Entanglement splitting of pure bipartite quantum states

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The concept of entanglement splitting is introduced by asking whether it is possible for a party possessing half of a pure bipartite quantum state to transfer some of his entanglement with the other party to a third party. We describe the unitary local transformation for symmetric and isotropic splitting of a singlet into two branches that leads to the highest entanglement of the output. The capacity of the resulting quantum channels is discussed. Using the same transformation for less than maximally entangled pure states, the entanglement of the resulting states is found. We discuss whether they can be used to do teleportation and to test the Bell inequality. Finally, we generalize to entanglement splitting into more than two branches.
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Entanglement is a resource in the physics of quantum information that has deserved and received increased attention over the recent years, as it is a main reason for fundamental differences with classical information: given a system that consists of n quantum bits, the total state is called separable if it can be decomposed as

$$\varrho_{\text{sep}}=\sum_i p_i\varrho_i^{(1)}\otimes\varrho_i^{(2)}\otimes\cdots\otimes\varrho_i^{(n)}, \tag{1}$$

where $\varrho_i^{(n)}$ is a density matrix for the nth subsystem, and the positive weights p_i obey $\Sigma_i p_i=1$. Otherwise the system is called entangled—a property that exists only for quantum states.

Several features of entanglement have been studied so far, e.g. how to concentrate entanglement of several pairs into more entangled fewer pairs (distillation or purification [1,2]) or how to transfer entanglement to pairs that have not been entangled before (swapping [3]). It is desirable to study more properties of entanglement, as we know that it is the central resource for speed-ups in quantum algorithms, but are still unable to fully classify multiparticle entanglement.

The purpose of this paper is to introduce the concept of entanglement splitting by answering the question whether the resource of entanglement can be shared, such that more than one party can profit from it. Apart from the fundamental aspect of this topic the setting has a direct relevance in teleportation [3]. This paper is restricted to entanglement splitting of two-dimensional states.

Let us consider the following situation: the two parties Alice and Bob are sharing a singlet,[1] i.e., the Bell state

$$|\psi^-\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle). \tag{2}$$

This resource can be used to do teleportation of a quantum state from one party to the other. The singlet can thus be seen as an ideal quantum channel along which quantum information can be sent. We ask the following question: let us imagine Bob has a brother Brian and wants to generously share his resource with him. There is a limit to his generosity, though—for example he does not want to simply swap the entanglement to Brian, because then he would be left without any entanglement himself. Can he design a unitary transformation acting on his and his brother's state (which is supposed to be in a prescribed state $|0\rangle$ at the beginning) and, if necessary, an ancilla, such that after the transformation both Bob and Brian have an identical amount of entanglement with Alice? Thus she could teleport imperfectly a quantum state to both of them. In other words, rather than using one perfect quantum channel, there would be two imperfect or noisy channels. To be precise, we would have created a channel bifurcation with one input side (Alice) and two output sides (Bob, Brian). Note that Bob's action is local in the sense that he does not act on Alice's side. It is not local with respect to Brian, though. Formally this transformation can be written as

$$|\psi\rangle^{\text{in}}=|\psi^-\rangle_{AB_1}|0\rangle_{B_2}|0\rangle_{\text{anc}}\rightarrow\mathbb{1}_A\otimes U_{B_1B_2\text{anc}}|\psi\rangle^{\text{in}}, \tag{3}$$

where Bob is abbreviated as B_1 and Brian as B_2. What is the highest entanglement that can remain between Alice and Bob after his transformation?

This question has not been addressed in work about related areas, namely about broadcasting of entanglement [4,5], where it was shown that an entangled pure state can be copied imperfectly by employing a unitary transformation on *both* subsystems and about telecloning [6], where the authors described a multiparticle state which allows one to do imperfect teleportation of a state from one party to several others. Transformations of a subsystem of a maximally entangled state have been used in [7] in the context of asymmetric cloning. Note that in this context one could generalize our arguments to the case of asymmetric entanglement splitting in a straightforward way.

In our scenario of entanglement splitting the emerging imperfect quantum channels can, in principle, be of any kind, depending on Bob's transformation. We do not attempt a complete study here, but restrict ourselves to the case where the ideal channel is split such that it can serve equally well for teleportation of any quantum state. The motivation for

[1] Any other Bell state would be equally good for our considerations.

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this is not to introduce any spatial asymmetry into the splitted channel. This means we only consider splitting of the singlet into two identical branches that act as depolarizing channels (i.e., with the same probability of error in the x, y, and z directions) when using the standard teleportation scheme. The equivalence between quantum channels and bipartite states has been studied in [8,9] where it was shown that a Werner state [i.e., a state that is with probability F_W in one of the Bell states and with equal probabilities $(1-F_W)/3$ in the remaining three Bell states], acts as a depolarizing channel when used for standard teleportation. In other words, we require the reduced density matrix of Alice and Bob after Bob's transformation to be a Werner state.

The most general transformation that Bob can perform is given by its action on the basis of his qubit (Brians qubit and the ancilla are initially fixed) and can be written as

$$U_{B_1B_2\text{anc}}|0\rangle|0\rangle|0\rangle = a|00\rangle|A\rangle + b(|01\rangle+|10\rangle)|B\rangle + c|11\rangle|C\rangle, \tag{4}$$

$$U_{B_1B_2\text{anc}}|1\rangle|0\rangle|0\rangle = \tilde{a}|11\rangle|\tilde{A}\rangle + \tilde{b}(|10\rangle+|01\rangle)|\tilde{B}\rangle + \tilde{c}|00\rangle|\tilde{C}\rangle. \tag{5}$$

where we have used the same notation as in [10] and implied symmetry under the exchange of Bob's and Brian's qubits. The coefficients and ancilla states are restricted by constraints from unitarity of $U_{B_1B_2\text{anc}}$. We also require symmetry under renaming the basis, namely exchange of $|0\rangle_{B_1}\leftrightarrow|1\rangle_{B_1}$. Inserting Eqs. (4) and (5) into Eq. (3) and tracing over B_2 one finds the following structure before tracing over the ancilla:

$$\begin{aligned}\rho_{AB_1} = \frac{1}{4}\text{Tr}_{\text{anc}}[&\{a(|\Psi^+\rangle+|\Psi^-\rangle)|A\rangle + b(|\Phi^+\rangle+|\Phi^-\rangle)|B\rangle \\ &-\tilde{b}(|\Psi^+\rangle-|\Psi^-\rangle)|\tilde{B}\rangle - \tilde{c}(|\Phi^+\rangle \\ &-|\Phi^-\rangle)|\tilde{C}\rangle\}\{\langle\ldots|\} + \{b(|\Psi^+\rangle+|\Psi^-\rangle)|B\rangle \\ &+c(|\Phi^+\rangle+|\Phi^-\rangle)|C\rangle - \tilde{a}(|\Psi^+\rangle-|\Psi^-\rangle)|\tilde{A}\rangle \\ &-\tilde{b}(|\Phi^+\rangle-|\Phi^-\rangle)|\tilde{B}\rangle\}\{\langle\ldots|\}].\end{aligned} \tag{6}$$

Here the notation $\{\langle\cdots|\}$ indicates that the whole ket-vector to the left now appears as bra-vector. We have used the customary definition for the Bell states, namely

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{7}$$

After tracing over the ancilla we have to set Eq. (6) equal to the Werner state we aim at, namely

$$\rho_{AB_1} = F_W|\Psi^-\rangle\langle\Psi^-| + \frac{1-F_W}{3}(|\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|). \tag{8}$$

The explicit calculation is laborious and, therefore, not shown explicitly. It yields some constraints for the parameters characterizing this transformation:

(i) $|a|^2-|c|^2=|\tilde{a}|^2-|\tilde{c}|^2$,
(ii) $|a|^2-|c|^2=\text{Re}[\tilde{b}^*a\langle\tilde{B}|A\rangle+\tilde{a}^*b\langle\tilde{A}|B\rangle]$,
(iii) $\text{Im}[\tilde{b}^*a\langle\tilde{B}|A\rangle+\tilde{a}^*b\langle\tilde{A}|B\rangle]=0$,
(iv) $b^*\tilde{c}\langle B|\tilde{C}\rangle+c^*\tilde{b}\langle C|\tilde{B}\rangle=0$,
(v) $b^*a\langle B|A\rangle+c^*b\langle C|B\rangle=0$,
(vi) $\tilde{b}^*\tilde{a}\langle\tilde{B}|\tilde{A}\rangle+\tilde{c}^*\tilde{b}\langle\tilde{C}|\tilde{B}\rangle=0$
(vii) $\tilde{c}^*a\langle\tilde{C}|A\rangle-\tilde{a}c^*\langle C|\tilde{A}\rangle=\tilde{b}b^*\langle B|\tilde{B}\rangle-b\tilde{b}^*\langle\tilde{B}|B\rangle$.

It turns out that these constraints coincide—apart from (vii)—with the ones the parameters of an isotropic and symmetric $1\rightarrow 2$ quantum cloning transformation have to fulfill. (These are given in Sec. II A of [10].) This is not astonishing, as in both scenarios we want to find a transformation with the highest possible degree of symmetry. Nevertheless is not obvious, either, as in this paper we perform a transformation on a part of an entangled state, i.e., a system in which the reduced density matrix is the identity and does not contain information about any direction of polarization, in contrast with the scenario in quantum cloning. As mentioned above, the sets of constraints in the two scenarios are not exactly identical.

An isotropic $1\rightarrow 2$ cloner was introduced by Bužek and Hillery [11] and shown to be optimal in [10,12]. A less than optimal but still isotropic cloning transformation can be constructed by varying not only the coefficients of the unitary transformation, but also the scalar products of the ancilla, such that the conditions given in [10] are preserved. (Note that for the case of less than optimal cloning the ancilla dimension has to be increased to at least two qubits.) This statement is best illustrated by providing an example. The fidelity $F_C=\langle\psi^{\text{in}}|\rho^{\text{out}}|\psi^{\text{in}}\rangle$ of a cloning transformation is a measure for how close the reduced density matrix ρ^{out} of a clone is to the input. With the following bad cloner we reach only $F_C=3/4$:

$$U^{\text{bad}}|0\rangle|0\rangle|0\rangle_{a1}|0\rangle_{a2} = \frac{1}{\sqrt{2}}|00\rangle|0\rangle_{a1}|0\rangle_{a2} + \tfrac{1}{2}(|01\rangle + |10\rangle)|1\rangle_{a1}|0\rangle_{a2},$$

$$\begin{aligned}U^{\text{bad}}|1\rangle|0\rangle|0\rangle_{a1}|0\rangle_{a2} = &\frac{1}{\sqrt{2}}|11\rangle|1\rangle_{a1}\frac{1}{\sqrt{2}}(|0\rangle_{a2}+|1\rangle_{a2}) \\ &+\tfrac{1}{2}(|01\rangle+|10\rangle)|0\rangle_{a1}\frac{1}{\sqrt{2}}(|0\rangle_{a2} + |1\rangle_{a2}),\end{aligned} \tag{9}$$

where the two first bits on the right-hand side are the clones and the subscripts $a1$, $a2$ denote the ancilla states.

Which of the many possible isotropic cloning machines optimizes the quality of the channels resulting from entanglement splitting? From Eq. (6) and the constraints given above one finds that the fraction of the singlet in our output channel is

$$F_W=\frac{1}{4}[3(|a|^2-|c|^2)+1], \tag{10}$$

which is related to the cloning fidelity F_C in a simple way, namely

$$F_W=\tfrac{1}{2}(3F_C-1). \tag{11}$$

Remember that our constraint (vii) is slightly different from the according constraint in cloning. When going through the maximization procedure, given in the appendix of [10], step by step, it turns out, however, that the same transformation that maximizes the cloning fidelity also maximizes F_W in our scenario. The upper bound of a universal cloner is given by $F_C\leqslant 5/6$, and therefore the Werner fidelity has to fulfill $F_W \leqslant 3/4$. The entanglement of a Werner state with $F_W\geqslant 1/2$ is an increasing function of F_W. In other words, the highest entanglement of the output is reached when Bob performs an optimal cloning transformation on his and Brian's qubit.

It is interesting to observe that our method provides a simple derivation for the upper limit of the quality of the universal NOT or spin-flip (which is equal to the lower bound on the cloning fidelity): as $F_W=\langle\Psi^-|\rho_{AB}|\Psi^-\rangle$ has to be positive, we arrive at $F_C\geqslant 1/3$ which corresponds to the upper bound for the universal NOT given in [13]. In other words, when applying our positive map it is enough to require positivity of ρ_{AB} in order to find an explicit bound for the best universal NOT (when the corresponding transformation acts on Bob's qubit it erases the singlet completely).

We have shown that the best way to split entanglement is to use the optimal universal cloner. Note that even a global unitary transformation acting on Alice's, Bob's, and Brian's bit cannot split the singlet into depolarizing channels with higher degree of entanglement because Alice could then teleport a state to both Bob and Brian with higher quality than the optimal cloning quality. It was shown in [10] that the quantum capacity of the depolarizing channels that can be reached in this scenario vanishes. In other words: it is impossible to bifurcate a perfect quantum channel into two depolarizing channels with nonvanishing capacity.

Note that this observation sets limits on the possible amount of entanglement of subsystems in multipartite states with the described structure, see also the generalization to splitting into more than two branches given below.

Let us now look at the case where Alice and Bob share a pure, but not maximally entangled state of the kind

$$|\psi^-(\alpha)\rangle=\alpha|01\rangle-\beta|10\rangle \quad \text{with} \quad |\alpha|^2+|\beta|^2=1. \tag{12}$$

Note that *every* pure bipartite state can be written in this way by using the Schmidt decomposition and naming the bases of the two systems accordingly. If Bob performs the same optimal cloning transformation as above (note that the original channel is not an ideal channel and the outcomes are not depolarizing channels, so our optimality arguments considered above do not apply), the reduced density matrix of Alice and Bob (or Alice and Brian) after this operation is

$$\rho_{AB}=\frac{1}{6}\begin{pmatrix} |\alpha|^2 & 0 & 0 & 0\\ 0 & 5|\alpha|^2 & -4\alpha\beta^* & 0\\ 0 & -4\alpha^*\beta & 5|\beta|^2 & 0\\ 0 & 0 & 0 & |\beta|^2 \end{pmatrix} \tag{13}$$

in the basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$. Is this matrix inseparable? Using the separability criterion established by Peres [14] and the Horodeckis [15] one has to check the partially transposed density matrix for positivity. Its eigenvalues are

$$\lambda_1=\frac{5}{6}|\alpha|^2,$$

$$\lambda_2=\frac{5}{6}|\beta|^2,$$

$$\lambda_{3,4}=\frac{1}{12}(1\pm\sqrt{1+60|\alpha|^2|\beta|^2}), \tag{14}$$

and as $\lambda_4<0$ for any α with $|\alpha|\neq 0,1$, the output is inseparable for *any* entangled input. So, no matter how small Bob's entanglement was at the beginning, he can still make a donation of it to Brian. Something like a smallest unit of entanglement, that could not be split, does not exist.

How much entanglement remains in the state of Alice and Bob after the transformation? The entanglement of formation for a given density matrix ρ was introduced in [16] and is given by

$$E(\rho)=-(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-C^2})\log_2(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-C^2}) -(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-C^2})\log_2(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-C^2}), \tag{15}$$

where the concurrence C is defined as

$$C(\rho)=\max\{0,\sqrt{\xi_1}-\sqrt{\xi_2}-\sqrt{\xi_3}-\sqrt{\xi_4}\} \tag{16}$$

with ξ_i being the eigenvalues of $\rho[(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)]$, ordered by size as $\xi_1\geqslant\xi_2\geqslant\xi_3\geqslant\xi_4$.

The entanglement of formation for the density matrix in Eq. (13), i.e., the density matrix of Alice and Bob or Alice and Brian after the transformation is found by calculating the concurrence as

$$C(\rho)=|\alpha\beta|. \tag{17}$$

Note that the concurrence of the original state was $C_{\text{orig}}=2|\alpha\beta|$. We observe that $C(\rho)>0$ for $0<|\alpha|<1$. A discussion about inequalities for the squares of the concurrences in the subsystems of three-particle states is presented in [17]. The curve for the entanglement of formation is shown in Fig. 1.

There is some loss of entanglement in the system of Alice, Bob, and Brian compared to the original entanglement because the unavoidable ancilla is entangled as well. Which subsystems of the total state after the transformation will be entangled with each other? By looking at the density matrices for the bipartite subsystems one finds the following: A is entangled with B_1 and with B_2. The ancilla is entangled with

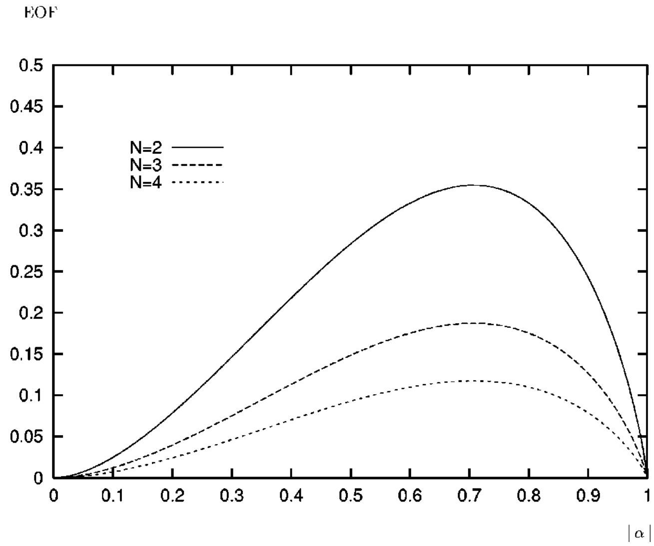

FIG. 1. Entanglement of formation (EOF) of any of the resulting branches as a function of initial parameter $|\alpha|$, see Eq. (12), for different numbers of branches N.

B_1 and with B_2. B_1 and B_2 are not entangled with each other. This can be visualized as in Fig. 2.

Is the remaining entanglement between Alice and each of the brothers enough so that she can do teleportation to both for *any* initial α? We learned in [18] that a mixed state is better for teleportation than classically whenever the following criterion for the maximally reachable fidelity is fulfilled:

$$F_{\max}=\frac{1}{2}\left(1+\frac{1}{3}\mathrm{Tr}\sqrt{T^{\dagger}T}\right)>\frac{2}{3}, \tag{18}$$

where the matrix T is defined by the general expansion of the density matrix

$$\rho=\frac{1}{4}(\mathbb{1}\otimes\mathbb{1}+\vec{s}^{(1)}\cdot\vec{\sigma}\otimes\mathbb{1}+\mathbb{1}\otimes\vec{s}^{(2)}\cdot\vec{\sigma}+T_{ij}\sigma_i\otimes\sigma_j), \tag{19}$$

and in our case given by

$$T=\frac{2}{3}\begin{pmatrix}-2\,\mathrm{Re}[\alpha\beta^*] & -2\,\mathrm{Im}[\alpha\beta^*] & 0\\ -2\,\mathrm{Im}[\alpha\beta^*] & -2\,\mathrm{Re}[\alpha\beta^*] & 0\\ 0 & 0 & -1\end{pmatrix} \tag{20}$$

in the basis $\{x,y,z\}$.

In order for the teleportation scheme to work we thus find the following window for the initial state:

$$\frac{1}{2}\left(1-\frac{\sqrt{15}}{4}\right)<|\alpha|^2<\frac{1}{2}\left(1+\frac{\sqrt{15}}{4}\right). \tag{21}$$

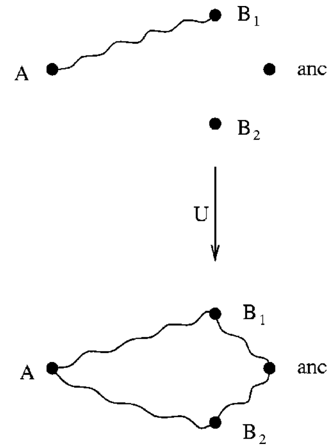

FIG. 2. Entanglement of the total state before and after splitting a pure entangled bipartite state into two branches. The wavy lines between two parties indicate that their reduced density matrix is entangled.

This is a very wide range for α, namely $0.008<\alpha<0.992$. Thus, even for small initial entanglement the teleportation fidelity of the splitted channel is still higher than the classical fidelity.

Could we also use the output state to do a three-party test of the Bell-CHSH inequality [19]? Using the criterion given in [20,21], we have to calculate the eigenvalues of $T^{\dagger}T$ where T is given in Eq. (20). The authors showed that the Bell inequality is equivalent to the sum of the two largest eigenvalues being smaller or equal to one. In our example this is the case for any α, so our outputs do not violate the Bell inequality. The output state is another example (for $|\alpha|\neq|\beta|$ distinct from the ones given in [21]) for the fact that separability is a different criterion from violation of Bell's inequalities.

In the remaining part of the paper we want to generalize our results to the case where Bob is willing to share part of his entangled state, given in Eq. (12), with more than one brother. Let us denote the number of brothers by N-1. Thus N refers to the number of output channels including Bob. Making use of the $1\rightarrow N$ cloning transformation given in [12] we can generalize entanglement splitting into N branches. In this case the ancilla swallows a higher amount of entanglement. Therefore, the interesting questions here are: is there a threshold number N where not every entangled input leads to entangled outputs? Is teleportation through the output channels still possible for any number N, and what is the condition the initial entanglement has to fulfill?

Explicitly we find the following results. The reduced density matrix of Alice and Bob (or any of his brothers) is given by

$$\rho_{AB}(N)=\frac{1}{3N}\begin{pmatrix}(N-1)|\alpha|^2 & 0 & 0 & 0\\ 0 & (2N+1)|\alpha|^2 & -(N+2)\alpha\beta^* & 0\\ 0 & -(N+2)\alpha^*\beta & (2N+1)|\beta|^2 & 0\\ 0 & 0 & 0 & (N-1)|\beta|^2\end{pmatrix}. \tag{22}$$

The concurrence of this density matrix is

$$C(\rho)=\frac{2}{N}|\alpha\beta|, \tag{23}$$

i.e., the output concurrences of Alice with all brothers sum up to the input concurrence between Alice and Bob. The according entanglement of formation of one of the N branches is shown in Fig. 1 for $N=3$ and $N=4$: we find that the output states are inseparable for *any* α with $|\alpha|\neq 0,1$ and for *any* N; Bob can give as many brothers as he wants some of his entanglement, no matter how much he had to start with. For $N\rightarrow\infty$ the output entanglement goes to zero for any α.

The condition the initial state has to fulfill in order to make teleportation through the output channels possible is

$$\frac{1}{2}\left(1-\frac{\sqrt{3(2N+1)}}{(N+2)}\right)<|\alpha|^2<\frac{1}{2}\left(1+\frac{\sqrt{3(2N+1)}}{(N+2)}\right). \tag{24}$$

The window for teleportation shrinks like $1/\sqrt{N}$ to zero width for $N\rightarrow\infty$.

We found that entanglement splitting is an ''easy'' task, in the sense that we only need to apply the optimal cloning transformation, which splits any initial pure state entanglement. Does the same statement also hold for an initial mixed state? It does not, as can be seen by giving a counter example. Let us consider the same scenario as above, but now Alice and Bob share an initial Werner state with fidelity F_W and of the form given in Eq. (8). If Bob simply applies the cloning transformation as for the case of a pure input, this leads to an output density matrix for Alice and Bob that is separable for $1/2\leqslant F_W\leqslant 5/8$, a parameter region in which the input was entangled. In this sense mixed-state entanglement is qualitatively different from pure state entanglement. It is still an open question, though, whether there exists a different transformation that could split any initial mixed state entanglement.

To summarize, we have introduced the concept of entanglement splitting. We have shown that the well-known optimal cloning transformation also maximizes the entanglement after symmetric splitting of a singlet into two branches. The capacity of the resulting depolarizing channels vanishes. We observed that *any* pure entangled state of two qubits can be split such that after the transformation there is some remaining entanglement. Teleportation, though, can only be performed better than classically if the entanglement of the original state exceeds a certain threshold. We have generalized the results to the case of splitting into more than one branch. In this scenario we still find nonvanishing resulting entanglement for any initial entanglement and *any* number of branches N, tending to zero for infinitely many branches. Teleportation is possible for any N, if the initial entanglement is higher than a threshold that depends on N. We hope that the ideas developed in this paper help the reader to understand some fundamental aspects of entanglement.

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