# Coherent population trapping in open systems: A coupled/noncoupled-state analysis

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Coherent population trapping (CPT) with losses towards off-resonant levels is analyzed on the basis of the coupled/noncoupled states. We derive that at a long interaction time the noncoupled state is depopulated at a rate that is decreasing for decreasing Raman detuning. This behavior produces a narrowing of the dark resonance with the interaction time. The asymptotic evolution of the atomic system is completely characterized by the effective width $\Gamma_{NC}$ of the noncoupled state. The analytic expression for this width evidences a quadratic dependence on the Raman detuning. This dependence implies that the width of the dark resonance decreases asymptotically with the inverse of the square root of the interaction time. This narrowing law, independent of the interaction scheme, has a general validity for CPT in open systems. Our analysis points out the differences between CPT with losses and CPT in closed systems. [S1050-2947(99)08707-7]

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## I. INTRODUCTION

Coherent population trapping (CPT) was discovered in the interaction of a sodium vapor with a multimode laser field [1]. In that experiment an inhomogeneous magnetic field was applied to modify the Zeeman splitting of the atomic levels along the laser path within the vapor. Sharp decreases of the sodium fluorescence (''dark resonances'') were observed whenever the frequency difference between two modes of the laser field matched the splitting between two Zeeman sublevels belonging to different ground-state hyperfine states. Dark resonances of the same nature were also observed in atomic-beam experiments [2].

Dark resonances are associated [2–4] with a trapping of the population in a coherent superposition of ground states: because of destructive quantum interference between different excitation pathways this ground-state coherent superposition (''dark state'') is decoupled from the laser light. In these early theoretical works the interaction atom-laser field was described by using a three-level atomic system composed of two ground states coupled to a common excited state by two laser fields ($\Lambda$ system). The system was assumed to be closed, i.e., the total population of the levels constituting the $\Lambda$ system was conserved. By analyzing the steady-state solution of the relevant optical Bloch equations (OBEs) it was shown that the stationary population of the excited state exhibits a sharp decrease in the correspondence of the two-photon resonance.

Later theoretical investigations clarified other aspects of the CPT phenomenon in a closed three-level $\Lambda$ system: the time evolution towards the nonabsorbing state was numerically studied and the typical time scale determined as a function of the interaction parameters [5], the effect of a finite interaction time examined by introducing in the OBE appropriate relaxation terms [6]. However, in most CPT experiments with alkali-metal atoms, including the first CPT observations [1,2], the $\Lambda$ system based on the two ground states forming the nonabsorbing superposition is not closed: the population can be in fact lost in other ground-state sublevels that are either out of resonance with the laser light or that cannot be excited due to selection rules.

The theoretical description of coherent population trapping in open systems is not a trivial extension of the early theoretical works on CPT, because in an open system the steady-state population of the excited state is zero independently of the value of the Raman detuning. In this case the transient regime is not negligible and a time-dependent analysis is therefore required to describe the properties of CPT in open systems. Recently coherent population trapping with losses towards levels out of resonance has been investigated, theoretically and experimentally, by analyzing the CPT schemes established on the sodium $D_1$ line in a Hanle effect configuration [7,8]. It was shown that both the contrast and the width of the dark resonance are strongly influenced by the population loss: the contrast is significantly reduced and the width of the dark resonance decreases without a lower limit for an increasing interaction time. A numerical analysis showed that the width of the dark resonance asymptotically scales with the inverse of the square root of the interaction time $t$ for all the CPT Hanle schemes examined. This is at variance with the case of a closed system [4], in which the optical pumping process into the dark state leads to an almost complete atomic preparation in the nonabsorbing superposition. There the narrowing of the dark resonance is limited by a power broadening term, and such a limit is reached with a $1/t$ law.

In this work coherent population trapping with losses is analyzed on the basis of the coupled/noncoupled states. The analysis is performed for a $J_g=1\rightarrow J_e=0$ transition interacting with two resonant $\sigma^+,\sigma^-$ laser fields. A static magnetic field is applied to shift the ground levels. We show that the long-interaction-time evolution of the atomic system can be completely characterized by the effective width $\Gamma_{NC}$ of the noncoupled state due to the light irradiation and to the applied magnetic field. Such a width is perturbatively calculated. From $\Gamma_{NC}$ the asymptotic law for the width of the dark resonance is derived. We clarify that this narrowing law has a general validity for CPT in open systems and is independent of the interaction scheme. Finally, we discuss formally

 

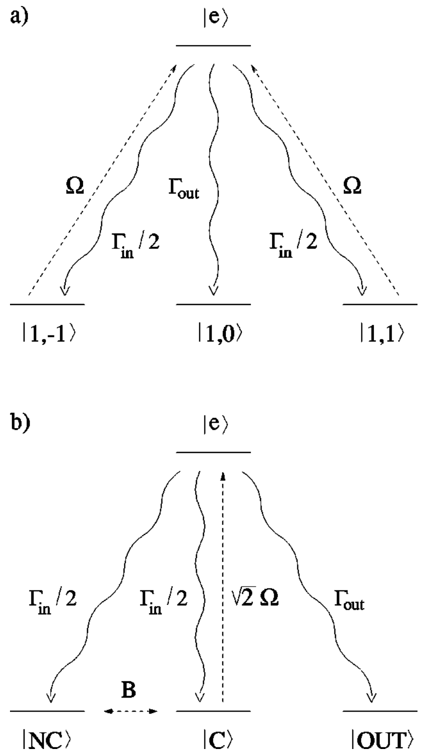


FIG. 1. Interaction scheme of the transition $|J_g=1\rangle \rightarrow |J_e=0\rangle$ resonant with two $\sigma^+, \sigma^-$ laser fields. For the ground state, in (a) the $\{|J_g, M\rangle\}$ basis is used, in (b) the basis of the coupled/noncoupled states is used.

where the sharp difference in the narrowing law of closed and open systems originates.

## II. INTERACTION SCHEME

As a simple system for the study of coherent population trapping with losses, we consider a transition $|J_g=1\rangle \rightarrow |J_e=0\rangle$ interacting resonantly with two $\sigma^+, \sigma^-$ laser fields [Fig. 1(a)]. The CPT system composed of the two ground states $|J_g=1, M=\pm 1\rangle$ and by the excited state $|J_e=0\rangle$ is open: the population can be lost in the ground level $|J_g=1, M=0\rangle$, in the following, indicated as $|\text{out}\rangle$. We indicate the partial decay rate $\Gamma_{|e\rangle \rightarrow |\text{out}\rangle}$ by $\Gamma_{\text{out}}$ and the decay rates $\Gamma_{|e\rangle \rightarrow |J_g=1, M=\pm 1\rangle}$, assumed to be equal, by $\Gamma_{\text{in}}/2$.

In our analysis the atom is at rest and recoil effects due to the absorption and emission of photons are not taken into account. The two $\sigma^+, \sigma^-$ laser fields are derived from linearly polarized laser light, i.e., they have equal intensity and propagation direction. The laser frequency $\omega$ coincides with the atomic transition frequency (which therefore will be referred to as $\omega$ too). A static magnetic field of strength $B$ is applied collinear to the laser field wave vector.

Under these assumptions, the Hamiltonian of the system is

$$H = H_0 + H_B + V_{\text{AL}}, \tag{1}$$

where $H_0$ is the Hamiltonian of the atomic system in zero magnetic field,

$$H_0 = \hbar\omega |e\rangle\langle e|. \tag{2}$$

$H_B$ describes the energy shifts due to the applied magnetic field

$$H_B = g\mu_B B(|g,+1\rangle\langle g,+1| - |g,-1\rangle\langle g,-1|), \tag{3}$$

where $g$ is the giromagnetic factor and $\mu_B$ the Bohr magneton. Finally, $V_{\text{AL}}$ represents the coupling atom-laser field, which in the dipole and rotating-wave approximation is

$$V_{\text{AL}} = \hbar\Omega \exp(-i\omega t)(|e\rangle\langle g,-1| + |e\rangle\langle g,+1|) + \text{H.c.} \tag{4}$$

A transparent description of the CPT phenomenon can be reached in the so-called coupled/noncoupled-state basis [4]. This basis is composed of the excited state $|e\rangle$ and of the two linear combinations of ground states

$$|\text{C}\rangle = (|g,-1\rangle + |g,+1\rangle)/\sqrt{2}, \tag{5a}$$

$$|\text{NC}\rangle = (|g,-1\rangle - |g,+1\rangle)/\sqrt{2}. \tag{5b}$$

The noncoupled state $|\text{NC}\rangle$ is decoupled from the laser field for any value of the applied magnetic field $B$:

$$V_{\text{AL}}|\text{NC}\rangle = 0. \tag{6}$$

On the other hand, the magnetic field induces a coupling between $|\text{C}\rangle$ and $|\text{NC}\rangle$,

$$\langle \text{C}|H_B|\text{NC}\rangle = -g\mu_B B. \tag{7}$$

Finally, the laser fields couple $|\text{C}\rangle$ to the excited state

$$V_{\text{AL}}|\text{C}\rangle = \sqrt{2}\hbar\Omega \exp(-i\omega t)|e\rangle. \tag{8}$$

In summary the couplings due to the Hamiltonian

$$H_{\text{I}} = H_B + V_{\text{AL}} \tag{9}$$

are, in the coupled/noncoupled state basis,

$$\langle \text{C}|H_{\text{I}}|\text{NC}\rangle = -g\mu_B B, \tag{10}$$

$$\langle e|H_{\text{I}}|\text{C}\rangle = \sqrt{2}\hbar\Omega \exp(-i\omega t), \tag{11}$$

as shown in Fig. 1(b).

## III. OPTICAL BLOCH EQUATIONS

Introducing the Raman detuning $\delta = 2g\mu_B B/\hbar$ and the new density-matrix elements $\tilde{\rho}_{\text{C},e} = \rho_{\text{C},e}\exp(-i\omega t)$, $\tilde{\rho}_{\text{NC},e} = \rho_{\text{NC},e}\exp(-i\omega t)$, the OBEs are

$$\dot{\rho}_{\text{NC,NC}} = \frac{\Gamma_{\text{in}}}{2}\rho_{e,e} + \delta\,\text{Im}\,\rho_{\text{NC,C}}, \tag{12a}$$

$$\dot{\rho}_{\text{C,C}} = \frac{\Gamma_{\text{in}}}{2}\rho_{e,e} - \delta\,\text{Im}\,\rho_{\text{NC,C}} + 2\sqrt{2}\Omega\,\text{Im}\,\tilde{\rho}_{e,\text{C}}, \tag{12b}$$

$$\dot{\rho}_{e,e} = -(1+\alpha)\Gamma_{\text{in}}\rho_{e,e} - 2\sqrt{2}\Omega\,\text{Im}\,\tilde{\rho}_{e,\text{C}}, \tag{12c}$$

$$\dot{\tilde{\rho}}_{e,\text{NC}} = -\frac{1+\alpha}{2}\Gamma_{\text{in}}\tilde{\rho}_{e,\text{NC}} - i\frac{\delta}{2}\tilde{\rho}_{e,\text{C}} - i\sqrt{2}\Omega\rho_{\text{C,NC}}, \tag{12d}$$

$$\dot{\tilde{\rho}}_{e,\mathrm{C}}=-\frac{1+\alpha}{2}\Gamma_{\mathrm{in}}\tilde{\rho}_{e,\mathrm{C}}-i\frac{\delta}{2}\tilde{\rho}_{e,\mathrm{NC}}-i\sqrt{2}\Omega(\rho_{\mathrm{C,C}}-\rho_{e,e}),\tag{12e}$$

$$\dot{\rho}_{\mathrm{NC,C}}=i\frac{\delta}{2}\rho_{\mathrm{C,C}}+i\sqrt{2}\Omega\tilde{\rho}_{\mathrm{NC},e}-i\frac{\delta}{2}\rho_{\mathrm{NC,NC}},\tag{12f}$$

where we indicated by $\alpha$ the branching ratio $\Gamma_{\mathrm{out}}/\Gamma_{\mathrm{in}}$. By introducing a vector $\vec{\rho}$ formed by the real and imaginary parts of the density matrix

$$\vec{\rho}\overset{\mathrm{def}}{=}(\rho_{\mathrm{NC,NC}},\rho_{\mathrm{C,C}},\rho_{e,e},\mathrm{Re}\,\tilde{\rho}_{e,\mathrm{NC}},\mathrm{Im}\,\tilde{\rho}_{e,\mathrm{NC}},\mathrm{Re}\,\tilde{\rho}_{e,\mathrm{C}},\times\mathrm{Im}\,\tilde{\rho}_{e,\mathrm{C}},\mathrm{Re}\,\rho_{\mathrm{NC,C}},\mathrm{Im}\,\rho_{\mathrm{NC,C}}),\tag{13}$$

the OBEs can be rewritten as

$$\dot{\vec{\rho}}=M\vec{\rho}.\tag{14}$$

In this way the problem of solving the OBEs is reduced to the problem of finding the eigenvalues $\lambda_i$ and the eigenvectors $\vec{v}_i$ of the real *Bloch matrix M*. Consistently with the definition (13), in the following we will indicate the first (second, third, ...) component(s) of the eigenvector $\vec{v}_k$ by $v_{k,\mathrm{NC,NC}}$ ($v_{k,\mathrm{C,C}},v_{k,e,e}$, ...).

## IV. SOLUTION OF THE OBE AT RAMAN RESONANCE

In this section we examine, in the low-intensity limit, the asymptotic evolution of the atomic system in the case of zero Raman detuning. At the steady state a fraction of the initial population of the CPT system is in the noncoupled state, the rest having been optically pumped into the $|\mathrm{out}\rangle$ state. The steady-state population $\rho_{\mathrm{NC,NC}}(\infty)$ of the noncoupled state is independent of the (nonzero) Rabi frequency and can be determined without solving the OBE. A straightforward calculation gives

$$\rho_{\mathrm{NC,NC}}(\infty)=\rho_{\mathrm{NC,NC}}(0)+\rho_{\mathrm{C,C}}(0)\frac{1}{2\alpha+1}.\tag{15}$$

To study the evolution towards the steady state, we determine the eigenvalues of the Bloch matrix. At Raman resonance the calculation is straightforward, and the exact expressions for the eigenvalues are easily found. However, for some eigenvalue this expression is very lengthy and will therefore be reported only to second order in $\Omega/\Gamma_{\mathrm{in}}$. The eigenvalues of the Bloch matrix are

$$\lambda_1=0,\tag{16a}$$

$$\lambda_2=-\frac{1+\alpha}{2}\Gamma_{\mathrm{in}},\tag{16b}$$

$$\lambda_{3,4}=-\frac{1+\alpha}{4}\Gamma_{\mathrm{in}}-\sqrt{\left(\frac{1+\alpha}{4}\Gamma_{\mathrm{in}}\right)^2-2\Omega^2},\tag{16c}$$

$$\lambda_{5,6}=-\frac{1+\alpha}{4}\Gamma_{\mathrm{in}}+\sqrt{\left(\frac{1+\alpha}{4}\Gamma_{\mathrm{in}}\right)^2-2\Omega^2},\tag{16d}$$

$$\lambda_7=-4\frac{1+2\alpha}{(1+\alpha)^2}\frac{\Omega^2}{\Gamma_{\mathrm{in}}}+O\left(\left(\frac{\Omega}{\Gamma_{\mathrm{in}}}\right)^4\right),\tag{16e}$$

$$\lambda_8=-(1+\alpha)\Gamma_{\mathrm{in}}+\frac{4(3+2\alpha)}{(1+\alpha)^2}\frac{\Omega^2}{\Gamma_{\mathrm{in}}}+O\left(\left(\frac{\Omega}{\Gamma_{\mathrm{in}}}\right)^4\right),\tag{16f}$$

$$\lambda_9=-\frac{1+\alpha}{2}\Gamma_{\mathrm{in}}-\frac{8}{(1+\alpha)^2}\frac{\Omega^2}{\Gamma_{\mathrm{in}}}+O\left(\left(\frac{\Omega}{\Gamma_{\mathrm{in}}}\right)^4\right).\tag{16g}$$

The zero eigenvalue corresponds to the eigenvector

$$\vec{v}_1=(1,0,0,0,0,0,0,0,0);\tag{17}$$

i.e., it expresses the stability of the dark state: an atom initially prepared in the dark state does not evolve.

In the low-intensity limit all the eigenvalues are real and, except for $\lambda_1$, negative. The smallest, in absolute value, nonzero eigenvalue determines the time scale for the system to reach the steady state. In the present case the smallest eigenvalues are

$$\lambda_{5,6}=\frac{-4}{1+\alpha}\frac{\Omega^2}{\Gamma_{\mathrm{in}}}+O\left(\left(\frac{\Omega}{\Gamma_{\mathrm{in}}}\right)^4\right).\tag{18}$$

By inspecting the corresponding eigenvectors $\vec{v}_5$ and $\vec{v}_6$, we found that the components $v_{5,\mathrm{NC,NC}},v_{5,\mathrm{C,C}},v_{5,e,e}$, and analogously for $\vec{v}_6$, are zero. Therefore the eigenvalues $\lambda_{5,6}$ do not determine the asymptotic evolution of the atomic populations that reach the steady state with a time scale smaller than $1/|\lambda_{5,6}|$. Among the nonzero eigenvalues, whose eigenvectors have nonzero NC,NC$-$,C,C$-$, and $e,e$-components, $\lambda_7$ is the one closest to zero. Therefore the populations approach the steady-state values with the following time dependences:

$$\rho_{\mathrm{NC,NC}}(t\to\infty)-\rho_{\mathrm{NC,NC}}(\infty)\propto\exp(-t/\tau),\tag{19a}$$

$$\rho_{\mathrm{C,C}}(t\to\infty)\propto\exp(-t/\tau),\tag{19b}$$

$$\rho_{e,e}(t\to\infty)\propto\exp(-t/\tau),\tag{19c}$$

with $\tau=1/|\lambda_7|$. In writing Eqs. (19) we took into account that at the steady-state the coupled and excited states are not populated.

### Discussion

Let us now discuss some features of the time scale $\tau$ characteristic of the asymptotic evolution of the populations. The important point for our analysis is that $\tau$ does not present any discontinuous behavior for $\alpha\to0$; i.e., at Raman resonance the time scale of the asymptotic evolution for a closed system ($\alpha=0$) and a weakly open one ($\alpha$ small) is substantially the same. It will be shown that this is at variance with the situation encountered for nonzero Raman detuning.

Consider now the dependence of $\tau=1/|\lambda_7|$ on the loss parameter $\alpha$. It may be surprising that, at small $\alpha,\tau$ is nearly constant, increasing $\alpha$. One would naively expect that an increase of the population loss would result in a decrease of the asymptotic time scale of the evolution of the populations;

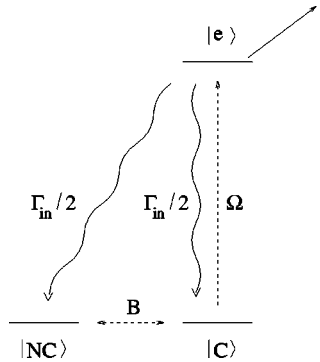


FIG. 2. Three-level Λ system in which the excited state is incoherently depopulated. Couplings due to the laser and to the applied magnetic field are shown in the coupled/noncoupled state basis.

a larger part of the population being pumped out of the system, the internal dynamics of the CPT system should be faster. In the system analyzed so far such behavior does not apply because a change in the loss parameter $\alpha$ directly affects also the evolution of the optical coherences. In fact, in the OBE (12) $\alpha$ enters not only in Eq. (12c) for $\rho_{e,e}$, but also in Eqs. (12d) and (12e) for $\tilde{\rho}_{e,\mathrm{NC}}$ and $\tilde{\rho}_{e,\mathrm{C}}$. To clarify this point, we show explicitly that in the case of a system in which the changes in the loss parameter $\alpha$ affect directly only the evolution of the populations, the time scale for the asymptotic evolution effectively decreases for increasing $\alpha$. An example of such a system is shown in Fig. 2. In that case the population is pumped out of the system not by spontaneous emission but by means of an incoherent coupling that extracts atoms from the excited state. The only differences between the OBE for the system of Fig. 2 and the ones previously examined, Eqs. (12), are in the evolution equations for the optical coherences, Eqs. (12d) and (12e). In fact, in the system of Fig. 2 the loss parameter does not enter in the equation for $\dot{\tilde{\rho}}_{e,\mathrm{NC}}$ and $\dot{\tilde{\rho}}_{e,\mathrm{C}}$, and the equations for this system are obtained from Eqs. (12) simply by dropping in Eqs. (12d) and (12e) the terms in $\alpha$. For the system of Fig. 2 in the case of weak depopulation pumping from the excited state ($\alpha$ small) and in the low-intensity limit the time scale for the asymptotic evolution of the populations is found to be

$$\tau \simeq \frac{1}{4(1+\alpha)}\,\frac{\Gamma_{\mathrm{in}}}{\Omega^2}, \tag{20}$$

which decreases for increasing $\alpha$, as expected in the case in which the loss mechanism affects only the total population.

## V. THE TRAPPING REGION

In this section we examine the long-interaction-time evolution of the atomic system in the case of nonzero Raman detuning. As before, the low-intensity limit is considered. For the study of the asymptotic evolution of the system, only the case of small Raman detunings is relevant. In fact, for small Raman detunings the nonabsorbing superposition is effectively a trap, and a significant fraction of the atomic population survives in the CPT system at a long interaction time. On the other hand, for larger Raman detuning the effect of the trapping of the population in the noncoupled state is absent and the population of the CPT system is quickly pumped into the $|\mathrm{out}\rangle$ state. Therefore we limit our analysis to the interval of small detunings for which the rate of transfer $|\mathrm{NC}\rangle \leftrightarrow |\mathrm{C}\rangle$ is much smaller than the transition rate $|\mathrm{C}\rangle \rightarrow |\mathrm{e}\rangle$; i.e., for Raman detuning $\delta$ much smaller than the Rabi frequency $\Omega$. In the following, the interval of detunings $|\delta| \ll \Omega$ will be called, for the above-mentioned properties, the *trapping region*.

As in the preceding section, to find the time scale of the asymptotic evolution of the atomic populations we determine the eigenvalues of the Bloch matrix. For nonzero Raman detuning they can be calculated using perturbation theory, obtaining a series in $\delta$ for each eigenvalue. Let us begin with $\lambda_1$, which will be shown to determine the time scale of the asymptotic evolution of the populations

$$\lambda_1 = -\frac{\alpha(1+\alpha)}{1+2\alpha}\,\frac{\delta^2}{4\Omega^2}\Gamma_{\mathrm{in}} + O(\delta^4) = -C\delta^2 + O(\delta^4), \tag{21}$$

with

$$C = \frac{\alpha(1+\alpha)}{1+2\alpha}\,\frac{\Gamma_{\mathrm{in}}}{4\Omega^2}. \tag{22}$$

For the other eigenvalues, it is clear from the expressions of Eqs. (16), for $\lambda_2, \ldots, \lambda_9$ at Raman resonance, that for $\delta$ small enough, (a) the real part of $\lambda_2, \ldots, \lambda_9$ is negative and (b) the eigenvalues $\lambda_2, \ldots, \lambda_9$ have real parts more negative than $\lambda_1$. The real parts of all the eigenvalues being negative, we conclude that the solution of the OBE approaches zero for $t \rightarrow +\infty$: owing to the population loss all the coherences and populations of the $\{|g,+1\rangle, |g,-1\rangle, |e\rangle\}$ system decay to zero.

Because of the inequality $\mathrm{Re}\,\lambda_j < \mathrm{Re}\,\lambda_1 < 0 (j=2, \ldots, 9)$, the time scale of the asymptotic solution is given by $\lambda_1$. To determine the expressions for the asymptotic evolution of the populations, we calculated, at the lowest nonzero order in $\delta$, the components $v_{1,\mathrm{NC,NC}}, v_{1,\mathrm{C,C}}, v_{1,e,e}$ of the eigenvector corresponding to $\lambda_1$. We found that $v_{1,\mathrm{NC,NC}}(\delta) = 1 + O(\delta^2), v_{1,\mathrm{C,C}}(\delta) \propto \delta^2 + O(\delta^3), v_{1,e,e} \propto \delta^2 + O(\delta^3)$. Thus the population of the noncoupled state evolves asymptotically as

$$\rho_{\mathrm{NC,NC}} \simeq c \exp(\lambda_1 t), \quad t \rightarrow \infty. \tag{23}$$

The above expression is valid for an interaction time $t$, such that $t\lambda_j \gg 1 \quad (j=2, \ldots, 9)$; therefore we can replace the constant $c$ with the steady-state population $\rho_{\mathrm{NC,NC}}(0,\infty)$ at Raman resonance.

To sum up, at long interaction times the population of the noncoupled state shows an exponential decay

$$\rho_{\mathrm{NC,NC}}(\delta,t) \simeq \rho_{\mathrm{NC,NC}}(0,\infty)\exp(-\Gamma_{\mathrm{NC}}t), \quad t \rightarrow \infty, \tag{24}$$

with $\Gamma_{\mathrm{NC}}$ the effective width of the noncoupled state due to the light irradiation and to the applied magnetic field,

$$\Gamma_{\mathrm{NC}} = C\delta^2. \tag{25}$$

Here we have implicitly assumed that terms of order $O(\delta^4)$ are neglected, as is also the case in the following.

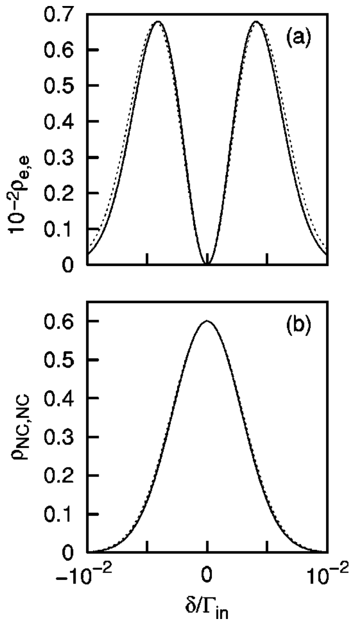


FIG. 3. Populations of the (a) excited and (b) noncoupled states as a function of the Raman detuning. Continuous lines refer to the analytic expressions, Eqs. (24) and (26), dashed ones to the numerical solution of the OBEs. The constant $c'$ for Eq. (26) has been adjusted so that the maxima of the analytic expression of $\rho_{e,e}$ and the maxima of the numerical solution coincide. The parameters of the calculation are $t=2000/\Gamma_{in}$, $\alpha=2$, and $\Omega=0.1\Gamma_{in}$. The initial condition is $\rho_{NC,NC}(0)=\rho_{C,C}(0)=1/2$, with all the other density-matrix elements equal to zero.

In the same way, the asymptotic evolution of the populations of the excited and coupled states is described by

$$\rho_{e,e}(\delta,t)\simeq c'\delta^2\exp(-\Gamma_{NC}t),\quad t\to\infty, \tag{26}$$

$$\rho_{C,C}(\delta,t)\simeq c''\delta^2\exp(-\Gamma_{NC}t),\quad t\to\infty, \tag{27}$$

where the factor $\delta^2$ derives from $v_{1,e,e}$ and $v_{1,C,C}$, and $c'$ and $c''$ are constants. The expressions (24), (26), and (27) demonstrate that for CPT in open systems, the asymptotic evolution of the atomic populations is completely characterized by the effective width $\Gamma_{NC}$.

In Fig. 3 the populations of the excited and noncoupled states, as seen from Eqs. (26) and (24), are plotted as a function of the Raman detuning. For comparison, results for $\rho_{e,e}$ and $\rho_{NC,NC}$ obtained by numerically solving the OBE are also reported. The very good agreement between the results of the two different approaches confirms the validity of our derivation of the expressions (24), (26), and (27) for the atomic populations.

### A. Linewidth narrowing

The population of the noncoupled state exhibits a narrow peak around $\delta=0$, as seen from Eq. (24) and Fig. 3(b). This narrow peak is the result of the depopulation of the noncoupled state with a rate that is decreasing for decreasing Raman detuning [9]. Correspondingly, the population of the excited state, Eq. (26) and Fig. 3(a), shows a dark resonance around $\delta=0$. For increasing Raman detuning there are two competing effects: on the one hand, the coupling $|NC\rangle\leftrightarrow|C\rangle$ increases; on the other hand, the decrease to zero of $\rho_{NC,NC}$ limits the possibility of populating the excited state. It is therefore clear that for CPT in open systems the extension of the dark resonance is not limited by the disappearance of the ground-state coherence effect, as for CPT in closed systems, but by the decrease to zero of the population of the noncoupled state.

The knowledge of the effective loss rate, $\Gamma_{NC}$, allows us to derive an analytic expression for the width of the dark resonance as a function of the interaction time. From Eq. (26) we derive straightforwardly the full width at half maximum

$$\Delta B=\frac{\xi\hbar}{g\mu_B}\sqrt{\frac{1}{Ct}}, \tag{28}$$

with [10] $\xi\simeq0.48$. In Ref. [8] the $1/\sqrt{t}$ narrowing law was found through a numerical calculation of the excited-state population for arbitrary Raman detuning and of the width for the dark resonance. The presented analysis provides an explanation for the general validity of the law $1/\sqrt{t}$ for the asymptotic narrowing of the dark resonance in an open system. Because for CPT in an open system the evolution at long interaction times is characterized by the effective width $\Gamma_{NC}$ of the noncoupled state, the asymptotic law $1/\sqrt{t}$ is determined by the quadratic dependence of $\Gamma_{NC}$ on the Raman detuning.

We can also clarify the origin of the sharp difference, found in Ref. [8], in the narrowing law for the closed system ($\alpha=0$) and open system ($\alpha\neq0$). In effect, because $\lambda_1\to0$ for $\alpha\to0$, in the limit $\alpha\to0$ the expression of Eq. (28) loses meaning. For $\alpha=0$ the eigenvalue $\lambda_1$ is zero because the system is closed: OBEs of Eqs. (12) are not linearly independent and therefore the characteristic polynomial of the associated Bloch matrix is zero independently of the Raman detuning. In this case the populations of the CPT system do not asymptotically approach zero and the expression (24) is no longer valid. Moreover, $\lambda_1$ being equal to zero, the asymptotic evolution of the system is determined by one of the eigenvalues $\lambda_2,\ldots,\lambda_9$, and precisely by the one with the real part closest to zero. Such an eigenvalue will be, for nonzero Raman detuning, of the form $a\Omega^2+b\delta^2$, with $a$ and $b$ nonzero constants. In conclusion, the two properties leading to the law $\Delta B\propto1/\sqrt{t}$ for an open system, i.e., the asymptotic exponential decay of the population of the noncoupled state and a decay rate proportional to the square of the Raman detuning, are not valid in the case of a closed system. In consequence a completely different narrowing law follows.

### B. Depopulation of the trapping region

As discussed above and in Ref. [8], the depopulation of the noncoupled state, with a rate that is decreasing for decreasing Raman detuning, produces the narrowing of the dark resonance with the interaction time. The theoretical limit of the width of the dark resonance for infinite interaction time is zero, and this corresponds to a complete depopulation of the trapping region (the point $\delta=0$ has null measure; therefore it does not contribute to the population of the trapping region). To complete our analysis, we show that the population of the trapping region decays to zero and we determine the relevant asymptotic law. Within the trapping region the populations of the coupled and excited states are much smaller than that of the noncoupled state. Therefore we can limit our analysis to the study of the fraction

$$f_{\text{trap}}=\int_{-\delta_0}^{+\delta_0}\rho_{\text{NC,NC}}\,d\delta, \tag{29}$$

where $\delta_0$ is an arbitrary Raman detuning within the trapping region. For large interaction time the population of the noncoupled state is given by Eq. (24); therefore,

$$f_{\text{trap}}\propto\int_{-\delta_0}^{+\delta_0}\exp(-Ct\delta^2)\,d\delta=\sqrt{\frac{\pi}{Ct}}\,\text{erf}(\sqrt{Ct}\,\delta_0), \tag{30}$$

where for the width $\Gamma_{\text{NC}}$ of the noncoupled state, Eq. (25) has been used. In the limit of large interaction time, we use the expansion for the error function

$$\text{erf}(x)\simeq 1-\frac{\exp(-x^2)}{\sqrt{\pi}x} \tag{31}$$

that is valid for large $x$. Thus we find that $f_{\text{trap}}$ follows the law

$$f_{\text{trap}}\propto\sqrt{\frac{\pi}{Ct}}\left(1-\frac{\exp(-Ct\delta_0^2)}{\sqrt{\pi Ct}\,\delta_0}\right). \tag{32}$$

Retaining only the leading term for large $t$, we get

$$f_{\text{trap}}\propto\frac{1}{\sqrt{t}}. \tag{33}$$

The asymptotic law (33) may also be determined by a numerical analysis of the CPT/Hanle effect [11]. As for the linewidth narrowing, the asymptotic law $1/\sqrt{t}$ for $f_{\text{trap}}$ has a general validity because it derives entirely from the quadratic dependence of the width of the noncoupled state on the Raman detuning.

## VI. CONCLUSIONS

We have considered CPT with losses towards levels not excited by the laser light because they are out of resonance or because of selection rules. The simple case of a $J_g=1\rightarrow J_e=0$ transition interacting with two $\sigma^+,\sigma^-$ laser fields has been studied. The analysis has been carried out in the coupled/noncoupled state basis. A detailed description of the asymptotic evolution of the atomic system has been given.

At Raman resonance the long-interaction-time evolution of closed and open systems is similar: the dynamics is dominated by the optical pumping process that empties the coupled state, with an asymptotic time scale proportional to $\Gamma_{\text{in}}/\Omega^2$ in both cases. For a nonzero Raman detuning the evolution of an open system is completely different from that of a closed system. In fact, for a closed system the steady-state populations of the noncoupled, coupled and excited states are smooth functions of the Raman detuning $\delta$, and also the time scale for the asymptotic evolution varies smoothly with $\delta$. On the other hand, in an open system the steady-state population of the noncoupled state is not a continuous function of $\delta$. At $\delta=0$ the final fraction of the atomic population $\rho_{\text{NC,NC}}$ prepared in the noncoupled state is in general significantly different from zero. For any small nonzero Raman detuning the noncoupled state can be completely depopulated and the corresponding steady-state population $\rho_{\text{NC,NC}}(\delta)$ is zero. Correspondingly, the time scale for the asymptotic evolution is not a continuous function of $\delta$.

The long-time evolution of an open system can be characterized by the effective width $\Gamma_{\text{NC}}$ of the noncoupled state induced by the laser and the applied magnetic fields. An analytic expression for $\Gamma_{\text{NC}}$ was derived. From this expression, the asymptotic law of the narrowing of the dark resonance was determined. We have shown that the quadratic dependence of $\Gamma_{\text{NC}}$ in the Raman detuning leads to an asymptotic law for the width of the dark resonance that is inversely proportional to the square root of the interaction time. This also explains why the $1/\sqrt{t}$ narrowing law has a general validity for CPT in open systems, independently of the details of the interaction scheme. Finally, we have clarified the origin of the sharp difference in the narrowing law for closed and open systems.

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