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# Light amplification without stimulated emission: Beyond the standard quantum limit to the laser linewidth

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The standard quantum limit to the linewidth of a laser for which the gain medium can be adiabatically eliminated is $l_0 = \kappa/2\bar{n}$. Here $\kappa$ is the intensity damping rate and $\bar{n}$ the mean photon number. This contains equal contributions from the loss and gain processes, so that simple arguments which attribute the linewidth wholly to phase noise from spontaneous gain are wrong. I show that an *unstimulated* gain process actually introduces no phase noise, so that the ultimate quantum limit to the linewidth comes from the loss alone and is equal to $l_{\rm ult} = \kappa/4\bar{n}$. I investigate a number of physical gain mechanisms which attempt to achieve gain without phase noise: a linear atom-field coupling with a finite interaction time, a nonlinear atom-field coupling, and adiabatic photon transfer using a counterintuitive pulse sequence. The first at best reaches the standard limit $l_0$, the second reaches $\frac{3}{4}l_0$, and the third reaches the ultimate limit of $l_{\rm ult} = \frac{1}{2}l_0$. [S1050-2947(99)03711-7]

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## I. INTRODUCTION

It is more than 40 years since Schawlow and Townes introduced the idea of an ''optical maser'' [1], now known of course as a laser. Probably the most famous result from this paper is the expression for the quantum-limited laser linewidth, their Eq. (17),

$$\Delta\omega_{\rm osc} = \frac{2\hbar\omega}{P_{\rm out}}(\Delta\omega)^2. \tag{1.1}$$

Here $\Delta\omega_{\rm osc}$ is the half-width at half maximum (HWHM) of the laser spectrum, $\Delta\omega$ is the HWHM of the spectrum of the relevant atomic transition, $P_{\rm out}$ is the output power, and $\omega$ is the frequency of the laser. Defining $l = 2\Delta\omega_{\rm osc}$ and $\gamma = 2\Delta\omega$, this expression can be rewritten

$$l_{\rm ST} = \frac{\hbar\omega}{P_{\rm out}}\gamma^2, \tag{1.2}$$

where ST stands for Schawlow-Townes. The derivation of this expression assumes that reabsorption of photons by atoms in the ground state of the relevant transition is negligible, and also ignores thermal photons and other extraneous noise sources.

To describe lasers accurately, a number of refinements must be made to the Schawlow-Townes expression [2]. These are discussed in the Appendix. This discussion, I believe, helps to put in perspective some of the past work on quantum limits to the laser linewidth. The end result is that a better expression for the standard quantum limit to the laser linewidth is

$$l_{\rm st} = \frac{l_{\rm bare}}{2\bar{N}} \leqslant \frac{\hbar\omega}{2P_{\rm out}}\frac{\gamma^2\kappa^2}{(\gamma+\kappa)^2}. \tag{1.3}$$

Here st stands for standard (quantum limit). As explained in the Appendix, $l_{\rm bare}$ is the bare linewidth, $\bar{N}$ is the number of coherent excitations stored in the laser mode and its gain medium, and $\kappa$ is the cavity linewidth. The inequality is an equality only for perfectly efficient output coupling.

In the limit $\gamma \gg \kappa$ the gain medium can be adiabatically eliminated, resulting in Markovian evolution for the laser mode. This means that $\bar{N}$ can be replaced by $\bar{n}$ (the mean photon number), and $l_{\rm bare}$ by $\kappa$, to give the standard Markovian limit as

$$l_0 = \frac{\kappa}{2\bar{n}}. \tag{1.4}$$

For the remainder of this paper I will assume the Markovian limit, and drop the adjective ''Markovian'' distinguishing $l_0$ from $l_{\rm st}$ when no confusion is likely to arise.

Most older textbooks [3–5] quote the result in Eq. (1.4), or one which reduces to it in the appropriate limit of neither reabsorption nor thermal photons. The first two of these [3,4] derive this result rigorously using a Fokker-Planck equation and quantum Langevin equations, respectively. All three attempt to explain it in terms of the noise added by the spontaneous contribution to the (mostly stimulated) gain of photons from the atomic medium. Loudon [5] even recommended the argument based on the uncertainty principle given by Weichel [6].

The argument of Weichel is as follows (in my notation). In a laser at steady state, the ratio of spontaneous emissions to total (spontaneous and stimulated) emissions is $1:\bar{n}+1$. Since the total gain rate must equal the total loss rate $\kappa\bar{n}$, the rate of spontaneous emissions is

$$A = \frac{\kappa\bar{n}}{1+\bar{n}} \simeq \kappa, \tag{1.5}$$

where it is assumed that $\bar{n} \gg 1$. Now the reciprocal of this, $\Delta t = 1/\kappa$, is [6] ''the average time between phase fluctuations caused by spontaneous emissions into the mode.'' Invoking the uncertainty principle

 

$$\Delta E \Delta t \geq \hbar/2, \quad (1.6)$$

and assuming that the energy uncertainty of the mode is $\Delta E \simeq \bar{n}\hbar \Delta\omega_{osc}$ gives

$$l = 2\Delta\omega_{osc} = \frac{1}{\bar{n}\Delta t} = \frac{\kappa}{\bar{n}}, \quad (1.7)$$

which agrees with the Schawlow-Townes result [Eq. (1.2)], with $\gamma$ replaced by $\kappa$ and $P_{out}$ by $\kappa\bar{n}$.

Almost every step in this argument is dubious, not the least being the starting assumption that phase diffusion is due solely to the gain mechanism. This is an artifact of thinking in terms of normally ordered operator products. That is, it results from using (implicitly in most cases) the Glauber-Sudarshan $P$ function [7–9] as a true representation of the fluctuations in the laser mode field. The $P$ function is of course no more fundamental than the $Q$ function [9], which is a representation based on antinormally ordered statistics. If one were to use the $Q$ function as an aid to intuition, one would find that it is the loss process that is wholly responsible for the phase noise. Of course the rate of phase diffusion would agree with that from the $P$ function, at least in steady state where loss and gain balance.

If one asks a question about phase diffusion, the only objective answer will come from using the phase basis itself. This is far more difficult than using the more familiar phase-space representations, but some approximate results have been obtained [10]. These show that, at steady state, the phase diffusion has equal contributions from the loss and gain process. The same result occurs from a Wigner function calculation [10]. This is not surprising since symmetrically ordered moments are known to closely approximate the true moments for the phase operator for states with well-defined amplitude [11].

The fact that phase diffusion comes equally from the loss and gain processes suggests that the standard quantum limit to the laser linewidth, $l_0$ of Eq. (1.4), may not be the ultimate quantum limit. The contribution from the loss mechanism is unavoidable. A laser, at least in useful definitions [12], requires a linear damping of the laser mode in order to form an output beam. However, it may be that the standard gain mechanism could be replaced by some other gain mechanism that causes less phase diffusion. The ultimate quantum limit to the laser linewidth could thus be as small as one-half of the standard limit.

In this paper I investigate various gain mechanisms in an attempt to find one which causes less phase diffusion than the standard gain mechanism. First, in Sec. II, I review the standard model for a laser, giving rise to the standard quantum limit $l_0$. Next, in Sec. III, I present gain without stimulated emission, which produces a linewidth of $l_0/2$, and discuss how this can be physically realized. In Secs. IV and V, I present models which attempt to approximate gain without stimulated emission, using a micromaserlike interaction and a nonlinear field-atom interaction, respectively, and discuss their success. After a comparison of these results in Sec. VI, I conclude in Sec. VII by returning to a derivation of the ultimate quantum limit to the laser linewidth using an uncertainty relation.

## II. STANDARD LASER

### A. Jaynes-Cummings coupling

The standard laser master equation results from just about any gain medium under the appropriate conditions in which noise due to thermal photons and photon reabsorption can be ignored. Here I will present probably the simplest derivation for this master equation in the limit far above threshold. In this model the gain is due to the coupling of the laser mode with a single transition in an atom. Ignoring the other levels in the atom, the interaction is governed by the usual Jaynes-Cummings Hamiltonian

$$H = i\Omega(\sigma a^\dagger - \sigma^\dagger a), \quad (2.1)$$

where $a$ is the annihilation operator for the cavity mode, $\sigma = |l\rangle\langle u|$ is the lowering operator for the atom, and $\Omega$ is the one-photon Rabi frequency.

Let the interaction time $\tau$ be such that $\epsilon\sqrt{\bar{n}} \ll 1$, where $\epsilon = \Omega\tau$ and $\bar{n}$ is the mean intracavity photon number. Then the unitary operator $\exp(-iH\tau)$ acting on the initially factorized state $R = \rho \otimes |u\rangle\langle u|$ can be expanded to second order in $\epsilon$ to give the entangled state for the atom and field

$$R = \rho \otimes |u\rangle\langle u| + \epsilon(a^\dagger \rho \otimes |l\rangle\langle u| + \text{H.c.}) + \epsilon^2(a^\dagger \rho a \otimes |l\rangle\langle l| - \tfrac{1}{2}\{aa^\dagger, \rho\} \otimes |u\rangle\langle u|). \quad (2.2)$$

Say there is a detector which detects the state of the atom immediately after it has interacted with the field. If the outgoing atom is detected in the upper state, then the conditioned state of the field (the norm of which represents the probability of this detection result) is, to first order in $\epsilon^2$,

$$\tilde{\rho}_u = \langle u|R|u\rangle = (1 - \epsilon^2 \mathcal{A}[a^\dagger])\rho = \exp(-\epsilon^2 aa^\dagger/2)\rho\exp(-\epsilon^2 aa^\dagger/2), \quad (2.3)$$

where the superoperator $\mathcal{A}$ is defined for an arbitrary operators $A$ and $B$ by

$$\mathcal{A}[A]B = \tfrac{1}{2}\{A^\dagger A, B\}. \quad (2.4)$$

If the atom is detected in the lower state (which happens rarely), the state is

$$\tilde{\rho}_l = \langle l|R|l\rangle = \epsilon^2 \mathcal{J}[a^\dagger]\rho, \quad (2.5)$$

where the superoperator $\mathcal{J}$ is defined by

$$\mathcal{J}[A]B = ABA^\dagger. \quad (2.6)$$

If this were all that there was to the model then the master equation would be found simply by averaging over the two results. If the entry of excited atoms into the cavity were a Poisson process with rate $\Gamma \ll \tau^{-1}$, the result would be

$$\dot{\rho} = \Gamma\epsilon^2 \mathcal{D}[a^\dagger]\rho + \kappa \mathcal{D}[a]\rho. \quad (2.7)$$

Here I have included linear loss (allowing the laser output) at rate $\kappa$, and I am using the notation

$$\mathcal{D}[A] \equiv \mathcal{J}[A] - \mathcal{A}[A]. \quad (2.8)$$

As long as $\Gamma\epsilon^2 < \kappa$, this master equation has a steady state. However it is not an appropriate steady state for the device to be a laser. As discussed in Ref. [12], it is necessary to have $\bar{n} \gg 1$ for the output of the device to be coherent (in a quantum-statistical sense). But in this limit, the stationary state of the master equation (2.7) has a photon number uncertainty $\sigma(n) \sim \bar{n}$. This leads to enormous low-frequency $(\sim \kappa/\bar{n})$ fluctuations in the intensity of the output beam. This ruins the second-order coherence of the device. Second-order coherence is ubiquitously recognized as a defining characteristic of a laser above threshold [3–5,9,16], and is included in the formal definition of a laser given in Ref. [12].

### B. Gain saturation

The origin of the problem with Eq. (2.7) is stimulated emission. Despite the fact that it is part of the acronym l.a.s.e.r., stimulated emission from an undepleted source (such as the source of excited atoms in the present case) leads directly to the unwanted intensity fluctuations inherent in Eq. (2.7). This is because such stimulated emission implies that, for $\bar{n} \gg 1$, the intensity gain is proportional to the intensity. Thus, if the intensity fluctuates above its mean value then that fluctuation will be reinforced by an increase in the gain, and if it fluctuates below the mean then the gain will correspondingly decrease. To avoid this, and hence obtain a second-order coherent output, one actually wants a photon gain which is a *nonlinear* function of intensity.

In most lasers, the nonlinearity of the gain as a function of intensity occurs automatically as $\bar{n}$ becomes very large because of *gain saturation*. This is not difficult to derive in the master equation approach [12]. Ignoring thermal photons and photon reabsorption as usual, the resulting master equation (including output loss) is

$$\dot{\rho} = G n_s \mathcal{D}[a^\dagger](\mathcal{A}[a^\dagger] + n_s)^{-1}\rho + \kappa\mathcal{D}[a]\rho. \tag{2.9}$$

Here $G$ is the ‘‘small signal gain,’’ which is the initial gain when the laser mode is begun in the vacuum state, and $n_s$ is the saturation photon number. Although it may be written in an unfamiliar form, this is the standard master equation for a laser with a saturable gain medium which can be adiabatically eliminated. For example, it is completely equivalent to the Fokker-Planck equation derived by Louisell [3], except for the thermal noise in the damping which he included. Above threshold $(G > \kappa)$, the mean photon number is approximately equal to

$$\bar{n} = (G/\kappa - 1)n_s. \tag{2.10}$$

The gain rate from this master equation varies as

$$\mathrm{Tr}[a^\dagger a \mathcal{D}[a^\dagger](\mathcal{A}[a^\dagger] + n_s)^{-1}\rho] = \left\langle \frac{aa^\dagger}{aa^\dagger + n_s} \right\rangle. \tag{2.11}$$

Stimulated (and spontaneous) emission is evident from the $aa^\dagger$ in the numerator, but the destabilizing effect of this is offset by the $aa^\dagger$ in the denominator, which is present because of gain saturation. Below threshold $(G/\kappa < 1)$, $\bar{n}$ is small compared to $n_s$ and the gain is approximately linear. Above threshold $(G/\kappa > 1)$, $\bar{n}$ is typically comparable to $n_s$ and the gain is quite nonlinear as a function of intensity. Far above threshold $(G/\kappa \gg 1)$, $n_s$ is negligible in comparison to $\bar{n}$ and the gain rate becomes essentially independent of intensity fluctuations. In the FAT (far above threshold) limit one can approximate the master equation (2.9) for a laser with a *saturable* gain medium by the master equation

$$\dot{\rho} = G n_s \mathcal{D}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}\rho + \kappa\mathcal{D}[a]\rho \tag{2.12}$$

for a laser with a *saturated* gain medium. In this limit the photon statistics of the laser mode become Poissonian (like a coherent state) so that it is usual to consider this limit [4,5,9]. For ease of expression I will call Eq. (2.9) the standard laser master equation, and Eq. (2.12) the FAT standard laser master equation.

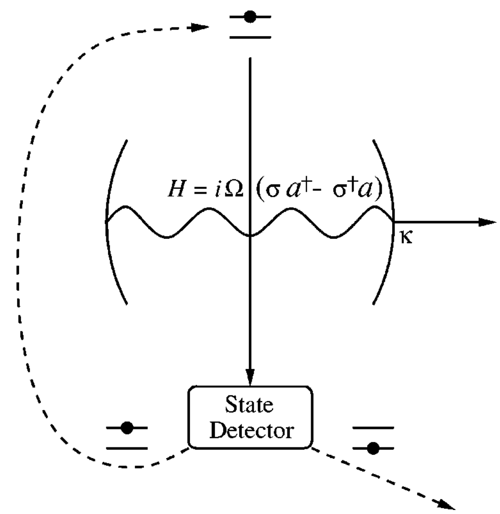


FIG. 1. Schematic of a simple gain mechanism which reproduces that of a standard laser far above threshold. An atom in the upper state passes through the cavity, and its state is then detected. If the atom remains in the upper state, the process is repeated until it is detected in the lower state. The time for this process (including repetitions) is assumed to be very short compared to the time between photon emissions from the cavity $(\kappa\mu)^{-1}$. Once the atom is detected in the lower state, a new upper-state atom is injected after a random waiting time $\tau$ having an exponential distribution $w(\tau) = \exp(-\kappa\mu\tau)$. Here $\mu$ is the desired mean number of photons in the cavity.

### C. FAT laser model

The FAT standard laser master equation can be derived easily within the current context of two-level atoms passing through a cavity. To make the gain independent of the photon number, it is simply necessary to ensure that each atom gives up exactly one quantum of energy to the field, regardless of the field state. This is achieved by the following procedure. If the outgoing atom is detected in the lower state, then the field has gained a photon and the process can stop. If it is detected in the upper state, one must try again with the same atom (or, more realistically, another excited atom). This process continues until the atom is detected in the ground state. This procedure is shown in Fig. 1.

Say $K$ atoms are required before the $(K+1)$th atom is detected in the lower state. From Eqs. (2.3) and (2.5), the unnormalized state matrix after the $(K+1)$th atom is

$$\tilde{\rho}_K = \epsilon^2 \mathcal{J}[a^\dagger]\exp[-K\epsilon^2 aa^\dagger/2]\rho \exp[-K\epsilon^2 aa^\dagger/2]. \tag{2.1}$$

The norm of this state matrix is equal to the probability that this many atoms are needed. Thus the average density operator, given that an atom is finally detected in the ground state, is

$$\rho' = \sum_{K=0}^{\infty} \tilde{\rho}_K. \tag{2.14}$$

Using the fact that $\epsilon^2$ is small, the sum in Eq. (2.14) can be converted to an integral by setting $\beta = \epsilon^2 K$:

$$\rho' = \mathcal{J}[a^\dagger]\int_0^\infty \exp(-\beta aa^\dagger/2)\rho \exp(-\beta aa^\dagger/2)d\beta. \tag{2.15}$$

This can be formally evaluated [13] as

$$\rho' = \mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}\rho. \tag{2.16}$$

The superoperator $\mathcal{A}[a^\dagger]^{-1}$ is well defined because $aa^\dagger$ is a strictly positive operator [14].

The action of the superoperator $\mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}$ is to add a photon to the system irrespective of its initial state. That is to say, it shifts the photon number distribution upwards by 1. If this addition of a photon is assumed to occur at Poisson-distributed times, with a rate $\Gamma \ll \Omega\epsilon\bar{n}$, then a Markovian master equation for the field results. If one also includes linear damping at rate $\kappa$ as above, and lets the gain (the rate of photon addition) be $\Gamma = \kappa\mu$, then one obtains

$$\begin{aligned}\kappa^{-1}\dot{\rho} &= \mu(\mathcal{J}[a^\dagger]\mathcal{A}[a^\dagger]^{-1} - 1)\rho + \mathcal{D}[a]\rho \\ &= \mu\mathcal{D}[a^\dagger]\mathcal{A}[a^\dagger]^{-1}\rho + \mathcal{D}[a]\rho.\end{aligned} \tag{2.17}$$

From Eq. (2.15) and the identity

$$1 = \int_0^\infty d\beta \exp(-\beta aa^\dagger/2)aa^\dagger \exp(-\beta aa^\dagger/2), \tag{2.18}$$

it is easy to see that the master equation (2.17) is of the required Lindblad form [15]. This equation was first derived in this explicit form in Ref. [13], but as noted above it is simply the far-above-threshold approximation (2.12) to the standard laser master equation. In this derivation it was assumed that the state preparation and detection are perfect. If instead one were to allow for an imperfect atomic state detector, for example, which has a probability $p \ll 1$ to incorrectly register an atom in the ground state, then one would obtain the standard laser master equation (2.9) with saturation photon number $n_s = p/\epsilon^2$.

### D. Stationary state

In the Fock basis the FAT standard laser master equation (2.17) is

$$\begin{aligned}\dot{\rho}_{n,m} &= \mu\left(\frac{2\sqrt{nm}}{n+m}\rho_{n-1,m-1} - \rho_{n,m}\right) - \frac{n+m}{2}\rho_{n,m} \\ &\quad + \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}.\end{aligned} \tag{2.19}$$

Here, as in the remainder of the paper, I have set $\kappa = 1$. Clearly the stationary state will be of the form $\rho_{n,m} = \delta_{n,m}P_n$. The equation of motion for $P_n$ is

$$\dot{P}_n = \mu(P_{n-1} - P_n) + (n+1)P_{n+1} - nP_n. \tag{2.20}$$

This has the stationary solution $P_n = e^{-\mu}\mu^n/n!$. That is, the intracavity photon statistics are exactly Poissonian.

The stationary state matrix can therefore be written

$$\rho_{ss} = \sum_n e^{-\mu}\frac{\mu^n}{n!}|n\rangle\langle n|. \tag{2.21}$$

Equivalently, it can be written

$$\rho_{ss} = \int_0^{2\pi}\frac{d\phi}{2\pi}|\alpha e^{i\phi}\rangle\langle\alpha e^{i\phi}|, \tag{2.22}$$

where $|\alpha| = \sqrt{\mu}$ and $|\alpha e^{i\phi}\rangle$ is a coherent state of amplitude $\alpha e^{i\phi}$. From either expression it is easy to verify that the mean number is $\mathrm{Tr}[a^\dagger a\rho_{ss}] = \mu$ and the mean amplitude $\mathrm{Tr}[a\rho_{ss}] = 0$.

### E. Calculating the linewidth

There are many different ways of calculating the linewidth of a laser from its master equation. One way is to covert the master equation into an approximate Fokker-Planck equation for a quasiprobability distribution function such as the $P$, $Q$, or $W$ function [16]. This is relatively straightforward for a master equation of the form of Eq. (2.17), despite the apparent awkwardness of the inverse superoperator $\mathcal{A}[a^\dagger]^{-1}$ [13]. However, for other master equations as I will consider later in this paper, the conversion is not so simple. Therefore, I will adopt a method using the Fock basis. The method is essentially a more rigorous version of that used by Sargent, Scully, and Lamb [4].

The linewidth $l$ of a laser I have taken to be the full width at half maximum (FWHM) of the power spectrum

$$P(\omega) \propto \int_0^\infty d\tau\, g^{(1)}(\tau)\cos\omega\tau, \tag{2.23}$$

where the normalized first-order coherence function is

$$g^{(1)}(\tau) = \langle a^\dagger(t+\tau)a(t)\rangle_{ss}/\langle a^\dagger a\rangle_{ss}. \tag{2.24}$$

If one represents the master equation (2.17) as $\dot{\rho} = \mathcal{L}\rho$, then one can write

$$g^{(1)}(\tau) = \mathrm{Tr}[a^\dagger e^{\mathcal{L}\tau}(a\rho_{ss})]/\mu. \tag{2.25}$$

Note that the stationary state matrix $\rho_{ss}$ is a mixture of coherent states, as in Eq. (2.22). Since $g^{(1)}(\tau)$ is invariant under a phase shift, Eq. (2.22) implies that in Eq. (2.25) one can take $\rho_{ss} = |\alpha\rangle\langle\alpha|$, with $|\alpha|^2 = \mu$. Then Eq. (2.25) becomes

$$g^{(1)}(\tau)=\mathrm{Tr}[a^{\dagger}\alpha\rho(\tau)]/\mu, \tag{2.26}$$

where $\rho(t)$ obeys the master equation (2.17), and

$$\rho(0)=|\alpha\rangle\langle\alpha|. \tag{2.27}$$

If one defines

$$f_n(t)=\sqrt{n}\rho_{n-1,n}(t)/\alpha^*, \tag{2.28}$$

then one can write

$$g^{(1)}(t)=\sum_n f_n(t). \tag{2.29}$$

Clearly if one can determine the evolution of $f_n(t)$, one can find $g^{(1)}(t)$ and hence the linewidth of the laser. From Eq. (2.17) one finds

$$\dot{f}_n=\mu\frac{2n}{2n-1}f_{n-1}-\mu f_n+nf_{n+1}-\frac{2n-1}{2}f_n. \tag{2.30}$$

Defining

$$r_n(t)=\frac{\mu f_n(t)}{nf_{n+1}(t)}, \tag{2.31}$$

one obtains

$$\dot{f}_n=\left[\frac{2n(n-1)}{2n-1}r_{n-1}-\mu+\frac{\mu}{r_n}-\frac{2n-1}{2}\right]f_n. \tag{2.32}$$

Now from definition (2.31), $r_n(0)\equiv 1$. Assuming that this ratio remains unity, we expand Eq. (2.32) to leading order in $1/\mu$ to obtain

$$\dot{f}_n\approx-\frac{1}{4n}f_n. \tag{2.33}$$

Solving this and substituting into Eq. (2.31) gives, to leading order,

$$r_n(t)\approx\exp\left(-\frac{t}{4n^2}\right)\approx 1-\frac{t}{4n^2}, \tag{2.34}$$

where the expansion to first order is valid for times much less than $\mu^2$. Since, as will be shown, the coherence time $\sim 2/l$ is of order $\mu$, it is quite safe to make this expansion even for times long compared to the coherence time.

Substituting this expression for $r_n(t)$ into Eq. (2.32) gives the more accurate expression

$$\dot{f}_n\approx-\frac{1}{4n}\left[1+\frac{n-\mu}{n^2}t\right]f_n. \tag{2.35}$$

Since the initial condition is

$$f_n(0)=e^{-\mu}\frac{\mu^{n-1}}{(n-1)!}, \tag{2.36}$$

the only significant contribution to sum (2.29) comes from $n$ such that $|n-\mu|\lesssim\sqrt{\mu}$. Also, as noted above, one can assume $t\lesssim n$. Then the correction term in Eq. (2.35) is of order $\mu^{-1/2}$ and can be ignored. One can thus return to the expression [Eq. (2.33)], which becomes (again ignoring corrections of order $\mu^{-1/2}$)

$$\dot{f}_n\approx-\frac{1}{4\mu}f_n. \tag{2.37}$$

The first-order coherence function is thus

$$g^{(1)}(\tau)=\exp(-\tau/4\mu), \tag{2.38}$$

so that the coherence time is $4\mu$ (which is of order $\mu$ as promised). The Fourier transform of this expression is a Lorentzian with the FWHM

$$l=\frac{1}{2\mu}. \tag{2.39}$$

This is the standard quantum limit $l_0$ of the linewidth for a laser.

## III. GAIN WITHOUT STIMULATED EMISSION

Since ‘‘stimulated emission of radiation’’ is part of the acronym for laser, it might be thought that stimulated emission is essential to produce a laser. A typical laser does rely upon stimulated emission to ensure that it runs single mode, and the stimulated emission is of course present in the standard laser master equation (2.9). However, the fact that the gain is independent of photon number in the FAT regime of the standard laser master equation (2.12) suggests that it may not be strictly necessary.

Of course if one were to consider a laser to be defined by the original acronym l.a.s.e.r., then a laser without stimulated emission would be an oxymoron. However, the word laser is no longer considered to be an acronym [17]. Also, it is now accepted usage to refer to a continuously out-coupled atomic condensate as an ‘‘atom laser,’’ which obviously cannot be encompassed within the original acronym. For this and other reasons I have argued elsewhere [12] for a general definition of a laser, based on the coherence properties of the output beam from the device. The gain of the device is not restricted to any particular mechanism (which seems wise given the inventiveness of laser physicists). On this basis, one can certainly conceive of a laser whose amplification does not rely on stimulated emission.

I will now show that stimulated emission is indeed *not* necessary to produce a device with the same coherence properties as a laser. Moreover, just as stimulated emission was to blame for the intensity noise in the linear amplifier, it is to blame for the phase noise in the laser gain. In other words, in a complete reversal of the laser physics folklore discussed in Sec. I, it is the *stimulated* emission, *not* the *spontaneous* emission, which causes the phase diffusion. Eliminating stimulated emission eliminates the amplification component of the phase diffusion and hence results in a narrower linewidth than the standard laser. To avoid contention, I will continue to refer to gain without stimulated emission, rather than a laser without stimulated emission.

Stimulated emission is a simple consequence of the linear coupling of the laser field to its source, as in Eq. (2.1). That is to say, Hamiltonian (2.1) is linear in the annihilation operator $a$ which, for classical fields, can be replaced by the $c$ number $\sqrt{n}e^{i\phi}$. According to Fermi's golden rule, a transition rate depends on the square of the Hamiltonian. Hence the fundamental gain rate from a linear coupling will vary as $n$, which is the so-called stimulated emission or Bose-enhancement factor. A fully quantum calculation of course gives spontaneous emission as well, and hence a gain rate proportional to $n+1$. Thus stimulated emission is still present in the model of Sec. II, even though the overall procedure illustrated in Fig. 1 leads to the addition of photons at a rate independent of the photon number in the cavity.

Since stimulated emission can be traced to the presence of $a$ in the coupling Hamiltonian, it can be removed by substituting for $a$ a different lowering operator, one whose classical analog does not increase with $n$. That is to say, in Eq. (2.1), replace

$$a=\sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n| \tag{3.1}$$

by the Susskind-Glogower [18] $e\equiv\widehat{e^{i\phi}}$ operator

$$e=(aa^{\dagger})^{-1/2}a=\sum_{n=1}^{\infty} |n-1\rangle\langle n|. \tag{3.2}$$

The new Hamiltonian would be extremely nonlinear if expressed as a power series in $a$ and $a^{\dagger}$, but it cannot be denied that it will not exhibit any stimulated emission.

Replacing $a$ by $e$ in Hamiltonian (2.1) presents no problems in the rest of the derivation in Sec. II. Moreover, it is not even necessary to assume that $\epsilon=\Omega\tau$ is very small. Instead, the result is independent of $\epsilon$, due to the fact that $ee^{\dagger}=1$. In particular, if one chooses $\epsilon=\pi/2$, the transformation effected on the field by one transit of the atom is semiunitary:

$$\exp\left[\frac{\pi}{2}(e^{\dagger}\sigma-\sigma^{\dagger}e)\right]|u\rangle|\psi\rangle=|l\rangle S|\psi\rangle. \tag{3.3}$$

Here $|\psi\rangle$ is the state of the field, and

$$S=e^{\dagger}=\sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{3.4}$$

The operator $S$ is semiunitary rather than unitary because $S^{\dagger}S=1$, but $SS^{\dagger}=1-|0\rangle\langle 0|$.

Surprisingly, the transformation $S$ can be achieved physically using only the usual electric-dipole coupling [19]. The trick is to use a three-level $\Lambda$ atom and another, classical field [20]. Then, using a counterintuitive pulse sequence, the atom is transferred from one lower state to the other, and one photon is created in the cavity field (with the energy lost from the classical field). Like the gain process in Sec. II, this adds precisely one photon to the field. The difference is that it does this without entangling the state of the field and the atom, and hence leaves the state of the field pure. The only approximation necessary to derive the semiunitary transformation $S$ from this technique is that the couplings be turned on and off sufficiently slowly for the total system to adiabatically follow the Hamiltonian. Specifically, the characteristic time for the photon addition $T$ has to satisfy

$$\Omega_{\rm cl},g\gg T^{-1}\gg\kappa\bar{n},\gamma, \tag{3.5}$$

where $\Omega_{\rm cl}$ is the Rabi frequency of the classical field, $g$ is a one-photon Rabi frequency [the equivalent of the $\Omega$ in Eq. (2.1)], and $\gamma$ is the spontaneous emission rate of the upper level of the atom. Note that these inequalities are consistent with the requirement that the gain rate $\Gamma$ (which must be smaller than $T^{-1}$) be equal to the loss rate $\kappa\bar{n}$. However, for large $\bar{n}$ the condition $g\gg\gg\kappa\bar{n}$ is much harder to satisfy than the usual strong coupling condition $g\gg\kappa$ in cavity quantum electrodynamics. Thus it would not be possible to produce a macroscopic field from this gain mechanism with current technology.

Ignoring these practical limitations, we can take the rate of addition of photons to the field to be $\Gamma=\kappa\mu$ as before; in place of Eq. (2.17), one obtains

$$\dot{\rho}=\mu\mathcal{D}[e^{\dagger}]\rho+\mathcal{D}[a]\rho. \tag{3.6}$$

As usual, time is being measured in units of $\kappa^{-1}$. In the Fock basis this becomes

$$\dot{\rho}_{m,n}=\mu(\rho_{n-1,m-1}-\rho_{m,n})-(n+m)\rho_{n,m}/2 + \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}. \tag{3.7}$$

This yields exactly the same equation for the diagonal elements (the photon number populations). Hence the unstimulated master equation (3.6) produces exactly the same photon number statistics as does the FAT standard laser master equation (2.17).

To calculate the linewidth, one can proceed as before. One finds the following equation for $f_n$, defined as in Sec. II:

$$\dot{f}_n=\mu\left(\sqrt{\frac{n}{n-1}}f_{n-1}-f_n\right)+nf_{n+1}-\frac{2n-1}{2}f_n \tag{3.8}$$

$$=\left[\sqrt{n(n-1)}\,r_{n-1}-\mu+\frac{\mu}{r_n}-\frac{2n-1}{2}\right]f_n. \tag{3.9}$$

Assuming $r_n\approx 1$ yields, as above, the self-consistent solution

$$\dot{f}_n\approx-\frac{1}{8\mu}f_n. \tag{3.10}$$

The first-order coherence function is therefore

$$g^{(1)}(\tau)=\exp(-\tau/8\mu), \tag{3.11}$$

so that the linewidth is

$$l=\frac{1}{4\mu}. \tag{3.12}$$

This is half the standard quantum limit $l_0$ of Eq. (1.4). As explained in Sec. I, the standard quantum limit for the laser phase diffusion rate contains equal contributions from the gain and loss processes. The gain process considered in this section does not introduce any phase noise; the operator $e^{\dagger}$ is more or less the exponentiation of the phase operator and so increases the photon number without affecting the phase distribution at all. Thus the phase diffusion in this model comes wholly from the loss process, and the rate is half the standard rate.

## IV. FINITE ATOM-FIELD INTERACTION TIME

Section III showed that an interaction in which the atom is sure to give up its quantum of energy to the field *from a single pass* results in a linewidth a factor of 2 smaller than the standard limit. It was noted there that this could be achieved using an adiabatic passage, but this has yet to be done experimentally. This suggests that it would be worth exploring other ways to mimic the unstimulated gain process.

In this section I investigate one idea, based upon the gain mechanism of a micromaser [21,22]. This utilizes the same Jaynes-Cummings coupling (2.1) as in Sec. II. The difference is that the scaled interaction time $\epsilon = \Omega\tau$ is not assumed to be small. This modifies the results of Sec. II as follows. The state of the field conditioned on the detection of an atom in the lower state is [21]

$$\tilde{\rho}_l = \mathcal{J}_l \rho, \tag{4.1}$$

where

$$\mathcal{J}_l = \mathcal{J}[e^{\dagger}\sin(\epsilon\sqrt{aa^{\dagger}})]. \tag{4.2}$$

The field state conditioned on an atom passing through and remaining in the upper state is [21]

$$\tilde{\rho}_u = \mathcal{J}_u \rho, \tag{4.3}$$

where

$$\mathcal{J}_u = \mathcal{J}[\cos(\epsilon\sqrt{aa^{\dagger}})]. \tag{4.4}$$

For states having a photon distribution localized around $\bar{n}$, if $\epsilon$ is such that $\epsilon\sqrt{\bar{n}} \approx \pi/2$, then it would seem that the action of the above superoperators could be approximated by

$$\mathcal{J}_l \approx \mathcal{J}[e^{\dagger}], \tag{4.5}$$

$$\mathcal{J}_u \approx 0. \tag{4.6}$$

That is, the atom would almost certainly come out in the lower state, having given up its quantum of energy to the field. This is the same situation as for the unstimulated gain as shown in Sec. III. This is why a finite interaction time $\epsilon$ might be expected to lead to a linewidth below the standard limit.

If atoms are injected at a Poissonian rate $\mu$ then the total master equation is the usual micromaser master equation

$$\dot{\rho} = \{\mu(\mathcal{J}_u + \mathcal{J}_l - 1) + \mathcal{D}[a]\}\rho. \tag{4.7}$$

Here linear damping at rate unity also has been included. This master equation has very complicated dynamics. For some values of $\epsilon$ and $\mu$ the stationary state does not have a well-defined intensity. That is, it is not the case that $\sigma(n) \ll \bar{n}$. Hence the device is not necessarily a true laser in the sense of Ref. [12].

To ensure that the a well-defined photon number distribution is produced, the same technique as in Sec. II can be used. That is, if an atom is detected still in the upper state it is sent through again until it is detected in the lower state. The resulting master equation is

$$\dot{\rho} = \left\{ \mu \mathcal{J}_l \sum_{k=0}^{\infty} \mathcal{J}_u^k + \mathcal{D}[a] \right\} \rho \tag{4.8}$$

$$= \{\mu \mathcal{J}_l (1 - \mathcal{J}_u)^{-1} + \mathcal{D}[a]\}\rho. \tag{4.9}$$

In the photon number basis

$$\dot{\rho}_{n,m} = \mu \frac{\sin(\epsilon\sqrt{n})\sin(\epsilon\sqrt{m})}{1 - \cos(\epsilon\sqrt{n})\cos(\epsilon\sqrt{m})} \rho_{n-1,m-1} - \mu\rho_{n,m} - (n+m)\rho_{n,m}/2 + \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}. \tag{4.10}$$

To find the linewidth, one proceeds as before to obtain the following equation for $f_n$:

$$\dot{f}_n = \mu \frac{\sqrt{n}\,\sin(\epsilon\sqrt{n-1})\sin(\epsilon\sqrt{n})}{\sqrt{n-1}\,[1 - \cos(\epsilon\sqrt{n-1})\cos(\epsilon\sqrt{n})]} f_{n-1} - \mu f_n + n f_{n+1} - \frac{2n-1}{2} f_n. \tag{4.11}$$

Using the parameter

$$\phi \equiv \epsilon\sqrt{\mu}, \tag{4.12}$$

one can continue the analysis as before, and eventually find

$$\dot{f}_n \approx -\frac{1}{8\mu}\left[1 + \frac{\mu^2 \sin^2(\phi/\mu)}{\sin^2\phi}\right] f_n. \tag{4.13}$$

That is, the linewidth of the laser is found to be

$$l = \frac{1}{4\mu}\left[1 + \left(\frac{\sin(\phi/\mu)}{(\sin\phi)/\mu}\right)^2\right]. \tag{4.14}$$

It is easy to verify that this expression has a global minimum

$$l = \lim_{\phi\to 0} \frac{1}{4\mu}\left[1 + \left(\frac{\sin(\phi/\mu)}{(\sin\phi)/\mu}\right)^2\right] = \frac{1}{2\mu}. \tag{4.15}$$

The limit $\phi \to 0$ is the limit of short interaction times in which the original model of Sec. II is recovered, and also the original linewidth $l_0$. That is, no linewidth narrowing is possible using a finite interaction time in preference to an infinitesimal interaction time, despite the fact that the former can deposit a photon in the cavity in a single pass of the atom with very high probability.

This line broadening is definitely not an artifact of the assumption that the atom is always put through again if it is detected still in its upper state; a similar result is obtained for the usual master micromaser equation with a single pass per atom [23]. The approach to calculating the linewidth used in Ref. [23] was similar to the one used here. A more accurate estimation of the linewidth for the usual micromaser has to take into account the fact that the intensity is not always well defined [24]. This yields some deviations from the simple theory of Ref. [23], but still never shows any line narrowing.

The reason that no linewidth narrowing occurs can be seen from the method of calculation I have employed. What turns out to be crucial is not to try to mimic the two terms in the unstimulated gain $\mathcal{D}[e^\dagger]$, namely,

$$\mathcal{J}[e^\dagger]; \quad \mathcal{A}[e^\dagger]=1, \tag{4.16}$$

but rather to mimic the following ratio of matrix elements involving these two terms:

$$\frac{\langle n-1|\{\mathcal{J}[e^\dagger]|n\rangle\langle n+1|\}|n\rangle}{\langle n|\{\mathcal{A}[e^\dagger]|n\rangle\langle n+1|\}|n+1\rangle}=1. \tag{4.17}$$

In the unstimulated case the ratio is unity, and the difference from unity in other cases is proportional to the contribution to the linewidth from the gain process. For the FAT standard laser,

$$\frac{\langle n-1|\{\mathcal{J}[a^\dagger]|n\rangle\langle n+1|\}|n\rangle}{\langle n|\{\mathcal{A}[a^\dagger]|n\rangle\langle n+1|\}|n+1\rangle}\approx 1-\frac{1}{8n^2}. \tag{4.18}$$

Multiplying the deviation from unity by the gain constant $\mu$ and replacing $n$ by the mean photon number $\mu$ gives $1/8\mu$. This is the standard contribution to the linewidth from the gain. For the above micromaser model,

$$\frac{\langle n-1|\{\mathcal{J}_l|n\rangle\langle n+1|\}|n\rangle}{\langle n|\{[1-\mathcal{J}_u]|n\rangle\langle n+1|\}|n+1\rangle}\approx 1-\frac{\sin^2(\phi/\mu)}{8\sin^2\phi}, \tag{4.19}$$

which again explains the result in Eq. (4.14).

## V. NONLINEAR ATOM-FIELD INTERACTION

With now a better understanding of how to reduce the gain-induced phase diffusion, I turn to a second method for trying to mimic unstimulated gain. As noted in Sec. III, the operator $e^\dagger$ would require an infinite series to be expressed in terms of powers of $a$ and $a^\dagger$. Any Hamiltonian containing infinite powers of the field is unlikely to be realizable in practice. However, nonlinear optical processes containing field powers greater than unity do occur. This suggests that it is worth considering the following approximation:

$$e^\dagger = a^\dagger(aa^\dagger)^{-1/2} = a^\dagger[\mu+(aa^\dagger-\mu)]^{-1/2} \tag{5.1}$$

$$\approx \frac{a^\dagger}{\sqrt{\mu}}\left(\frac{3}{2}-\frac{1}{2}\frac{aa^\dagger}{\mu}\right). \tag{5.2}$$

That is, I wish to consider a nonlinear Jaynes-Cummings Hamiltonian of the form

$$H=i\Omega[\sigma a^\dagger(3-aa^\dagger/\mu)+(3-aa^\dagger/\mu)a\sigma^\dagger], \tag{5.3}$$

which I expect to be useful when the photon number is approximately $\mu$.

Physically, this Hamiltonian means that there are two processes which can excite the atom. The first is the usual linear dipole coupling to the field. The second is a three-photon process, whereby a photon is virtually absorbed and re-emitted before finally being absorbed by the atom. The Hamiltonian matrix element for the second process is much smaller (for $\mu \gg 1$), which is physically reasonable, and is of the opposite sign. It is doubtful that such a Hamiltonian could be achieved simply using a two-level atom. However, it is possible that an effective Hamiltonian of this form could be achieved using a multilevel atom, and other fields. I will not further discuss the feasibility of producing this Hamiltonian, as my chief concern is with the question of principle: how well can the nonlinear Hamiltonian (5.3) reproduce the results of the model with unstimulated gain?

Assuming, as in previous sections, that the atoms are initially in the upper state and that any atom which exits the cavity still in the upper state is put through again, one can derive, following the method of Sec. II, the following master equation for the cavity mode:

$$\dot{\rho}=\mu\mathcal{D}[a^\dagger(3-aa^\dagger/\mu)]\mathcal{A}[a^\dagger(3-aa^\dagger/\mu)]^{-1}\rho+\mathcal{D}[a]\rho. \tag{5.4}$$

This has the same Poissonian mixture of number states as in the FAT standard laser, and is amenable to the same method of calculating the linewidth. The result is

$$l=\frac{3}{8\mu}. \tag{5.5}$$

That is, the contribution from the gain is $1/8\mu$, which is half the standard result and half the contribution of $1/4\mu$ from the loss (which is of course unchanged). This result can again be understood from the ratio

$$\frac{\langle n-1|\{\mathcal{J}[a^\dagger(3-aa^\dagger/\mu)]|n\rangle\langle n+1|\}|n\rangle}{\langle n|\{\mathcal{A}[a^\dagger(3-aa^\dagger/\mu)]|n\rangle\langle n+1|\}|n+1\rangle}\approx 1-\frac{1}{16n^2}. \tag{5.6}$$

## VI. DISCUSSION

The standard quantum limit to the laser linewidth is not the ultimate quantum limit, even for the Markovian case in which the gain medium is eliminated from the equations of motion of the laser mode. Hidden within the standard Markovian expression

$$l_0=\frac{\kappa}{2\bar{n}} \tag{6.1}$$

are equal contributions of $\kappa/4\bar{n}$ from the gain and loss mechanisms for the laser. The latter contribution is a fundamental limit because linear loss is necessary for a coherent output beam to form. However the former results from a particular (extremely reasonable) assumption about the gain mechanism for laser action, that is, that it comes from a weak linear coupling between the field and the gain medium.

These arguments suggest that a different sort of gain mechanism could produce a laser with a linewidth up to 50% below the standard quantum limit. As I have shown above, this ultimate Markovian limit

$$l_{\rm ult}=\frac{\kappa}{4\bar{n}} \tag{6.2}$$

can be achieved with a gain mechanism in which stimulated emission into the cavity mode is eliminated. This requires that the matrix element for the addition of a photon to the cavity mode be independent of the number of photons in the mode. As discussed, this could be physically achieved with adiabatic transfer of photons from another field using a counterintuitive pulse sequence.

I also examined two other gain mechanisms with similarities to the nonstimulated gain, to see if they also produced linewidth narrowing. The first, using the usual Jaynes-Cummings Hamiltonian but with a finite interaction time (as in the micromaser), did not. The second, using a nonlinear Jaynes-Cummings Hamiltonian involving three-photon as well as one-photon processes, produced a linewidth of

$$l=\frac{3\kappa}{8\bar{n}}. \tag{6.3}$$

That is, the phase diffusion due to the gain was reduced by 50% from the standard limit, resulting in an overall reduction of 25% in the linewidth. Presumably higher-order nonlinear optical processes could more closely approach the ultimate limit. However, the difficulty in producing such nonlinear optical processes, and the fact that even a third-order nonlinearity goes only halfway to the ultimate limit, suggests that the adiabatic transfer method is a better experimental option for probing toward the ultimate quantum limit to the laser linewidth.

The ultimate limit for the rate of phase diffusion attained by eliminating gain noise can also be obtained, for short times, by instead eliminating loss noise. This can be achieved by coupling the laser output into a squeezed vacuum rather than a normal vacuum [25,26]. This only works for short times because it requires a specific phase relation between the squeezed vacuum and the coherent field in the laser, which will not remain valid since the laser phase continues to diffuse. It was suggested in Ref. [25] that it might be possible to produce the squeezed vacuum by driving the squeezing device with the laser itself. In this case the whole squeezing device should really be considered as part (an internal absorber, in fact) of the laser, so that $\bar{n}$ in the original laser cavity should no longer be used as a good measure of the total stored excitation. Similar comments could be made about the proposal of Ghosh and Agarwal [27], who also misquoted the expression for the standard quantum limit given in Ref. [4] by a factor of 2 as their Eq. (18). I believe that a rigorous analysis of these proposals would reveal no reduction below the standard quantum limit.

## VII. CONCLUSION

In Sec. I, I reproduced a simple argument purporting to use the time-energy uncertainty principle to derive the standard laser linewidth as a consequence of phase diffusion due to the gain process. The results of this paper show that any such simple argument is untenable since the gain process contributes only half of the standard phase diffusion rate. To compensate for disposing of this simple argument, I will conclude this paper with a not quite so simple (but much more rigorous) argument deriving the ultimate Markovian quantum limit $l_{\rm ult}$ from another uncertainty principle argument.

Instead of the energy-time relation, which is of doubtful content, I will use the quadrature uncertainty relation

$$V(X)V(Y)\geqslant 1, \tag{7.1}$$

where $X/2$ and $Y/2$ are the real and imaginary components of the laser mode amplitude $a$. Clearly the vacuum state is rotationally symmetric with

$$V(X)=V(Y)=1, \tag{7.2}$$

and this holds also for a coherent state, which is the state the laser mode can be assumed to be in [see Eq. (2.22)].

Let the mean amplitude of the coherent state be real and positive, so that $\bar{X}/2=\sqrt{\bar{n}}$ and $\bar{Y}=0$. The phase variance is

$$V(\phi)=V\left(\arctan\frac{Y}{X}\right)\simeq\frac{V(Y)}{\bar{X}^2}=\frac{1}{4\bar{n}} \tag{7.3}$$

for $\bar{n}\gg 1$. Now the effect of linear damping for an infinitesimal time $dt$ is to reduce the mean photon number of the coherent state from $\bar{n}$ to $\bar{n}(1-\kappa dt)$. Thus the change in the phase variance is

$$dV(\phi)=\frac{\kappa dt}{4\bar{n}}. \tag{7.4}$$

A noiseless gain process will return the mean photon number to $\bar{n}$ without increasing the phase noise. Therefore, the phase variance increases at least as

$$V(\phi)\sim\frac{\kappa t}{4\bar{n}}. \tag{7.5}$$

The linewidth is defined from the two-time correlation function

$$\langle a^\dagger(t)a(0)\rangle\sim\bar{n}\langle e^{i\phi(t)}\rangle\sim\bar{n}e^{-V(\phi)/2}\sim\bar{n}e^{-\kappa t/8\bar{n}}. \tag{7.6}$$

The Fourier transform of this expression is a Lorentzian with a FWHM of

$$l=\frac{\kappa}{4\bar{n}}, \tag{7.7}$$

which is the ultimate quantum limit to the laser linewidth, as claimed.

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## APPENDIX: REFINING THE SCHAWLOW-TOWNES LIMIT

The Schawlow-Townes expression

$$l_{\rm ST}=\frac{\hbar\omega}{P_{\rm out}}\gamma^2 \tag{A1}$$

was derived in the days before good optical cavities, and hence implicitly assumes that the atomic linewidth $\gamma$ is much smaller than the (FWHM) cavity linewidth $\kappa$. With $\kappa \lesssim \gamma$, it is necessary to replace $\gamma$ by the bare linewidth of the laser $l_{\rm bare}$. This is the frequency spread the output would have if the pump were suddenly turned off and all of the energy allowed to escape. For a large class of line shapes, it can be shown that a reasonable approximation to the bare linewidth including contributions from the atomic (or other gain) medium and the cavity is

$$l_{\rm bare}^{-1}=\gamma^{-1}+\kappa^{-1}. \tag{A2}$$

For instance, this expression agrees with that given by Haken (p. 103 of Ref. [2]) for the case where $\kappa \gtrsim \gamma$. In the other cases, where $\kappa \ll \gamma$, the $\gamma$ in the Schawlow-Townes expression is simply replaced by $\kappa$ [2], which also agrees with Eq. (A2). The corrected Schawlow-Townes expression is thus

$$l'_{\rm ST}=\frac{\hbar\omega}{P_{\rm out}}l_{\rm bare}^2=\frac{\hbar\omega}{P_{\rm out}}\frac{\gamma^2\kappa^2}{(\gamma+\kappa)^2}. \tag{A3}$$

The second correction which must be made to the Schawlow-Townes linewidth relates to its use of the output power. Say, for argument's sake, that one has a laser with a linewidth given by the Schawlow-Townes limit, with all of the power coming out of one mirror. Then say that the mirror is replaced by one of the same reflectance, but with larger internal absorption. Then the power loss per round trip is identical, so the laser dynamics remain the same and the linewidth would remain the same. But the power out would be reduced because the transmittance is reduced. Therefore, the Schawlow-Townes formula would now predict an increased linewidth, which does not occur. In other words, the actual new linewidth would be *less* than the quantum limit set by the Schawlow-Townes formula. It is obviously inappropriate that a quantum limit can be surpassed by building a worse device.

The resolution to this problem with the Schawlow-Townes linewidth is to eliminate $P_{\rm out}$ from the expression by recognizing that

$$\frac{P_{\rm out}}{l_{\rm bare}} \tag{A4}$$

is an *upper* bound on the mean energy $\bar{E}$ stored as coherent excitations in the laser system. If all of the stored coherent excitation eventually makes it into the output beam of the laser then the bare linewidth $l_{\rm bare}$ is due wholly to the output coupling, and $P_{\rm out}=l_{\rm bare}\bar{E}$. In general $P_{\rm out}$ is less than this. Reducing the output coupling efficiency (as discussed in the preceding paragraph) will not affect $\bar{E}$ so it is the correct parameter to use, rather than $P_{\rm out}$. The doubly corrected Schawlow-Townes limit is thus

$$l''_{\rm ST}=\frac{l_{\rm bare}\hbar\omega}{\bar{E}}\leqslant\frac{\hbar\omega}{P_{\rm out}}\frac{\gamma^2\kappa^2}{(\gamma+\kappa)^2}, \tag{A5}$$

where the inequality becomes an equality only for perfectly efficient output coupling.

It is convenient to define the number of quanta of coherent excitation, $\bar{N}=\bar{E}/\hbar\omega$. For the case $\kappa \gg \gamma$ the excitation stored in the gain medium is negligible and $\bar{N}=\bar{n}$, where the latter represents the mean *photon* number in the cavity. If the gain medium cannot be adiabatically eliminated then $\bar{N}$ must include the excitations stored coherently in the gain medium as well. If $\gamma \ll \kappa$, as in the original Schawlow-Townes expression, these excitations in the gain medium will be the dominant ones.

The final correction which needs to be made to the Schawlow-Townes linewidth is to insert a factor of $\frac{1}{2}$. The Schawlow-Townes limit without this factor is appropriate to a laser below threshold in which the complex amplitude of the field undergoes large slow fluctuations (for $\bar{N} \gg 1$, which is the limit in which the Schawlow-Townes equation is valid). Above threshold, the laser intensity fluctuations are almost eliminated [2], leaving only phase fluctuations. This increases the coherence time by a factor of 2, so that the final corrected expression for the laser linewidth is

$$l_{\rm st}=\frac{l_{\rm bare}}{2\bar{N}}\leqslant\frac{\hbar\omega}{2P_{\rm out}}\frac{\gamma^2\kappa^2}{(\gamma+\kappa)^2}. \tag{A6}$$

Here st stands for standard (quantum limit) as opposed to ST, which stands for Schawlow-Townes.

In the limit $\kappa \ll \gamma$, which applies for many modern lasers, and which allows the gain medium to be adiabatically eliminated from the field equations, one obtains

$$l_0=\frac{\kappa}{2\bar{n}}\leqslant\frac{\hbar\omega}{2P_{\rm out}}\kappa^2. \tag{A7}$$

This result has often (including by myself [12]) been quoted as the Schawlow-Townes limit, despite the obvious differences from Eq. (A1). Here I will call it instead the *standard Markovian quantum limit* to the laser linewidth. "Markovian" refers to the fact that the equations of motion for the laser mode, including gain and loss, are well approximated by Markovian equations. For the gain process this is a consequence of adiabatically eliminating the gain medium. For the loss process, it is simply a consequence of assuming a high-$Q$ cavity. Corrections (upwards) for non-Markovian loss (low-$Q$ cavities) are discussed, for example, in Ref. [28], but here I will always assume a high-$Q$ cavity.

Obviously for $\gamma \lesssim \kappa$, the linewidth of Eq. (A6) will be less than the standard Markovian quantum limit of Eq. (A7). That is a reflection of the fact that in this case the bare linewidth

$l_{\rm bare}$ is less than $\kappa$, and also that the gain medium is an extra reservoir of energy (coherent with the laser mode) so that $\overline{N}$ is greater than $\overline{n}$. A linewidth which, in the absence of other noise sources, reduces to expression (A6) for the standard quantum limit $l_{\rm st}$ was recently derived in Ref. [29], for a laser with $\gamma \lesssim \kappa$. These authors claimed that this was ''reduced compared to the Schawlow-Townes limit'' because they followed the common (but, in my opinion, erroneous) practice of identifying $l_0$ as the Schawlow-Townes limit. To me this seems to be an example of imprecise terminology obscuring an otherwise valuable contribution to fundamental laser physics. In this paper I always work with models in which the gain medium can be adiabatically eliminated, so that $l_0 = l_{\rm st}$.


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