Bose-Einstein condensation of a finite number of particles trapped in any-dimensional space

Mingzhe Li,[1,2] Lixuan Chen,[1,2] Jincan Chen,[1,2] Zijun Yan,[1,2] and Chuanhong Chen[1]
[1]*Department of Physics, Xiamen University, Xiamen 361005, China*
[2]*CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China*
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Bose-Einstein condensation of an ideal Bose gas trapped in any-dimensional space is studied for the finite particle number effects. The corrections of statistical properties due to the finite particle number effects are obtained. We find that these corrections are more significant for the low-energy excited states which are the focus of our interest. We also find that the asymmetry of the external potential intensifies these effects. [S1050-2947(99)05910-7]

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Bose-Einstein condensation (BEC) was realized in ultracold trapped atomic gases in 1995 [1–3]. The new achievements have created a renewed interest in the theoretical study of this phenomenon. BEC is interesting because it provides us with a macroscopic phenomenon to study the quantum nature of matter; another exciting prospect is that it may allow us to create an atom laser, which is expected to have broad applications.

Theoretical studies have revealed that space dimensionality has significant effects on the properties of the system [4,5]. Moreover, there is evidence that effects of an external potential may be expressed as effects of a different space dimensionality. Furthermore, the occurrence of BEC in dimensionalities different from 3 is interesting in superfluidity, superconductivity, electroweak phase transition, and superfluidity in neutron stars. Therefore, it is interesting to study BEC in various space dimensionalities. In experiments, the system has a finite particle number N whose effects on BEC are measurable. References [6,7] have investigated these corrections, but have not discussed the conclusions presented in this paper. We focus on a harmonic potential for its relevance to experiments and for analytical results. Considering an ideal Bose gas trapped in an anisotropic harmonic potential in d-dimensional space, we shall derive analytical results for the finite-N corrections and discuss their determinants.

From the first principle of statistical mechanics, we have the Bose-Einstein distribution for a bosonic system,

$$n(E_l)=\frac{1}{e^{(E_l-\mu)/kT}-1}=\sum_{j=1}^{\infty} z^j e^{-jE_l/kT}, \tag{1}$$

where $z=\exp(\mu/kT)$ is the fugacity. In an anisotropic harmonic potential with ω_i as its frequency of each axis in d-dimensional space, each particle has an energy of $E_l=\Sigma_{i=1}^{d} n_{l,i}\hbar\omega_i$. Summing over the energy levels, we may obtain the total particle number

$$N=\sum_{l=0}^{\infty}\sum_{j=1}^{\infty} z^j e^{-jE_l/kT}=\sum_{j=1}^{\infty}\left[z^j\prod_{i=1}^{d}(1-e^{-j\hbar\omega_i/kT})^{-1}\right] \tag{2}$$

$$=\frac{z}{1-z}+g_d(z)\left(\frac{kT}{\hbar}\right)^d\frac{1}{\prod_{i=1}^{d}\omega_i}+\frac{1}{2}g_{d-1}(z)\left(\frac{kT}{\hbar}\right)^{d-1}\frac{\sum_{i=1}^{d}\omega_i}{\prod_{i=1}^{d}\omega_i}, \tag{3}$$

where $g_\nu(x)=\Sigma_{l=1}^{\infty} l^{-\nu}x^l$, and an approximation has been made by retaining the two highest-order terms in $kT/\hbar\omega$. As the density of states is introduced by

$$N-N_0=\int_0^{\infty}\rho(E)n(E)dE, \tag{4}$$

where N_0 is the ground-state population, Eq. (3) gives the density of states

$$\rho(E)=\frac{1}{(d-1)!}\frac{1}{\hbar^d\prod_{i=1}^{d}\omega_i}E^{d-1}+\frac{1}{2(d-2)!}\frac{\sum_{i=1}^{d}\omega_i}{\hbar^{d-1}\prod_{i=1}^{d}\omega_i}E^{d-2}. \tag{5}$$

In the case of an isotropic harmonic potential with a frequency ω, Eqs. (3) and (5) become

$$N=\frac{z}{1-z}+g_d(z)\left(\frac{kT}{\hbar\omega}\right)^d+\frac{d}{2}g_{d-1}(z)\left(\frac{kT}{\hbar\omega}\right)^{d-1} \tag{6}$$

and

$$\rho(E)=\frac{1}{(d-1)!}\frac{E^{d-1}}{(\hbar\omega)^d}+\frac{d}{2(d-2)!}\frac{E^{d-2}}{(\hbar\omega)^{d-1}}. \tag{7}$$

In fact, Eq. (7) may be obtained by using this traditional approach: the degeneracy of the state with energy $n\hbar\omega$ is a problem of distributing n identical balls into d different boxes; the combination number is

$$q_d(E)=\frac{(n+d-1)(n+d-2)\cdots(n+1)}{(d-1)!}$$
$$=\frac{n^{d-1}+[d(d-1)/2]n^{d-2}+\cdots}{(d-1)!}, \tag{8}$$

considering $E=n\hbar\omega$ and the energy level spacing $\hbar\omega$, a better approximation of the density of states is therefore readily available to be Eq. (7). By integrating the density of states Eq. (7), we may obtain an expression of the total particle number N which is exactly Eq. (6). This shows that there is no fundamental difference between the use of discrete sums and a continuous spectrum if the density of states is correctly approximated. We start with the method of discrete sums, because the degeneracy of states may not be directly perceived through the senses in the case of an anisotropic harmonic potential.

Equation (5) may be written as

$$\rho(E)=\frac{E^{d-1}}{(d-1)!\hbar^d\prod_{i=1}^{d}\omega_i}\left(1+\frac{d-1}{2}\frac{\hbar\sum_{i=1}^{d}\omega_i}{E}\right). \tag{9}$$

Comparing Eq. (9) with the traditional result

$$\rho(E)=\frac{E^{d-1}}{(d-1)!\hbar^d\prod_{i=1}^{d}\omega_i}, \tag{10}$$

we find that, under a constant external potential, the correction of the density of states due to finite N is proportional to $1/E$. Thus this effect is more significant for the low-energy excited states which are the focus of our attention. The correction is also a function of the external potential. If the density of states Eq. (5) is formulated as

$$\rho(E)=\frac{E^{d-1}}{(d-1)!(\hbar\Omega)^d}+\gamma\frac{E^{d-2}}{(\hbar\Omega)^{d-1}}, \tag{11}$$

where

$$\Omega=\left(\prod_{i=1}^{d}\omega_i\right)^{1/d}, \tag{12}$$

$$\gamma=\frac{1}{2(d-2)!}\frac{\sum_{i=1}^{d}\omega_i}{\left(\prod_{i=1}^{d}\omega_i\right)^{1/d}}, \tag{13}$$

the coefficient γ may be used to express the intensity of the finite-N correction versus the external potential. We find that γ reaches its minimum value of $d/2(d-2)!$ for an isotropic harmonic potential, and the asymmetry of the external potential intensifies this finite-N effect. Reference [7] also gives γ for an anisotropic harmonic potential in three-dimensional space by means of numerical calculations and conforms to our analytical result.

Compared with the usual result of transition temperature

$$T_c^0=\frac{\hbar}{k}\left[\frac{N}{g_d(1)}\prod_{i=1}^{d}\omega_i\right]^{1/d}, \tag{14}$$

Eq. (3) results in a transition temperature

$$\frac{T_c}{T_c^0}=1-\frac{g_{d-1}(1)}{2d[g_d(1)]^{1-1/d}}\frac{\sum_{i=1}^{d}\omega_i}{\left(\prod_{i=1}^{d}\omega_i\right)^{1/d}}N^{-1/d}. \tag{15}$$

Equation (15) gives a decrease in the transition temperature when the finite-N effect is taken into account. This is sensible for the decrease is proportional to $N^{-1/d}$, and vanishes as $N\to\infty$. For the transition temperature, we may obtain the intensity of the finite-N correction versus the external potential, similar to what we do for the density of states. Equation (15) may also be used to describe the case of an isotropic harmonic potential so long as ω_i is substituted with ω.

The condensate fraction at temperatures $T<T_c$ may be derived from Eqs. (3) and (14) to be

$$\frac{N_0}{N}=1-\left(\frac{T}{T_c^0}\right)^d\left[1+\frac{g_{d-1}(1)}{2g_d(1)}\frac{\sum_{i=1}^{d}\omega_i}{kT}\right]. \tag{16}$$

For convenience, we discuss the fraction of excited states

$$\frac{N_e}{N}=\left(\frac{T}{T_c^0}\right)^d\left[1+\frac{g_{d-1}(1)}{2g_d(1)}\frac{\sum_{i=1}^{d}\omega_i}{kT}\right]. \tag{17}$$

Compared with the usual result of the fraction of excited states

$$\frac{N_e}{N}=\left(\frac{T}{T_c^0}\right)^d, \tag{18}$$

Eq. (17) gives an increase of the fraction of excited states caused by the finite-N effect, which corresponds to the increase of the density of states illustrated by comparing Eq. (5) with Eq. (10).

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