8] of the description of stable relativistic particles by an UIR of  $\mathcal{P}$ , with Gamow's idea of describing decaying particles by eigenvectors with complex energy. Therefore, we need in the rest frame basis vectors with complex energy, i.e., the  $m$  (and the  $\mathbf{s} = m^2$ ) in (2.6) or in (2.14) has to be continued to complex values e.g., to  $\mathbf{s} = (M_R - i\Gamma/2)^2$ . This will result in a continuation of the momenta  $p_\mu$  to complex values as well and can lead to an enormous complication of the Poincaré group representations (see e.g., [5]). We want to do this analytic continuation in the invariant mass  $\mathbf{s}$  such that the  $p_\mu$  are continued to complex values in such a way that the  $\hat{p}_\mu = p_\mu / \sqrt{\mathbf{s}}$  remain real. Then, we obtain a smaller class of complex mass representations of  $\mathcal{P}$  which are as similar in property as possible to Wigner's UIR  $(m, j)$ . These are the minimally complex-mass representations which we shall denote by  $(\mathbf{s}, j)$ .

For this minimal analytic continuation to be possible, it must be compatible with the boost (2.7d) and (2.17b). The crucial observation is that the boosts  $L(p)$  are in fact, according to (2.4) only functions of  $\hat{p}_\mu = p_\mu / \sqrt{\mathbf{s}}$ ;  $L(p) = L(\hat{p})$ . As a consequence, the operators representing the boost  $\mathcal{U}(L(p)) = \mathcal{U}(L(\hat{p}))$  are functions of the real parameters  $\hat{p}$  and not of complex parameters  $p$ . This means they are the same operator functions in all the subspaces of the direct sum  $\sum_{m_b, j} \oplus \mathcal{H}(m_b, j)$  and of the continuous direct sum

$$\sum_{j,n} \int_{m_0^2}^{m_1^2} \oplus \mathcal{H}^n(\mathbf{s}, j) d\mu(\mathbf{s}) \quad (2.22)$$

of the irreducible representations

$$\mathcal{H}(\mathbf{s}, j), \quad \mathbf{s} = p_\mu p^\mu = E - \mathbf{p}^2. \quad (2.23)$$

If we consider in (2.22) only (continuous) direct sums with the same value for  $j = j_R$  then  $\mathcal{U}(\Lambda)$  for any Lorentz transformation  $\Lambda$  is, according to (2.7c), the same operator function of the six parameters which are given by the three  $\hat{p}^m$  or the three  $\mathbf{v}^m$ :

$$\begin{pmatrix} \hat{p}^0 \\ \hat{p}^m \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \\ \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \mathbf{v}^m \end{pmatrix} \quad (2.24)$$

and the three rotation angles (e.g., Euler angles in the rest frame). The analytic continuation in  $\mathbf{s}$  can therefore be accomplished without affecting the Lorentz transformations. The Lorentz transformations in the minimally-complex mass representation are represented unitarily by the same operators  $\mathcal{U}(\Lambda)$  as in Wigner's UIR  $(m, j_R)$ . At rest, on  $|\mathbf{0}, j_3(\mathbf{s}, j_R)\rangle$ , only the time translations of  $\mathcal{P}$  will be represented non-unitarily for complex values of  $\mathbf{s}$ . And using (2.17b) only the label  $\mathbf{s}$  in the velocity basis  $|\hat{\mathbf{p}}, j_3(\mathbf{s}, j_R)\rangle$  is complex. The basis vector decomposition (2.21) using the velocity basis,

$$\phi = \sum_{j_3} \int d\mu(\mathbf{s}) \int d\mu(\hat{\mathbf{p}}) |\hat{\mathbf{p}}, j_3(\mathbf{s}, j)\rangle \langle (\mathbf{s}, j) j_3, \hat{\mathbf{p}} | \phi \rangle$$

for  $\phi \in \Phi \subset \mathcal{H}(\mathbf{s}, j)$ , (2.25)

is therefore more suitable than (2.8) that uses the momentum basis, because  $\hat{\mathbf{p}}$  is independent of  $\mathbf{s}$  while  $\mathbf{p} = \sqrt{\mathbf{s}}\hat{\mathbf{p}}$  is not. If we deform the contour of integration for  $\mathbf{s}$  from the real axis as in (2.22) into the complex  $\mathbf{s}$ -plane then the integral over  $d\mu(\hat{\mathbf{p}})$  in (2.25) remains unaffected.

### III. RELATIVISTIC KINEMATICS FOR (TWO-PARTICLE) RESONANCE SCATTERING

Continuous direct sums like (2.22) appear in the case of scattering experiments of two relativistic particles like e.g., the process

$$e^+ e^- \rightarrow \rho^0 \rightarrow \pi^+ \pi^-, \quad (3.1a)$$

or the more theoretical process

$$\pi^+ \pi^- \rightarrow \rho^0 \rightarrow \pi^+ \pi^-. \quad (3.1b)$$

These processes predominantly happen in the  $j^P = 1^-$  partial amplitude if the  $\rho$ -meson mass region is selected for the invariant mass square

$$\mathbf{s} = (p_1 + p_2)^2 = E_\rho^2 + \mathbf{p}_\rho^2, \quad E_\rho = E_1 + E_2, \quad \mathbf{p}_\rho = \mathbf{p}_1 + \mathbf{p}_2, \quad (3.2)$$

where  $p_1$  and  $p_2$  are the momenta of the two pions  $\pi^+$ ,  $\pi^-$ .<sup>4</sup> The relativistic one particle states are given by an irreducible representation space  $\mathcal{H}^{n_i}(m_i, s_i)$  of the Poincaré group  $\mathcal{P}$ . The independent, *interaction-free* two-particle states (or  $n$  particle states)—like the  $\pi^+\pi^-$  system in (3.1b)—are given by the direct product of the irreducible representation spaces  $\mathcal{H}(m_1, s_1)$  and  $\mathcal{H}(m_2, s_2)$ :  $\mathcal{H}^{n_1}(m_1, s_1) \otimes \mathcal{H}^{n_2}(m_2, s_2) \equiv \mathcal{H}$ . Empirical evidence suggests that the resonances in processes like (3.1) appear in one partial amplitude with a given value of resonance spin  $j_R$  (e.g.,  $j_\rho^P = 1^-$ ). Therefore, the first problem is the reduction of the direct product  $\mathcal{H}(m_1, s_1) \otimes \mathcal{H}(m_2, s_2)$  into a direct sum of  $\mathcal{H}^n(\mathbf{s}, j)$ ; the second problem is how to go from the free two-particle system to the interacting two-particle system.

The first problem has been solved in general [2–4]

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}^{n_1}(m_1, s_1) \otimes \mathcal{H}^{n_2}(m_2, s_2) \\ &= \int_{(m_1+m_2)^2}^{\infty} d\mu(\mathbf{s}) \sum_{nsl} \sum_j \oplus \mathcal{H}^{nsl}(\mathbf{s}, j). \end{aligned} \quad (3.3)$$

The sums in (3.3) extend over

$$j = \begin{array}{lll} 0 & 1 & \dots \quad \text{if } s_1 + s_2 = \text{integer} \\ 1/2 & 3/2 & \dots \quad \text{if } s_1 + s_2 = \text{half integer} \end{array}$$

and the degeneracy indices  $(l, s)$  for a given  $j$  are summed over

$$\begin{aligned} s &= s_1 + s_2, \quad s_1 + s_2 - 1, \quad \dots |s_1 - s_2| \\ l &= j + s, \quad j + s - 1, \quad j + s - 2, \quad \dots j - s. \end{aligned}$$

Here  $j$  represents the total angular momentum of the combined  $\pi^+\pi^-$  system; one of these values will be the resonance spin  $j_R$ . The degeneracy indices  $(s, l)$  for each fixed value of  $j$  are the total spin angular momentum and the total orbital angular momentum of the two  $\pi$ , respectively. The quantum number  $n$  is summed over all channel numbers that can be obtained by combining the species numbers  $n_1$  and  $n_2$  of the two  $\pi$ .

Instead of the invariant mass square  $\mathbf{s} = p_\mu p^\mu = E^2 - \mathbf{p}^2$  that we have used in (3.3) one often uses  $w = \sqrt{\mathbf{s}}$ , the invariant mass or the energy in the center of mass system of the two particles  $n_1, n_2$  [2–4]. The choice of the measure

$$d\mu(\mathbf{s}) = \rho(\mathbf{s}) d\mathbf{s}, \quad (3.3a)$$

(or if one uses  $w$ , of  $d\mu(w) = \rho(w) dw$ )

depends upon the normalization of the system of generalized basis vectors of (3.3). We shall use

$$\rho(\mathbf{s}) = 1, \quad \text{and then } \rho(w) = 2w \quad (3.3b)$$

<sup>4</sup>Though our discussions apply with obvious modifications to the general case of

$$1+2+3+\dots \rightarrow R_i \rightarrow 1'+2'+3'+\dots$$

these generalizations lead to enormously more complicated equations. For the sake of simplicity, we shall therefore consider a resonance scattering process like (3.1).

if we label the basis by  $w$  so that we do not change the “normalization” of the kets. The resonance space will be related (but will not be identical) to a subspace of (3.3) with a definite value of angular momentum  $j$  (e.g.,  $j = j_3^P = 1^-$  in case of the  $\rho$ -resonance of (3.1)). This is based on empirical evidence; resonances appear in one particular partial amplitude with a particular value of resonance spin  $j = j_R$  (though it may happen that there are more than one resonance in the same partial amplitude, but at different resonance energy  $\mathcal{S}_{R_1}, \mathcal{S}_{R_2}, \dots$ ). We will therefore single out a particular subspace

$$\mathcal{H}^{nls} = \int_{(m_1+m_2)^2}^{\infty} d\mathbf{s} \oplus \mathcal{H}^{nls}(\mathbf{s}, j) \quad (3.4)$$

with definite degeneracy or/and channel quantum numbers  $\eta = ls, n$ .

The reduction (3.3) is usually done using the Wigner momentum kets (2.6) in which the Clebsch-Gordan coefficients are given by [2–4]:

$$\langle p_1 s_1 p_2 s_2 [m_1 s_1, m_2 s_2] | p j_3 [w j], \eta \rangle, \quad (3.5)$$

where  $\eta$  now denotes  $\eta = n, l, s$ .

For the reasons mentioned above we want to work with the 4-velocity eigenkets  $|\hat{p}, j_3 [w, j], \eta\rangle$  which are eigenvectors of the operators

$$\hat{P}_\mu = (P_\mu^{(1)} + P_\mu^{(2)}) M^{-1}, \quad M^2 = (P_\mu^{(1)} + P_\mu^{(2)})(P^{(1)\mu} + P^{(2)\mu}) \quad (3.6)$$

with eigenvalues

$$\hat{p}^\mu = \begin{pmatrix} \hat{E} = \frac{P^0}{w} = \sqrt{1 + \hat{\mathbf{p}}^2} = \hat{p}^0 \\ \hat{\mathbf{p}} = \frac{\mathbf{P}}{w} \end{pmatrix} \quad (3.7)$$

and eigenvalues  $w^2 = \mathbf{s}$ . In here,  $\hat{P}_\mu^{(i)}$  are the 4-velocity operators in the one particle spaces  $\mathcal{H}^{n_i}(m_i, s_i)$  with eigenvalues  $\hat{p}_\mu^i = p_\mu^i / m_i$ . The Clebsch-Gordan coefficients are the transition coefficients  $\langle \hat{p}_1 \hat{p}_2 s_{13} s_{23} [m_1 s_1, m_2 s_2] | \hat{p} j_3 [w j], \eta \rangle$  between the direct product basis

$$|\hat{p}_1 s_{13} m_1 s_1\rangle \otimes |\hat{p}_2 s_{23} m_2 s_2\rangle \equiv |\hat{p}_1 \hat{p}_2 s_{13} s_{23} [m_1 s_1, m_2 s_2]\rangle \quad (3.8)$$

and the angular momentum basis  $|\hat{p} j_3 [w j], \eta\rangle$ .

To obtain the Clebsch-Gordan coefficients, one follows the same procedure as given in the classic papers [2–4] for the Clebsch-Gordan coefficients (3.5). This will be done in the Appendix, where the general case will be discussed. Here we shall restrict ourselves to the special case  $s_1 = 0, s_2 = 0$  to avoid the inessential complications due to the  $SO(3)$  Clebsch-Gordan coefficients for the angular momentum couplings  $s_1 \otimes s_2 \rightarrow s$ ,  $s \otimes l \rightarrow j$  and the occurrence of the Wigner rotations  $R(L^{-1}(\hat{p}), \hat{p}_i)$  of the inverse boost  $L^{-1}(\hat{p})$  which will enter in (3.5). Also for the process (3.1b) this is suffi-

cient, since  $s_{\pi^+} = s_{\pi^-} = 0$ . There is no degeneracy of the angular momentum basis vectors in this case and  $|\hat{p}j_3[wj]\rangle$  is given in terms of (3.8) by

$$|\hat{p}j_3[wj]\rangle = \int \frac{d^3\hat{p}_1}{2\hat{E}_1} \frac{d^3\hat{p}_2}{2\hat{E}_2} |\hat{p}_1\hat{p}_2[m_1m_2]\rangle \times \langle \hat{p}_1\hat{p}_2[m_1m_2] | \hat{p}j_3[wj] \rangle \quad (3.9)$$

for any  $(m_1 + m_2)^2 \leq w^2 < \infty \quad j=0,1,\dots$

The choice of the measure  $d^3\hat{p}_i/2\hat{E}_i(\hat{p}_i) = d^3p_i/m_i^2 2E_i$  is the same as (2.15a).

From the 4-translation invariance (conservation of 4-momentum) it follows that the Clebsch-Gordan is of the form

$$\langle \hat{p}_1\hat{p}_2 | \hat{p}j_3[wj] \rangle = \delta^4(p-r) \langle \langle \hat{p}_1\hat{p}_2 | \hat{p}j_3[wj] \rangle \rangle, \quad \text{where } r \equiv p_1 + p_2. \quad (3.10)$$

The reduced matrix element in the center-of-mass is in analogy to the non-relativistic case given by [11]

$$\langle \langle \hat{p}_1^{cm} \hat{p}_2^{cm} | \mathbf{0}j_3[wj] \rangle \rangle = Y_{jj_3}(\mathbf{e}) \tilde{\mu}_j(w, m_1, m_2), \quad (3.11)$$

where  $\tilde{\mu}_j(w, m_1, m_2)$  is a function of  $w$  (or  $\mathbf{s}$ ) which depends upon our choice of ‘‘normalization’’ for the basis vectors  $|\hat{p}j_3[wj]\rangle$  in (3.9). The equations (3.10) and (3.11) are combined into

$$\langle \hat{p}_1\hat{p}_2 | \hat{p}j_3[wj] \rangle = 2\hat{E}(\hat{\mathbf{p}}) \delta^3(\mathbf{p}-\mathbf{r}) \delta(w-\epsilon) Y_{jj_3}(\mathbf{e}) \mu_j(w, m_1, m_2) \quad \text{with } \epsilon^2 = r^2 = (p_1 + p_2)^2, \quad (3.12)$$

where again  $\mu_j(w, m_1, m_2)$  is a function that fixes the  $\delta$ -function ‘‘normalization’’ of  $|\hat{p}j_3[wj]\rangle$ . The unit vector  $\mathbf{e}$  in (3.11) is chosen to be in the c.m. frame the direction of  $\hat{\mathbf{p}}_1^{cm} = -m_2/m_1 \hat{\mathbf{p}}_2^{cm}$ . In general it is obtained from the relative ‘‘4-momentum’’  $q_\mu$  of Michel and Wightman [2] by  $e_i = L^{-1}(p)_i{}^\mu q_\mu$ . The  $\mu_j(w, m_1, m_2)$  and  $\tilde{\mu}_j(w, m_1, m_2)$  are some weight functions which are determined from the required ‘‘normalization’’ of the 4-velocity kets (3.9). Since for a fixed value of  $[wj]$  these generalized eigenvectors are the basis of the irreducible representation space  $\mathcal{H}(w, j)$  of the Poincaré group, we want them to be normalized like (2.15b), which in (3.9) has been already assured by the choice of the invariant measure  $d^3\hat{p}_i/2\hat{E}_i$ . Therefore, in analogy to (2.15c), we take for the normalization of the basis vectors (3.9) to be

$$\langle \hat{p}'j'_3[w'j'] | \hat{p}j_3[wj] \rangle = 2\hat{E}(\hat{\mathbf{p}}) \delta^3(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) \delta_{j'_3j_3} \delta_{j'_j j_j} \delta(\mathbf{s} - \mathbf{s}'),$$

$$\text{where } \hat{E}(\hat{\mathbf{p}}) = \sqrt{1 + \hat{\mathbf{p}}^2} = \frac{1}{w} \sqrt{w^2 + \mathbf{p}^2} \equiv \frac{1}{w} E(\mathbf{p}, w). \quad (3.13)$$

The  $\delta$ -function normalization  $\delta(\mathbf{s}' - \mathbf{s}) = 1/2w \delta(w - w')$  in (3.13) is a consequence of the choice (3.4) for the measure.

After we have chosen the normalization as in (3.13), one determines the weight function  $\mu_j(w, m_1, m_2)$  using (3.9). The result is:

$$|\mu_j(w, m_1, m_2)|^2 = \frac{2m_1^2 m_2^2 w^2}{\sqrt{\lambda \left( 1, \left( \frac{m_1}{w} \right)^2, \left( \frac{m_2}{w} \right)^2 \right)}}, \quad (3.14)$$

where  $\lambda$  is defined by [2]:

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac).$$

Except for the normalization factor  $\mu$ , which follows from our chosen normalization (3.13), the values of the Clebsch-Gordan coefficients (3.12) is quite obvious.<sup>5</sup> It expresses momentum conservation and the only factor that one may be puzzled about is that it should be consistent with the 4-velocity normalization expressed by the  $\delta^3(\hat{\mathbf{p}}' - \hat{\mathbf{p}})$ ,  $\hat{\mathbf{p}} = \mathbf{p}/w$  in (3.13). Therewith, we have obtained by (3.9) with (3.8) and (3.12) a system of basis vectors for the space (3.3) (with  $s_1 = s_2 = 0$ ) which is the representation space of scattering processes like (3.1b). As expected, the basis vectors are outside the Hilbert space;  $|\hat{p}j_3[wj]\rangle \in \Phi^\times \supset \mathcal{H} \supset \Phi$ . They have definite values of angular momentum  $j$  and invariant mass  $w \equiv \sqrt{\mathbf{s}}$ ;<sup>6</sup> we shall define the Gamow vectors (describing  $\rho^0$ ) in terms of linear combinations of these c.m.-energy eigenvectors with a definite value of  $j$ . However, since the resonances form and decay under the influence of an interaction and the  $|\hat{p}j_3[wj]\rangle$  are interaction-free eigenvectors of the ‘‘free-particle’’ Hamiltonian

$$K = P_1^0 + P_2^0 \quad (3.15)$$

we have to go from the free-particle basis vectors (3.9) to the interaction-basis vectors. This can be done in analogy to the non-relativistic case and may be justified in two ways:

(1) One assumes that the time translation generator for the interaction system has two terms ([7] Ch. 3),  $H_0$  and the interaction  $V$

$$H = H_0 + V \quad (3.16)$$

in such a way that to each eigenvector of  $H_0$  with eigenvalue  $E = w \sqrt{1 + \hat{\mathbf{p}}^2}$ ,

$$H_0 |\hat{p}j_3[wj]\rangle = E |\hat{p}j_3[wj]\rangle, \quad (3.17)$$

there correspond eigenvectors of  $H$  with the same eigenvalue

$$H |\hat{p}j_3[wj]^{\pm int}\rangle = E |\hat{p}j_3[wj]^{\pm int}\rangle. \quad (3.18)$$

<sup>5</sup>A formula like (3.12) is also given and explained in section 3.7 of [7] which for  $s=0, s_1=s_2=0$  agrees with (3.12) except for the normalization factor (3.14). For  $s \neq 0, s_i \neq 0$ , see Appendix.

<sup>6</sup>Written in terms of Hilbert spaces,  $d\mu(\mathbf{s})$  means Lebesgue integrations. However, within the rigged Hilbert space mathematics, one can choose for  $\langle \phi | \hat{p}j_3[wj] \rangle$  a smooth function and use Riemann integration and assign to each vector a well defined value  $w$  (not just up to a set of measure zero)

Since vectors are not completely defined by the requirement that they be eigenvectors of an operator with a given eigenvalue [but may differ by a phase factor (phase shifts) or unitary transformation (S-matrix) in case of degeneracy] we have added the additional label *int*. This additional specification of the eigenvectors can be chosen in a variety of ways that are connected with the spaces  $\Phi$  that one admits, i.e., with initial and final boundary conditions (as explained for the non-relativistic case in [11]). Since (3.16) may be a questionable hypothesis in relativistic physics a second justification does not make use of the existence of the Hamiltonian splitting (3.16).

(2) One assumes the existence of an *S*-operator and of Møller operators  $\Omega^+$  and  $\Omega^-$ .  $\Omega^+$  transforms non-interacting states  $\phi^{in}$  which are prepared by an apparatus far away from the interaction region into exact state vectors  $\phi^+$ ,

$$\Omega^+ \phi^{in} = \phi^+, \quad \phi^+(t) = e^{-iHt} \phi^+, \quad (3.19)$$

which evolve with the exact time-evolution operator *H*.  $\Omega^-$  transforms observables  $|\psi^{out}\rangle\langle\psi^{out}|$  registered by the detector placed far away from the interaction region into the vectors  $\psi^-$  which evolve with the exact *H* in the interaction region:

$$\Omega^- \psi^{out} = \psi^-, \quad \psi^-(t) = e^{iHt} \psi^+, \quad (3.20)$$

where *t* is the time in the c.m. frame. The basis vectors for the free-particle space and the interaction-basis vectors are then assumed to be related by<sup>7</sup>

$$|\hat{p}j_3[wj]^\pm\rangle = \Omega^\pm |\hat{p}j_3[wj]\rangle. \quad (3.21)$$

If (3.16) also holds then the symbol  $\Omega^\pm$  at the center-of-mass is given by the solution of the Lippmann-Schwinger equation

$$|\mathbf{0}j_3[wj]^\pm\rangle = \left(1 + \frac{1}{w - H \pm i\epsilon} V\right) |\mathbf{0}j_3[wj]\rangle. \quad (3.22)$$

The vectors  $|\hat{p}j_3[wj]^\pm\rangle$  are obtained from the basis vectors at rest  $|\mathbf{0}j_3[wj]^\pm\rangle$  by the boost (rotation-free Lorentz transformation)  $\mathcal{U}(L(\hat{p}))$  whose parameters are the  $\hat{p}^m$  and whose generators are the interaction-incorporating observables

$$P_0 = H, \quad P^m, \quad J_{\mu\nu}, \quad (3.23)$$

i.e., the exact generators of the Poincaré group ([7] section 3.3). These vectors (3.22), which for a fixed value of  $[wj]$  span an irreducible representation space of the Poincaré group with the ‘‘exact generators,’’ will be used for the definition of the relativistic Gamow vectors. The values of *j* and  $\mathbf{s} = w^2$  are *j* = integer (for  $s_1 = s_2 = 0$  otherwise also half integer) and  $(m_1 + m_2)^2 \leq \mathbf{s} < \infty$ . The value of *j* will be fixed and represents the resonance spin; the same we do with parity and the degeneracy quantum numbers (*n*,  $\eta$ ). The values

<sup>7</sup>In nonrelativistic scattering off a fixed target one assumes that the  $|\mathbf{p}^+\rangle$  related by (3.21) to the  $|\mathbf{p}\rangle$  are not eigenvectors of *P* since  $[V, P_i] \neq 0$ .

of *s* we shall continue from the physical values into the complex plane of the relativistic *S*-matrix.

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## APPENDIX: REDUCTION OF THE DIRECT PRODUCT OF TWO ONE-PARTICLE UIR OF $\mathcal{P}$

The discussion here follows the one in Ref. [4] with the difference that here the two particle irreducible representation spaces are labeled by the square of the total invariant mass *s* instead of  $w = \sqrt{\mathbf{s}}$ ; and the velocity basis are used instead of the momentum basis.

We discuss here the reduction of the direct product of two one-particle irreducible representation spaces of the Poincaré group  $[m_1, s_1] \otimes [m_2, s_2]$  into a continuous direct sum of irreducible representation (irrep) spaces  $[\mathbf{s}, j]$  of invariant mass squared *s* and spin *j*. This has been done in [2–4] using the Wigner basis systems of momentum eigenvectors. Here we shall do it using the 4-velocity basis vectors of the Poincaré group  $\mathcal{P}$  and obtain the Clebsch-Gordan coefficients of  $\mathcal{P}$  for the velocity basis. For the one particle spaces, we choose the c.s.c.o. (2.13) with the generalized eigenvectors (2.14). Thus, the one particle spaces  $\mathcal{H}(m, j)$  are labeled by the mass *m* and the spin *j* of the particle. In analogy to the case of one-particle, a two-particle irrep space is labeled by the square of the total invariant mass  $\mathbf{s} = (p_1 + p_2)^2$  and the total angular momentum *j* of the two particles. The two-particle irrep space is denoted by  $\mathcal{H}_n^\eta(\mathbf{s}, j)$ , where  $\eta$  is a degeneracy label and *n* is a particle species label. Thus the reduction problem is written as

$$\mathcal{H}(m_1, s_1) \otimes \mathcal{H}(m_2, s_2) = \sum_{j\eta} \int_{(m_1+m_2)^2}^{\infty} \oplus \mathcal{H}_n^\eta(\mathbf{s}, j) d\mathbf{s}. \quad (A1)$$

As in (2.14), the two-particle basis vectors of  $\mathcal{H}_n^\eta(\mathbf{s}, j)$  have as the only continuous variables the total four velocity of the two particles and the square of the total invariant mass of the two particles. These basis vectors are denoted by:

$$|\hat{p}\sigma[\mathbf{s}j]\eta, n\rangle \quad (A2)$$

with the normalization:

$$\langle \hat{p}'\sigma'[\mathbf{s}'j']\eta', n' | \hat{p}\sigma[\mathbf{s}j]\eta, n \rangle = 2\hat{p}_0 \delta_{nn'} \delta_{jj'} \delta_{\sigma\sigma'} \delta_{\eta\eta'} \delta^3(\hat{\mathbf{p}} - \hat{\mathbf{p}}') \delta(\mathbf{s} - \mathbf{s}'), \quad (A3)$$

where  $\sigma$  is the three-component of the total angular momentum *j*. We denote the basis vectors of  $\mathcal{H}(m_1, s_1) \otimes \mathcal{H}(m_2, s_2)$  by:

$$\begin{aligned} & |\hat{p}_1\sigma_1[m_1s_1]\rangle \otimes |\hat{p}_2\sigma_2[m_2s_2]\rangle \\ & \equiv |\hat{p}_1\sigma_1[m_1s_1], \hat{p}_2\sigma_2[m_2s_2]\rangle, \end{aligned} \quad (A4)$$

where  $\sigma_1, \sigma_2$  are the three-components of the spins  $s_1, s_2$  respectively. In order to obtain the Clebsch-Gordan coefficients,

$$\langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2] | \hat{p} \sigma [Sj] \eta, n \rangle, \quad (\text{A5})$$

of the reduction (A1), we start by relabeling the basis vectors in (A4) by using  $\mathbf{s}$ ,  $\hat{p}$  and the unit vector  $\hat{\mathbf{n}} = (\mathbf{p}_1 - \mathbf{p}_2) / |\mathbf{p}_1 - \mathbf{p}_2|$  as continuous parameters (we note that both sets,  $\{\hat{p}_1, \hat{p}_2\}$  and  $\{\hat{p}, \hat{\mathbf{n}}, \mathbf{s}\}$  consist of six independent parameters). Thus, we can write:

$$|\hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2]\rangle \equiv |\hat{p} \hat{\mathbf{n}} \mathbf{s}, \sigma_1 [m_1 s_1] \sigma_2 [m_2 s_2]\rangle. \quad (\text{A6})$$

In the rest frame of both particles, i.e., for

$$\hat{p} = \hat{p}_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we can expand the unit vector  $\hat{\mathbf{n}}$  in terms of orbital angular momentum basis vectors:

$$|\hat{\mathbf{n}}\rangle = \sum_{l_3} |ll_3\rangle \langle ll_3 | \hat{\mathbf{n}} \rangle = \sum_{l_3} |ll_3\rangle Y_{l_3}^*(\hat{\mathbf{n}}). \quad (\text{A7})$$

We can further use the angular momentum Clebsch-Gordan coefficients to combine the two spins,  $s_1$  and  $s_2$ , to give a total spin  $s$  with three component  $\mu$ , which in turn is added to the orbital angular momentum  $l$  with three component  $l_3$  to form a total angular momentum  $j$  with three component  $\sigma$ . This gives the basis vector for the two-particle irrep space

$$|\hat{p} \sigma [Sj] l s; m_1 s_1, m_2 s_2\rangle. \quad (\text{A8})$$

Thus, the degeneracy label  $\eta$  in (A2) designates the total spin  $s$  and the total orbital angular momentum  $l$  of both particles; and the masses  $m_1, m_2$  and spins  $s_1, s_2$  of both particles are included in the particle species label  $n$ . Thus, (A1) can be rewritten in more details as:

$$\mathcal{H}(m_1, s_1) \oplus \mathcal{H}(m_2, s_2) = \sum_{jls} \int_{(m_1+m_2)^2}^{\infty} \oplus \mathcal{H}_n^{ls}(\mathbf{s}, j) d\mathbf{s}, \quad (\text{A9})$$

where  $s = |s_1 - s_2|, |s_1 - s_2| + 1, \dots, s_1 + s_2$

$$j = |l - s|, |l - s| + 1, \dots, l + s.$$

With (A6) and (A7), we deduce that in the rest frame, the Clebsch-Gordan coefficients of (A9) are given by:

$$\begin{aligned} & \langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n | \hat{p}_R \sigma [Sj], \eta, n' \rangle \\ & = 2N_n(\mathbf{s}) \delta_{nn'} \theta(\mathbf{s} - (m_1 + m_2)^2) \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \\ & \quad \times \delta(\mathbf{s} - (p_1 + p_2)^2) \\ & \quad \times \sum_{l_3 \mu} C_{s_1 s_2}(s \mu, \sigma_1 \sigma_2) C_{sl}(j \sigma, \mu l_3) Y_{l_3}(\hat{\mathbf{n}}), \end{aligned} \quad (\text{A10})$$

where  $N_n(\mathbf{s})$  is a normalization factor. Having obtained the Clebsch-Gordan coefficients in the rest frame (A10), we can use the boost operator (2.17b) to obtain the Clebsch-Gordan coefficients in a general frame:<sup>8</sup>

$$\begin{aligned} & \langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n | \hat{p} \sigma [Sj] \eta, n' \rangle \\ & = \langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n | \mathcal{U}(L(p)) | \hat{p}_R \sigma [Sj] \eta, n' \rangle \\ & = 2\hat{p}_0 N_n(\mathbf{s}) \delta_{nn'} \theta(\mathbf{s} - (m_1 + m_2)^2) \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \\ & \quad \times \delta(\mathbf{s} - (p_1 + p_2)^2) \sum_{\sigma'_1 \sigma'_2} D_{\sigma'_1 \sigma_1}^{s_1^*} [R(L^{-1}(p), p_1)] \\ & \quad \times D_{\sigma'_2 \sigma_2}^{s_2^*} [R(L^{-1}(p), p_2)] \\ & \quad \times \sum_{l_3 \mu} C_{s_1 s_2}(s \mu, \sigma'_1 \sigma'_2) C_{sl}(j \sigma, \mu l_3) Y_{l_3}(\mathbf{e}), \end{aligned} \quad (\text{A11})$$

where  $R(\lambda, p)$  is the Wigner rotation given in (2.7c) and

$$\mathbf{e} = \frac{\overrightarrow{L^{-1}(p)(p_1 - p_2)}}{|\overrightarrow{L^{-1}(p)(p_1 - p_2)}}.$$

The normalization factor  $N_n(\mathbf{s})$  depends upon our normalization choice (A3). Before discussing how to obtain it, let us first introduce the following notations:

$$\begin{aligned} \Gamma(s_1 \sigma_1, s_2 \sigma_2, s \mu) & = \sum_{\sigma'_1 \sigma'_2} D_{\sigma'_1 \sigma_1}^{s_1^*} [R(L^{-1}(p), p_1)] \\ & \quad \times D_{\sigma'_2 \sigma_2}^{s_2^*} [R(L^{-1}(p), p_2)] \\ & \quad \times C_{s_1 s_2}(s \mu, \sigma'_1 \sigma'_2), \end{aligned} \quad (\text{A12a})$$

$$Y_{j \sigma l s}(\mathbf{e}, \mu) = \sum_{l_3} C_{sl}(j \sigma, \mu l_3) Y_{l_3}(\mathbf{e}). \quad (\text{A12b})$$

With the above notations, (A11) is written as

<sup>8</sup>Formula (3.7.5) in [7], which corresponds to (A11) but for different choices of basis and normalizations, is missing the rotation matrices factors that appear in the Clebsch-Gordan coefficients away from the rest frame, as exhibited in (A11).

$$\begin{aligned} \langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n | \hat{p} \sigma [s_j] \eta, n' \rangle &= 2 \hat{p}_0 N_n(\mathbf{s}) \delta_{nn'} \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta(\mathbf{s} - (p_1 + p_2)^2) \\ &\times \sum_{\mu} \Gamma(s_1 \sigma_1, s_2 \sigma_2, s \mu) Y_{j \sigma s}(\mathbf{e}, \mu). \end{aligned} \tag{A13}$$

In order to obtain the normalization factor  $N_n(\mathbf{s})$ , we insert a complete set of basis vectors (A4) in  $\langle \hat{p}' \sigma' [s' j'] \eta', n' | \hat{p} \sigma [s_j] \eta, n \rangle$  and use (A13). Upon doing so, we obtain:

$$\begin{aligned} \langle \hat{p}' \sigma' [s' j'] \eta', n' | \hat{p} \sigma [s_j] \eta, n \rangle &= \sum_{n'' \sigma_1 \sigma_2} \int \frac{d^3 \hat{p}_1}{2 \hat{p}_1^0} \frac{d^3 \hat{p}_2}{2 \hat{p}_2^0} \langle \hat{p}' \sigma' [s' j'] \eta', n' | \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n'' \rangle \\ &\times \langle \hat{p}_1 \sigma_1 [m_1 s_1], \hat{p}_2 \sigma_2 [m_2 s_2], n'' | \hat{p} \sigma [s_j] \eta, n \rangle \\ &= (2 \hat{p}_0)^2 |N_n(\mathbf{s})|^2 \delta_{nn'} \delta^3(\mathbf{p} - \mathbf{p}') \delta(\mathbf{s} - \mathbf{s}') \\ &\times \sum_{\sigma_1 \sigma_2} \sum_{\mu \mu'} \int \frac{d^3 \hat{p}_1}{2 \hat{p}_1^0} \frac{d^3 \hat{p}_2}{2 \hat{p}_2^0} \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta(\mathbf{s} - (p_1 + p_2)^2) \\ &\times \Gamma^*(s_1 \sigma_1, s_2 \sigma_2, s' \mu') \Gamma(s_1 \sigma_1, s_2 \sigma_2, s \mu) Y_{j' \sigma' \eta'}^*(\mathbf{e}, \mu') Y_{j \sigma \eta}(\mathbf{e}, \mu). \end{aligned} \tag{A14}$$

Using the unitarity of the rotation matrices:

$$\sum_{\sigma} D_{\sigma' \sigma}^{*j} D_{\sigma'' \sigma}^j = \delta_{\sigma' \sigma''}$$

and the identity

$$\sum_{\sigma_1 \sigma_2} C_{s_1 s_2}(s \mu, \sigma_1 \sigma_2) C_{s_1 s_2}(s' \mu', \sigma_1 \sigma_2) = \delta_{s s'} \delta_{\mu \mu'},$$

we find that

$$\sum_{\sigma_1 \sigma_2} \Gamma^*(s_1 \sigma_1, s_2 \sigma_2, s' \mu') \Gamma(s_1 \sigma_1, s_2 \sigma_2, s \mu) = \delta_{s s'} \delta_{\mu \mu'}. \tag{A15}$$

With the identity (A15), (A14) can be written as:

$$\begin{aligned} \langle \hat{p}' \sigma' [s' j'] \eta', n' | \hat{p} \sigma [s_j] \eta, n \rangle &= (2 \hat{p}_0)^2 |N_n(\mathbf{s})|^2 \delta_{nn'} \delta^3(\mathbf{p} - \mathbf{p}') \delta(\mathbf{s} - \mathbf{s}') \delta_{s s'} \\ &\times \sum_{\mu l_3 l'_3} C_{s l' \mu} (j' \sigma', \mu l'_3) C_{s l \mu} (j \sigma, \mu l_3) \\ &\times \int \frac{d^3 \hat{p}_1}{2 \hat{p}_1^0} \frac{d^3 \hat{p}_2}{2 \hat{p}_2^0} \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \\ &\times \delta(\mathbf{s} - (p_1 + p_2)^2) Y_{l'_3}^*(\mathbf{e}) Y_{l_3}(\mathbf{e}). \end{aligned} \tag{A16}$$

In order to solve the integration in (A16), namely

$$\begin{aligned} I &= \int \frac{d^3 \hat{p}_1}{2 \hat{p}_1^0} \frac{d^3 \hat{p}_2}{2 \hat{p}_2^0} \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \\ &\times \delta(\mathbf{s} - (p_1 + p_2)^2) Y_{l'_3}^*(\mathbf{e}) Y_{l_3}(\mathbf{e}) \\ &= \frac{1}{m_1^2 m_2^2} \int \frac{d^3 p_1}{2 p_1^0} \frac{d^3 p_2}{2 p_2^0} \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \\ &\times \delta(\mathbf{s} - (p_1 + p_2)^2) Y_{l'_3}^*(\mathbf{e}) Y_{l_3}(\mathbf{e}), \end{aligned} \tag{A17}$$

we perform the change of variables (as in equation (4.9) in [2]):

$$\begin{aligned} p_1 &= \frac{(\mathbf{s} + m_1^2 - m_2^2)}{2 \mathbf{s}} r + \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{2 \sqrt{\mathbf{s}}} q \\ p_2 &= \frac{(\mathbf{s} - m_1^2 + m_2^2)}{2 \mathbf{s}} r - \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{2 \sqrt{\mathbf{s}}} q \end{aligned} \tag{A18}$$

where

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc).$$

With these new variables, we find that

$$\begin{aligned} \delta(p_1^2 - m_1^2) \delta(p_2^2 - m_2^2) \delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2) \delta(\mathbf{s} - (p_1 + p_2)^2) \\ = \frac{4 \mathbf{s}^{3/2}}{\lambda^{3/2}(\mathbf{s}, m_1^2, m_2^2)} \frac{1}{2 p_0} \delta(q^2 + 1) \delta(r \cdot q) \delta^4(r - p) \end{aligned} \tag{A19a}$$

$$d^4 p_1 d^4 p_2 = \frac{\lambda^2(\mathbf{s}, m_1^2, m_2^2)}{16 \mathbf{s}^2} d^4 r d^4 q \tag{A19b}$$

and

$$\mathbf{e} = \overrightarrow{L^{-1}(p)q}. \quad (\text{A19c})$$

Using (A19), the integration (A17) becomes:

$$I = \frac{1}{m_1^2 m_2^2} \frac{1}{2p_0} \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{4\sqrt{\mathbf{s}}} \int d^4q \delta(q^2 + 1) \delta(p \cdot q) \\ \times Y_{l'_1 l'_3}^* \overrightarrow{L^{-1}(p)q} Y_{l_3} \overrightarrow{L^{-1}(p)q}. \quad (\text{A20})$$

Performing the change of variable  $\mathbf{e} = L^{-1}(p)q$  in (A20), we obtain:

$$I = \frac{1}{m_1^2 m_2^2} \frac{1}{2p_0} \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{8\mathbf{s}} \int d\Omega(\mathbf{e}) Y_{l'_1 l'_3}^*(\mathbf{e}) Y_{l_3}(\mathbf{e}) \\ = \frac{1}{m_1^2 m_2^2} \frac{1}{2p_0} \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{8\mathbf{s}} \delta_{l'l'} \delta_{l_3 l'_3}. \quad (\text{A21})$$

Using (A21) and the identity

$$\sum_{\mu l_3} C_{sl}(j' \sigma', \mu l_3) C_{sl}(j \sigma, \mu l_3) = \delta_{jj'} \delta_{\sigma\sigma'},$$

(A16) finally becomes:

$$\langle \hat{p}' \sigma' [s' j'] \eta', n' | \hat{p} \sigma [s j] \eta, n \rangle \\ = (2\hat{p}_0) |N_n(\mathbf{s})|^2 \frac{1}{m_1^2 m_2^2} \frac{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}{8\mathbf{s}^3} \\ \times \delta_{nn'} \delta_{jj'} \delta_{\sigma\sigma'} \delta_{\eta\eta'} \delta^3(\hat{\mathbf{p}} - \hat{\mathbf{p}}') \delta(\mathbf{s} - \mathbf{s}'). \quad (\text{A22})$$

Comparing (A22) with (A3), we find that:

$$|N_n(\mathbf{s})|^2 = \frac{8m_1^2 m_2^2 \mathbf{s}^3}{\lambda^{1/2}(\mathbf{s}, m_1^2, m_2^2)}. \quad (\text{A23})$$

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