Floquet theory of intracavity laser frequency modulation

S. Longhi and P. Laporta
Istituto Nazionale di Fisica per la Materia, Dipartimento di Fisica del Politecnico di Milano and CEQSE-CNR, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
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The theory of laser oscillation with an intracavity sinusoidal modulation of the optical frequency is revisited and analyzed in the framework of general principles governing the properties of time-dependent periodic systems. It is shown that the two traditional and complementary descriptions of frequency modulation (FM) laser oscillation and pulsed FM mode-locking [S.E. Harris and O.P. McDuff, IEEE J. Quantum Electron. **QE**-1, 245 (1965); D.J. Kuizenga and A.E. Siegman, *ibid.* **QE-6**, 694 (1970)] can be unified by means of a more general approach based on a Floquet analysis of the laser equations in presence of a periodic phase perturbation. Starting from a spatially extended model of intracavity laser frequency modulation for a homogeneously broadened two-level ring laser, the relevant Floquet modes and corresponding Floquet exponents governing the stability properties of the nonlasing state are derived as solutions of a nonlinear eigenvalue problem. Resonance phenomena, which occur when the modulation frequency is made close to an integer multiple of the cavity axial mode separation, explain the onset of FM laser oscillation and the transition to the pulsed FM mode locking closer to the synchronous modulation. In particular, the transition from FM laser oscillation to the pulsed FM mode locking is shown to be sharp and due to a crossing of the threshold curves of two distinct Floquet modes. The role of cavity dispersion on the transition is also investigated.
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I. INTRODUCTION

Intracavity modulation of the phase (or frequency) of the electromagnetic field in optical cavities and lasers is known to profoundly affect the properties of the emitted light in different fashions, and a wide variety of effects, sometimes unpredictable at first sight, have been investigated both theoretically and experimentally in different contexts. Among others, these include the generation of highly coherent frequency modulated signals in lasers [1–4], laser frequency switching [5], ultrashort pulse generation [6–8], and generation of squeezed states of light (see, e.g., Refs. [9,10], and references therein). In case of a periodic (sinusoidal) phase perturbation, resonance phenomena are known to appear whenever the modulation period is equal to (or is an integer fraction of) the cavity photon transit time in such a way that even a small phase perturbation can strongly affect the dynamical properties of the system. A well-known manifestation of this resonance is the transition of the operational regime of an internally frequency-modulated laser which occurs when approaching the synchronous modulation condition [1,11]. In fact, when the intracavity phase perturbation is driven at a frequency which is approximately but not exactly equal to the cavity longitudinal mode separation or to one of its harmonics, the laser usually oscillates in the so-called FM regime. In this case the output field is basically an almost ideal frequency-modulated signal with an effective modulation index which is strongly enhanced with respect to that of the applied perturbation due to a cavity effect [1–4]. However, as the modulation frequency is made closer to synchronism, strong distorsions from the ideal FM operation take place, with the appearance of deep amplitude modulations (AM) superimposed to the pure phase modulation; further closer to resonance the laser switches into a different regime of operation, the pulsed FM mode-locking regime, in which the laser radiation consists of a train of short mode-locked pulses with a repetition frequency equal to the modulation frequency [6–8]. The theoretical framework of laser oscillation with an intracavity frequency modulation relies traditionally on two complementary approaches, namely the frequency-domain (or coupled-mode) method as originally proposed by Harris and McDuff for inhomogeneously broadened lasers [1], and the time-domain method, the simplest form of which was pioneered in an important series of papers by Kuizenga and Siegman on the theory of FM mode-locking for homogeneously broadened lasers [7,8,12]; an excellent and comprehensive review of these methods is given, for instance, in Refs. [11,13]. Although the two mentioned descriptions are capable of providing the most important physical insights into the problem, a general and detailed analysis of the stability properties of a laser subjected to an intracavity periodic phase perturbation seems not to be available in the literature yet, perhaps due to the complexity of the analysis when adding time-dependent perturbations to the laser equations in the physical space-time variables. Such an analysis, however, would be of upmost interest for several reasons. First, a linear stability analysis of the nonlasing state is capable of predicting the nature of the most unstable perturbations that spontaneously will grow from noise as the laser gain parameter is increased beyond threshold, and hence should be able to predict the characteristics of the bifurcating lasing state and, in particular, the occurrence of resonance phenomena in a very general fashion. Second, a bifurcation analysis should clarify the onset of transition from FM laser operation to pulsed FM mode locking which occurs very close to the synchronous modulation condition, an aspect of the problem which is hard to investigate in detail within the most traditional descriptions and that has not re-

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ceived, as a consequence, an adequate attention. Third, a complete linear stability analysis could be useful to investigate interesting transient phenomena, including transient built-up of FM laser oscillation, providing an initial basis to explain noisy phenomena and other varieties of unstable behaviors observed in phase-modulated lasers when operated close to the transition region which separates FM oscillation and FM mode-locking [11,14].

In this paper we present a detailed and comprehensive linear stability analysis of a homogeneously broadened two-level ring laser with an intracavity periodic phase perturbation based on a Floquet analysis of time-dependent Maxwell-Bloch laser equations. Due to the periodic time dependence of parameters in the laser equations introduced by the phase modulation, any initial field perturbation in the cavity can be decomposed as a superposition of Floquet solutions, which evolve (according to the Floquet theorem [15]) as independent modes. A Floquet mode is composed basically by a periodic time-dependent part, with periodicity equal to the modulation period, and by an exponential term $\sim\exp(\mu t)$, where the characteristic exponent μ determines the growth or decay of the mode. We show that the problem of determining the characteristic Floquet exponents and corresponding periodic Floquet modes for the spatially extended laser cavity can be reduced to the solution of a nonlinear eigenvalue problem involving solely the time variable. The solution to this problem clearly reveals the existence of resonances and explains the onset of FM oscillation, the appearance of distorted FM modes and the transition to the pulsed FM mode locking regime. In particular, a major result regards the transition from FM oscillation to FM mode-locking. We show, in fact, that this transition is not smooth, as one could expect, but it is abrupt and corresponds to the intersection of the threshold curves of two distinct Floquet modes. The role of cavity dispersion and frequency pulling effects of the gain medium on the transition point are also discussed.

The paper is organized as follows. In Sec. II the model of intracavity laser frequency modulation is reviewed, and the linear stability analysis of the nonlasing solution, based on a Floquet analysis of the laser equations, is presented. A detailed analysis of different regimes of operation is developed in Sec. III, with special emphasis to the transition between the FM regime and the pulsed FM mode locking. Finally, in Sec. IV the main conclusions are outlined.

II. DESCRIPTION OF THE MODEL AND FLOQUET ANALYSIS

A. The model

The starting point of our analysis is provided by a rather general model of frequency modulation inside a laser cavity. The system under investigation is schematically depicted in Fig. 1 and it consists of a ring cavity of geometrical length L containing a gain medium composed by a collection of two-level homogeneously-broadened atoms, a traveling-wave longitudinal electro-optic phase modulator, and a filter and/or a dispersive line which account for finite gain-bandwidth and/or dispersive effects in the cavity.

The dynamical equations for the field variables inside the optical cavity are represented by the Maxwell-Bloch laser

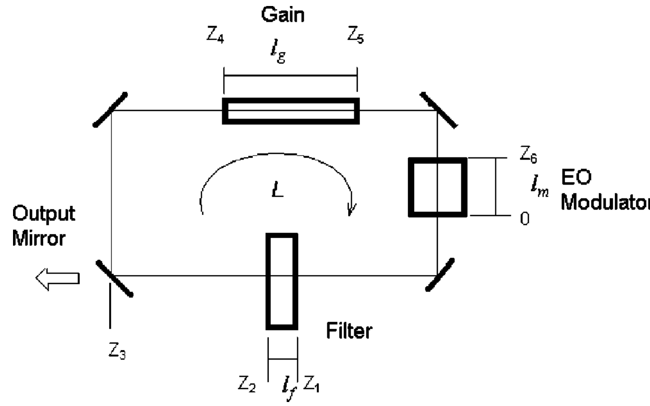

FIG. 1. Schematic diagram of the frequency-modulated laser cavity.

equations for a homogeneously broadened two-level laser [16], extended to include the parametric term of polarization due to the phase modulator and the linear dispersive or absobptive effects of the filter and of the dispersive elements possibly present in the cavity. The presence of such elements is here simulated by free-propagation of the field, for a distance l_f, through a linear dispersive and absorptive medium. In the plane-wave and rotating-wave approximations the equations of motion read

$$\frac{1}{c}\partial_t F = -\partial_z F - \alpha P + iSF\cos[\omega_m(t-z/c)+\phi] - \rho F + \beta\partial_t^2 F, \tag{1}$$

$$\partial_t P = -\gamma_\perp[FN+(1+i\delta)P], \tag{2}$$

$$\partial_t N = -\gamma_\parallel\left[N-1-\frac{1}{2}(PF^*+FP^*)\right], \tag{3}$$

where z is the longitudinal spatial variable along the ring cavity, c is the group velocity of light at each plane inside the optical cavity, F and P are the normalized slowly varying envelopes of the electric and polarization fields, respectively, the carrier frequency of which has been chosen equal to the eigenfrequency ω_c of the empty cavity closest to the atomic transition frequency ω_{at}, $\delta=(\omega_{at}-\omega_c)/\gamma_\perp$ is the atomic detuning parameter, N is the normalized population inversion of the two-level atoms, α is the small-signal gain per unit length in the active medium, $\rho=\rho(z)$ accounts for both distributed and lumped cavity losses, S is proportional to the parametric contribution to the polarization inside the electro-optic modulator, ω_m is the modulation frequency, and ϕ an arbitrary phase delay. The last term in Eq. (1) describes the propagation of the field through the dispersive and absorptive medium at leading order, and simulates the effects of filtering $[\mathrm{Re}(\beta)]$ and dispersion $[\mathrm{Im}(\beta)]$ inside the cavity. In writing Eq. (1) we have also assumed that the electro-optic phase modulator operates in a traveling-wave configuration with exact matching between the phase velocities of light and of the modulation signal [17]. The boundary conditions imposed by the ring cavity are given by

$$F(L,t)=F(0,t), \tag{4}$$

where the origin of the longitudinal coordinate z has been taken at the exit of the modulator (see Fig. 1). It should be noticed that, in Eqs. (1)–(3), the variables α, S, and β must be regarded as functions of z, assuming constant values inside the gain medium, the modulator and the filter, respectively, and vanishing outside these three regions. Likewise, cavity losses due to output coupling at the partially trasmitting mirror, located at $z=z_3$ (see Fig. 1), can be simulated in Eq. (1) by assuming $\rho(z)=l\,\delta(z-z_3)$, where $l=-\ln r$ and r is the field reflectivity of the output mirror.

The system of Eqs. (1)–(3), together with the boundary conditions (4), represents our basic model of intracavity laser frequency modulation and provides the starting point of the following analysis. It is worth pointing out that a time-domain approach to the intracavity frequency modulation problem, close to the most familiar descriptions of mode locking and related forms of mode coupling in lasers [11,13,18], could be derived by transformation of the boundary-value problem, expressed by Eqs. (1)–(4), into a propagative problem. In this case a differential-delayed equation for the field envelope at a reference plane inside the cavity is derived to simulate the successive transits of the field in the cavity (see, for instance, Ref. [19]). Although the propagative model provides a more direct and physical picture of field propagation inside the cavity than the spatially extended model [Eqs. (1)–(4)], powerful mathematical tools useful for the present physical problem (such as the Floquet theory of periodic equations [15]) are most easily applicable to partial differential equations than to differential-delayed equations. In the following, therefore, we will develop our analysis making use of the full spatially extended model.

B. Linear stability analysis of nonlasing solution: Floquet analysis

The basic properties of the laser emission in the presence of an intracavity phase perturbation depend upon the nature of the solutions that the laser selects when the nonlasing state becomes unstable. These solutions are found by linearizing the laser equations (1)–(3) around the nonlasing solution $F=P=0$, $N=1$ and looking for the growth of perturbations. The relevant equations in the linearized dynamics are those for the electric and polarization fields, which, after introduction of the auxiliary variable $D=\beta\partial_t F$, can be cast in the normal form

$$\partial_t \mathbf{v}=\mathcal{L}\mathbf{v}, \tag{5}$$

where $\mathbf{v}(z,t)\equiv(F,D,P)^T$ contains the field variables, and $\mathcal{L}(t)$ is a time-dependent periodic operator given by

$$\mathcal{L}=\begin{pmatrix} 0 & 1/\beta & 0 \\ \rho+\partial_z-iS\cos[\omega_m(t-z/c)+\phi] & 1/\beta c & \alpha \\ -\gamma_\perp & 0 & -\gamma_\perp(1+i\delta) \end{pmatrix}. \tag{6}$$

Since the linear operator $\mathcal{L}(t)$ is invariant under the discrete time translation $t\to t+T_m$, where T_m is the modulation period, the Floquet theorem [15] applies to the corresponding partial differential equations, and the relevant Floquet solutions are functions of the type

$$\mathbf{v}(z,t)=\mathbf{u}_\mu(z,t)\exp(\mu t), \tag{7}$$

where $\{\mu\}$ are the characteristic Floquet exponents and $\mathbf{u}_\mu(z,t)=\mathbf{u}_\mu(z,t+T_m)$ the corresponding periodic Floquet modes, which are eigenvalues and eigenfunctions, respectively, of the Floquet operator $\mathcal{L}-\partial_t$, i.e.,

$$[\mathcal{L}(t)-\partial_t]\mathbf{u}_\mu(z,t)=\mu\mathbf{u}_\mu(z,t). \tag{8}$$

In the eigenvalue equation (8), the functional space is defined by the class of functions that are periodic in t with period T_m and in z with period L. The temporal periodicity is a consequence of the Floquet theorem, whereas the spatial periodicity is due to the ring cavity boundary conditions [Eq. (4)]. In view of the identity

$$\exp(\mu t)\mathbf{u}_\mu(z,t)=\exp(\tilde{\mu}t)\tilde{\mathbf{u}}_\mu(z,t), \tag{9}$$

where $\tilde{\mu}=\mu+ik\omega_m$, $k=0,\pm1,\pm2,\ldots$, $\tilde{\mathbf{u}}_\mu(z,t)=\mathbf{u}_\mu(z,t)\exp(-ik\omega_m t)$, we observe that the characteristic Floquet exponents $\{\mu_n\}$ and corresponding periodic Floquet modes can be defined only mod$(i\omega_m)$, and therefore a degree of freedom is left in the choice of the imaginary part of Floquet exponents. Conversely, the real part of μ is uniquely determined, and it governs the linear stability of the nonlasing state. In fact, due to the periodicity of $\mathbf{u}_\mu(z,t)$ with respect to t, from Eq. (7) it is clear that the nonlasing solution is linearly stable provided that $\mathrm{Re}(\mu_n)<0$ for any Floquet exponent μ_n, and that an instability (laser threshold) arises when at least one of the Floquet exponents crosses the imaginary axis. The determination of the Floquet exponents $\{\mu_n\}$ requires to solve a two-dimensional linear eigenvalue problem [Eq. (8)]. As it will be shown below, such problem is equivalent to a one-dimensional nonlinear eigenvalue problem. In fact, after setting $\mathbf{u}_\mu=(E_\mu,W_\mu,V_\mu)^T$, from Eq. (8) it follows that $W_\mu=\beta(\mu+\partial_t)E_\mu$ and $V_\mu=-\chi(\mu+\partial_t)E_\mu$, where $\chi(\omega)=1/(1+i\delta+\omega/\gamma_\perp)$ is the complex Lorentzian function of the two-level gain medium, so that the following boundary-value problem for $E_\mu(z,t)$ is obtained:

$$\partial_z E_\mu=\left(\mathcal{G}_\mu(z,t)-\frac{\mu}{c}\right)E_\mu, \tag{10}$$

where

$$\mathcal{G}_\mu(z,t) = -\rho - \frac{1}{c}\partial_t + \alpha\chi(\mu+\partial_t) + \beta(\mu+\partial_t)^2 + iS\cos[\omega_m(t-z/c)+\phi]$$

and $E_\mu(L,t)=E_\mu(0,t)$. The solution of Eq. (10) can be written as

$$E_\mu(L,t) = \exp\left[\int_{L-l_m}^{L}\mathcal{G}_\mu dz\right]\exp\left[\int_0^{L-l_m}\mathcal{G}_\mu dz\right] \times \exp(-\mu L/c)E_\mu(0,t), \tag{11}$$

where, for the sake of convenience, the field propagation inside the phase modulator has been evidenced through the operator $\exp(\int_{L-l_m}^{L}\mathcal{G}_\mu dz)$. After observing that

$$\exp\left[\int_{L-l_m}^{L}\mathcal{G}(z,t)dz\right] = \exp\{i\Delta\cos[\omega_m(t-L/c)+\phi]\} \times \exp\left(-\frac{l_m}{c}\partial_t\right),$$

where $\Delta = Sl_m$ is the single-pass modulation index, using the boundary condition $E_\mu(L,t)=E_\mu(0,t)$ and assuming $\phi = \omega_m L/c$, we finally obtain the following nonlinear eigenvalue equation for $E_\mu(t)\equiv E_\mu(0,t)$:

$$\mathcal{Q}(\mu,t)E_\mu = \exp(2\pi\mu/\omega_c)E_\mu, \tag{12}$$

where $\omega_c = 2\pi c/L$ is the free-spectral range of the laser cavity and $\mathcal{Q}(\mu,t)$ is a time-dependent operator, defined on the space of functions that are periodic in the interval $[0,T_m]$, given by

$$\mathcal{Q}(\mu,t) = \exp[i\Delta\cos(\omega_m t)]\exp[D_f(\partial_t+\mu)^2 + g_0\chi(\partial_t+\mu) - l]\exp(\delta T\partial_t). \tag{13}$$

In Eq. (13), $D_f=\beta l_f$ is the filter parameter, $\delta T = T_m - T_R$ is the time detuning between the cavity round-trip time $T_R = L/c$ and the modulation period $T_m = 2\pi/\omega_m$, $l=\int_0^L \rho dz$ are the single-pass cavity losses, and $g_0 = \alpha l_g$ is the gain parameter. The nonlinear eigenvalue equation (12) is the basic result of this subsection and allows one to determine the set of Floquet exponents $\{\mu_n\}$ and corresponding periodic Floquet modes $\{E_n\}$ at the reference plane $z=0$. It is important to point out that, although in general the form of periodic Floquet modes along the cavity depends on the position of the various elements inside the cavity (i.e., on the cavity topology), the Floquet exponents $\{\mu_n\}$, and therefore the threshold conditions for the various Floquet modes, turn out to be independent of cavity topology, as it should be. Due to the nonlinear dependence of the operator $\mathcal{Q}$ on the Floquet exponent μ, the problem of determining eigenfunctions and corresponding eigenvalues of $\mathcal{Q}$ is in general challenging. However, an effective iterative procedure can be used whenever any frequency shift and line broadening effect induced by the gain medium and by the filter on the longitudinal modes of the empty cavity is small. This case applies, for instance, to laser systems with a moderate gain or loss and with gain and dispersion lines much broader than both the modulation frequency and the free spectral range of the laser cavity. These conditions are typically fullfilled, for instance, in case of intracavity frequency modulation of broad-band solid-state lasers, where the effective cavity bandwidth, limited by either the atomic transition linewidth (as in Nd:YAG lasers, see Ref. [4]) or intracavity birefringent tuners or etalons (as in FM-operated Ti:sapphire or Er-Yb:glass lasers and in erbium-doped fiber lasers, see Refs. [20–22]), can vary typically from few hundreds of GHz up to a few THz, and is therefore much larger than modulation frequencies achievable with electro-optic phase modulators. In this case, a leading order approximation to the Floquet exponents can be obtained by an analysis of the lossless empty cavity; this is done by assuming in Eq. (12) $\mathcal{Q}\sim\exp(\delta T\partial_t)$, which yields $\exp(2\pi\mu/\omega_c)\sim\exp(i\alpha\omega_m\delta T)$, $\alpha=0,\pm1,\pm2,\ldots$. Since μ is defined mod(ω_m), we can satisfy this condition by assuming

$$\mu_n^{(0)} = in\omega_c, \tag{14}$$

where n is an arbitrary integer. Once we have a first approximation to the Floquet exponents $\mu_n^{(0)}$, we can use them as a first trial for a self-consistent calculation of both periodic Floquet modes and characteristic exponents by successive iterations. For each value of $\mu_n^{(0)}$ given by Eq. (14), the linear eigenvalue problem

$$\mathcal{Q}(\mu_n^{(0)},t)E_\mu = \Lambda_n E_\mu \tag{15}$$

can be solved and, since $\Lambda_n = \exp(2\pi\mu_n/\omega_c)$, the new value μ_n closest to $\mu_n^{(0)}$ can be determined. With this better value of μ_n one can proceed to update the operator Q and find a new estimate to the characteristic exponent and corresponding Floquet mode. Note that, due to the periodicity of E_μ, the problem of determining the eigenvalues of Eq. (15) is equivalent to the computation of the eigenvalues of the infinite-dimensional matrix associated to the periodic operator $\mathcal{Q}$, which can be easily done by standard numerical methods. The iteration can be repeated until convergence is reached within a fixed precision level. It should be noticed that, in the case of small frequency shift effects, an accurate approximation to the characteristic exponents and corresponding periodic modes is obtained at the first iteration. Furthermore, if the modulation frequency is close to an integer multiple of ω_c (which is indeed the case of major interest in practice), due to the invariance $\mu\rightarrow\mu+in\omega_m$ from Eq. (14) it follows that the characteristic exponents can be organized in N groups of nearly degenerate modes with values close to

$$\mu_n^{(0)} \sim 0, i\omega_c, 2i\omega_c, \ldots, (N-1)i\omega_c, \tag{16}$$

where N is the integer closest to ω_m/ω_c. These groups of modes correspond to the so called supermodes (or hypermodes) of the harmonic FM laser (see, for instance, Ref. [23]). Note that, due to the near degeneracy of modes within each set, a leading order approximation of eigenmodes for any set is at once obtained by solving the linear eigenvalue problem [Eq. (15)] when the operator $\mathcal{Q}$ is evaluated in correspondence of the various values of $\mu_n^{(0)}$ given by Eq. (16).

In particular, in the fundamental FM operation, i.e., if $\omega_m \sim \omega_c$, there is only one set of eigenmodes corresponding to $\mu_n^{(0)} \sim 0$.

III. ANALYSIS OF DIFFERENT REGIMES OF OPERATION BY PERTURBATION THEORY OF FLOQUET MODES

It is known that the frequency modulation introduced by the modulator inside the laser cavity can strongly influence the laser threshold as well as the spectral and temporal behavior of the selected lasing state near threshold even if the phase perturbation is arbitrarily small. This singular behavior is related to a resonance phenomenon which occurs when the modulation frequency is made close to an integer multiple of the cavity axial mode separation, by means of which efficient mode coupling of laser modes can take place [11]. This resonant phenomenon is the basic physical principle underlying the appearance of the two major regimes of operation of a phase-modulated laser, namely, frequency modulation oscillation (where the laser emits a nearly constant amplitude, phase modulated beam) and FM mode locking (corresponding to a pulsed regime of operation). In this section we show that the Floquet theory developed in the previous section is capable of providing a unified and comprehensive view of the basic physical phenomena occuring in a phase-modulated laser, namely, the existence of resonances, the onset of frequency modulation regime and the transition to FM mode locking. In particular, it shows that the transition from FM oscillation to FM mode-locking regimes is sharp and corresponds to an intersection in the phase plane of two different bifurcating Floquet modes. In Sec. III A we develop a perturbative theory of Floquet modes starting from the modes of the nonmodulated empty laser cavity, and show the failure of the perturbation analysis and the appearance of secular resonances for modulation frequencies approaching the cavity resonance frequencies. In Sec. III B we improve the perturbation analysis by considering a near resonant frequency modulation, and show that the most natural basis for the asymptotic expansion is provided in this case by the Bessel modes of the phase-modulated empty cavity. This analysis again fails when the frequency detuning parameter is made closer to zero and the bandwidth of the Bessel modes becomes comparable to the gain bandwidth of the laser. In this case a transition from FM oscillation to the pulsed FM mode-locking takes place, which is analyzed in details in Sec. III C.

A. The off-resonance regime

Let us assume that the longitudinal modes of the nonmodulated, lossless empty cavity are weakly perturbed by the presence, inside the cavity, of the various elements depicted in Fig. 1. As we will show below, this is indeed what happens whenever the modulation frequency ω_m is far away from the resonance cavity eigenfrequencies. We can formally state our perturbation idea by introducing a bookeeping parameter ϵ, which provides the smallness of phase modulation, gain, loss and cavity dispersion effects, and we set

$$\mathcal{Q}(\mu,t)=\exp[i\epsilon\Delta\cos(\omega_m t)]\exp[\epsilon\mathcal{B}(\mu)]\exp(\delta T\partial_t), \tag{17}$$

where

$$\mathcal{B}(\mu)=D_f(\partial_t+\mu)^2+g_0\chi(\partial_t+\mu)-l. \tag{18}$$

Then we look for a solution of the nonlinear eigenvalue equation (12) as an asymptotic expansion in ϵ by setting

$$E(t)=E^{(0)}+\epsilon E^{(1)}+\epsilon^2 E^{(2)}+\cdots, \tag{19}$$

$$\mu=\mu^{(0)}+\epsilon\mu^{(1)}+\epsilon^2\mu^{(2)}+\cdots. \tag{20}$$

After introduction of expansions (19),(20) into Eq. (12) and using Eqs. (17),(18), a hierarchy of equations at successive orders in ϵ is obtained. The eigenvalue equation at leading order, $O(\epsilon^0)$, is solved by the following set of eigenfunctions and corresponding eigenvalues:

$$E_n^{(0)}=\exp(in\omega_m t), \tag{21}$$

$$\mu_n^{(0)}=in\gamma\omega_c, \tag{22}$$

where $n=0,\pm1,\pm2,\ldots$ is the mode index and

$$\gamma=1-\omega_m/\omega_c. \tag{23}$$

Note that, at this order, the complete Floquet modes $E_n(t)\exp(\mu_n t)$ reduce to the longitudinal modes of the lossless, nonmodulated empty cavity, and that all these modes are neutrally stable at this order. The next oder correction to the Floquet exponents is obtained from the solvability condition at $O(\epsilon^2)$ in the asymptotic expansion, which yields

$$\mu_n^{(1)}=\frac{\omega_c}{2\pi}[-D_f n^2\omega_c^2+g_0\chi(in\omega_c)-l] \tag{24}$$

and the solution at this order is given by

$$E_n^{(1)}=A_+\exp[i(n+1)\omega_m t]+A_-\exp[i(n-1)\omega_m t], \tag{25}$$

where

$$A_\pm=-\frac{i\Delta\exp(-2\pi in\gamma)}{2(\exp(\pm i2\pi\gamma)-1)}. \tag{26}$$

Equations (25),(26) show the formation, at the leading order in the perturbation expansion, of sideband modes at frequencies $\pm\omega_m$ around the cavity axial mode induced by the phase modulator, whereas Eq. (24) determines at leading order the growth rates, and hence the threshold condition, of the various modes. It should be noticed that the asymptotic expansion given by Eqs. (19),(20) is meaningful provided that any term in the expansion is of increasing order in ϵ. An inspection of Eqs. (25),(26) reveals that $E_n^{(1)}$ becomes of order ~ 1 whenever γ gets close to an integer by less than $\sim\Delta$, i.e., whenever the modulation frequency is close to an integer multiple of the cavity free spectral range. This case, which corresponds to a near-resonant modulation of the optical field, leads to a secular growth of generated sideband modes and indicates strong coupling among the longitudinal modes of the lossless, nonmodulated empty cavity.

B. The near-resonant regime: FM laser oscillation

When the modulation frequency ω_m is close to an integer multiple of the cavity free-spectral range ω_c, the analysis developed in Sec. III A indicates that even a small value of Δ can drastically change the periodic Floquet modes from those corresponding to the nonmodulated empty cavity. In this case, the laser operates in the so-called frequency modulation regime, where, at leading order, the natural ''laser modes'' are Bessel modes of the phase-modulated empty cavity [1,11]. For the sake of clearness, we will limit our analysis to the case $\omega_m \sim \omega_c$, i.e., to the fundamental FM operation, although the present analysis could be extended to a modulation at harmonic orders. To study the near-resonant regime, it is convenient to follow a different strategy from that used in the previous section by including at the zeroth order approximation in the asymptotic expansion the phase modulation term, regardless of the smallness of Δ. This circumstance is related to the fact that, as we will show below, the natural modes of the laser for a near resonant modulation are those of the phase-modulated empty cavity. In order to proceed in the analysis, we assume $\gamma \sim O(\epsilon)$ [see Eq. (23)], where ϵ is a bookeeping parameter which organizes the asymptotic expansion, and we set

$$\mathcal{Q}(\mu,t)=\mathcal{Q}_0(t)\exp[\epsilon^2\mathcal{B}(\mu)] \tag{27}$$

where $\mathcal{Q}_0(t)=\exp[i\Delta\cos(\omega_m t)]\exp(\delta T\partial_t)$ and $\mathcal{B}(\mu)$ is given by Eq. (18). Note that in writing Eq. (27) we have assumed $\mathcal{B}\sim\epsilon^2$; the choice of this scaling will be clearer later and it turns out to be satisfied in the range of parameters where the laser operates in the undistorted FM regime. The nonlinear eigenvalue equation (12) is solved by an asymptotic expansion in ϵ of eigenvalues and eigenfunctions, as given by Eqs. (19),(20). At zeroth order the eigenvalue problem reads

$$\exp[i\Delta\cos(\omega_m t)]E^{(0)}(t+\delta T)=\exp\left(2\pi\frac{\mu^{(0)}}{\omega_c}\right)E^{(0)}(t) \tag{28}$$

whose eigenfunctions and corresponding eigenvalues are

$$E_n^{(0)}\equiv|n\rangle=\exp(in\omega_m t)\exp[i\Gamma\cos(\omega_m t+\varphi)], \tag{29}$$

$$\mu_n^{(0)}=in\gamma\omega_c, \tag{30}$$

where $n=0,\pm1,\pm2,\ldots$, and

$$\Gamma=\frac{\Delta}{2\sin(\omega_m\delta T/2)}\sim\frac{\Delta}{2\pi\gamma}, \tag{31}$$

$$\varphi=-\frac{\omega_m\delta T}{2}+\frac{\pi}{2}. \tag{32}$$

Notice that at this order the complete Floquet modes $E_n(t)\exp(i\mu_n t)$ are given by $\exp(in\omega_c t)\exp[i\Gamma\cos(\omega_m t+\varphi)]$, i.e., they correspond to the well-known Bessel modes of a phase-modulated optical cavity. Note also that, as $\gamma\sim\epsilon$ is small, for a given value $\overline{n}$, all modes with $n-\overline{n}\sim O(1)$ are degenerate at leading order with eigenvalue $\mu^{(0)}\sim i\overline{n}\gamma$. To correctly perform a perturbative analysis we must therefore assume as a solution at leading order an arbitrary linear superposition of nearly degenerate modes $|n\rangle$, the coefficients of the superposition at any order being determined by the solvability conditions at the next order. If we limit our analysis by considering FM modes that are close to the center of the laser gain line, we may assume at $O(\epsilon^0)$:

$$E^{(0)}=\sum_n c_n^{(0)}|n\rangle, \quad \mu^{(0)}=0, \tag{33}$$

where the sum in Eq. (33) is extended over the modes with $n\sim O(1)$ and $c_n^{(0)}\sim O(1)$. At order $\sim\epsilon$ in the perturbation expansion, we obtain

$$(\mathcal{Q}_0-1)E^{(1)}=\sum_n\left(-2\pi in\gamma+\frac{2\pi}{\omega_c}\mu^{(1)}\right)c_n^{(0)}|n\rangle. \tag{34}$$

Since we require that $E^{(1)}\sim O(\epsilon)$, the right hand side in Eq. (34) must vanish, i.e.,

$$\left(-2\pi in\gamma+\frac{2\pi}{\omega_c}\mu^{(1)}\right)c_n^{(0)}=0. \tag{35}$$

Equation (35) can be satisfied by assuming

$$c_n^{(0)}=\delta_{n,\overline{n}}, \quad \mu^{(1)}=i\gamma\overline{n}\omega_c \tag{36}$$

and the solution at $O(\epsilon)$ is given by

$$E^{(1)}=\sum_{n\ (n\neq\overline{n})}c_n^{(1)}|n\rangle. \tag{37}$$

Equation (36) shows that, at leading order, the Bessel modes of the modulated empty cavity are not mixed, i.e., they represent the natural modes of the laser system for the chosen scaling in the perturbation expansion. It may be noticed also that at this order any solution corresponding to different values of $\overline{n}$ is neutrally stable, so that we need to push the perturbation analysis to the second order to remove the degeneracy of $\mathrm{Re}(\mu)$. At $O(\epsilon^2)$ we get

$$(\mathcal{Q}_0-1)E^{(2)}=-2\pi i\gamma\sum_{n\ (n\neq\overline{n})}(n-\overline{n})c_n^{(1)}|n\rangle+\left[\frac{2\pi}{\omega_c}\mu^{(2)}-\mathcal{Q}_0\mathcal{B}(0)\right]|\overline{n}\rangle. \tag{38}$$

The solution $E^{(2)}$ to Eq. (38) is of order $\sim\epsilon^2$ provided that the right hand side in the equation be zero. Since the Bessel modes $|n\rangle$ form a set of orthogonal functions, this condition gives

$$\mu^{(2)}=\frac{\omega_c}{2\pi T_m}\langle\overline{n}|\mathcal{Q}_0\mathcal{B}(0)|\overline{n}\rangle, \tag{39}$$

$$c_n^{(1)}=\frac{i}{2\pi T_m\gamma(n-\overline{n})}\langle n|\mathcal{Q}_0\mathcal{B}(0)|\overline{n}\rangle \quad (n\neq\overline{n}), \tag{40}$$

where $\langle f|g\rangle=\int_0^{T_m}f^*(t)g(t)dt$ denotes the usual scalar product. To evaluate the integrals in Eqs. (39),(40), we consider the case, usually satisfied in practice, where the complex Lorentzian function $\chi(\omega)$ of the gain medium can be expanded up to the second order in ω and the laser cavity is tuned to resonance with the atomic transition frequency ($\delta \sim 0$); after setting

$$\chi(\partial_t)=(1+\partial_t/\gamma_\perp)^{-1}\sim 1-\frac{1}{\gamma_\perp}\partial_t+\frac{1}{\gamma_\perp^2}\partial_t^2$$

in Eq. (18), we obtain

$$\mu^{(2)}=\frac{\omega_c}{2\pi}\left[-\left(\overline{n^2}\omega_m^2+\frac{\Gamma^2\omega_m^2}{2}\right)\left(D_f+\frac{g_0}{\gamma_\perp^2}\right)+g_0-l-\frac{i\overline{n}\omega_m g_0}{\gamma_\perp}\right] \quad (41)$$

and

$$c_{\overline{n}\pm 1}^{(1)}=\mp\frac{\Gamma\omega_m}{4\pi\gamma}\left[\frac{g_0}{\gamma_\perp}-i\left(D_f+\frac{g_0}{\gamma_\perp^2}\right)(2\overline{n}\pm 1)\omega_m\right], \quad (42)$$

$$c_{\overline{n}\pm 2}^{(1)}=\mp\frac{i}{4\pi\gamma}\frac{\Gamma^2\omega_m^2}{4}\left(D_f+\frac{g_0}{\gamma_\perp^2}\right), \quad (43)$$

$$c_n^{(1)}=0 \quad \text{for} \quad n\neq\overline{n}\pm 1, \quad \overline{n}\pm 2. \quad (44)$$

Equation (41) can be used to determine the threshold condition of the various FM modes, whereas Eqs. (37),(42)–(44) provide the leading order correction to the ideal FM modes and show the appearance of amplitude modulations (AM) at frequencies ω_m and $2\omega_m$ superimposed to the pure FM signal. This is in agreement with the previous analysis done by Harris and McDuff [1] and explains the onset of AM oscillations observed in experiments on FM operated lasers [3,19]. By setting $\mathrm{Re}(\mu^{(2)})=0$ in Eq. (41), the threshold conditions $g_{th}(\overline{n})$ for the various Bessel modes are found to be

$$g_{th}(\overline{n})=\frac{l+D_g(\omega_m^2\overline{n^2}+\Gamma^2\omega_m^2/2)}{1-(\omega_m^2\overline{n^2}+\Gamma^2\omega_m^2/2)/\gamma_\perp^2}, \quad (45)$$

where $D_g=\mathrm{Re}(D_f)$. From Eq. (45) it follows that the resonant FM mode, corresponding to $\overline{n}=0$, has the lowest threshold, and Eqs. (42),(43) indicates that for this mode AM oscillations at frequency ω_m vanish if the cavity dispersion is negligible [i.e., if $\mathrm{Im}(D_f)\sim 0$]. This result is in agreement with the analysis previously given in Ref. [19], where it was also experimentally observed that AM oscillations at frequency $2\omega_m$, i.e., at a frequency twice the modulation frequency, are dominant over AM oscillations at the modulation frequency. The asymptotic expansion based on the scaling used in Eq. (27) and corresponding to the FM regime of operation of the laser is valid provided that the bandwidth of the FM spectrum, given approximately by $\sim 2\Gamma\omega_m$, remains much smaller than the gain bandwidth of the cavity, determined either by the gain medium or by the filter. This can be seen, for instance, by an inspection of Eq. (43), which reveals that $E^{(1)}$ remains small and of order $\sim\epsilon$ provided that

$$\Gamma^2\omega_m^2(D_f+g_0/\gamma_\perp^2)\sim\epsilon^2. \quad (46)$$

It should be noticed that, when the detuning parameter $|\gamma|$ governing the effective modulation index Γ [see Eqs. (23),(31)] is so close to zero that the left hand side in Eq. (46) is still small, but of order $\sim\epsilon$, a different scaling for $\mathcal{B}$ should be used in Eq. (27), namely, $\mathcal{B}\sim\epsilon$. In this case, it turns out that the effects of finite gain bandwidth and cavity dispersion would appear already at $O(\epsilon)$ in the asymptotic expansion, and strong distorsions of FM modes would be therefore present at leading order. This situation foresees the transition from the FM regime to the pulsed mode-locking regime, which is analyzed in the next subsection and in the Appendix.

C. Transition from the FM regime to pulsed FM mode locking

Break up of the FM regime described in previous section occurs when the modulation frequency is made close to the axial mode separation (or to an integer multiple of it) by an amount such that condition (46) is violated. Conversely, it is known that at resonance (or for a modulation frequency detuned from exact synchronism by a small amount) the laser operates in the FM mode-locking regime, which is characterized by the formation of a periodic train of short pulses which pass through the modulator in correspondence to either the maxima or minima of the phase perturbation. Although the theory of FM mode locking for homogeneously broadened lasers, including detuning effects, has been widely studied in literature (see, for instance, Refs. [7,12,24]), for the sake of completeness it is reviewed in the framework of the Floquet theory in the Appendix. The analysis of the laser behavior in the detuning range connecting the exact synchronous or slightly detuned operation, corresponding to FM mode locking, and the moderate detuning region, corresponding to the FM oscillation regime, is challenging and no completely satisfactory analysis seems to be available yet. In this subsection we show that the FM theory based on the Floquet analysis is capable of providing a satisfactory and comprehensive understanding of the transition from FM oscillation to FM mode locking; in particular, we show that the transition is sharp and due to an intersection of two Floquet modes in the plane (γ,g_{th}). For the sake of simplicity, we will focus our analysis to the case of frequency modulation at the fundamental harmonic, i.e., $\omega_m\sim\omega_c$. Furthermore, we will be mainly concerned with the case where cavity dispersion is negligible and the overall gain bandwidth of the cavity is determined by the filter, leaving a discussion about cavity dispersion effects and finite bandwidth of the gain medium to the end of this paragraph. We thus set in Eq. (13) $\chi\sim 1$ and $D_f=D_g+iD_i\sim D_g$, where the real parameters D_g and D_i describe finite gain badwidth and cavity dispersion effects, respectively. In this case, the dimensionless free parameters that govern the properties of Floquet modes and characteristic exponents are the normalized frequency detuning parameter $\gamma=1-\omega_m/\omega_c$, the single-pass modulation depth Δ, and the parameter $N=1/\omega_m\sqrt{D_g}$, which corresponds roughly to the number of cavity axial modes that fall under the gain curve of the cavity. Figure 2 shows a typical behavior of the threshold curve $g_{th}-l$ of the laser as a func-

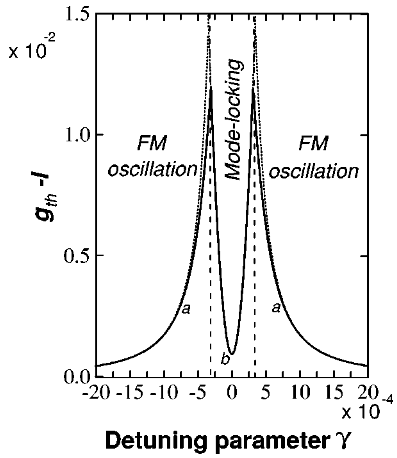

FIG. 2. Laser threshold $g_{th}-l$ as a function of the normalized frequency detuning parameter $\gamma=1-\omega_m/\omega_c$ for parameter values: $\Delta=0.04$, $N\equiv 1/(\omega_m\sqrt{D_g})=108$, and $D_f=1/\gamma_\perp=0$. The vertical dashed lines in the figure separate the regions of FM oscillation and of pulsed FM mode locking, whereas the dotted line, partially overlapped with the continuous curve, is the threshold curve as predicted by Eqs. (45),(A26).

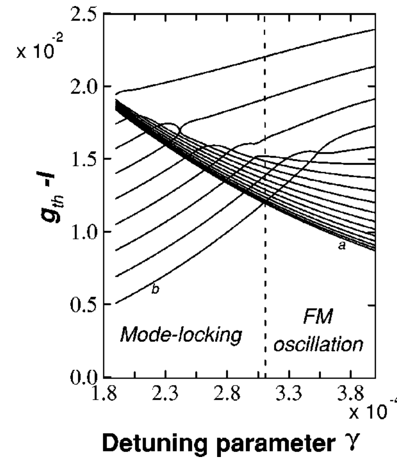

FIG. 3. Threshold curves of a few low-order Floquet modes as a function of the frequency detuning parameter γ for the same parameter values as in Fig. 2.

tion of the normalized detuning parameter γ, as obtained by numerical solution of the eigenvalue equation (12) using the technique described at the end of Sec. II B. The number of modes considered in the numerical discretization of the operator $\mathcal{Q}$ has been chosen, for each value of the frequency detuning parameter, sufficiently large to contain all nonvanishing modes. As it can be seen from Fig. 2, the threshold curve is symmetric around resonance, and it results from the intersection of two different curves *a* and *b* that correspond to the laser oscillating in the FM regime (far from resonance) and in the pulsed FM mode locking (close to the resonance). Far away from resonance, the threshold for laser oscillation approaches the value $g_{th}-l\sim 0$ of the free-running laser. As the detuning parameter γ is moved toward the resonance from both sides (curve *a* in the figure), there is an increase of the threshold, which is due to the spectral broadening of the central Bessel mode of the phase modulated cavity (see Sec. III B); the dotted lines in the figure show the threshold condition for this mode as predicted by the perturbation analysis developed in Sec. III B and given by Eq. (45) with $\overline{n}=0$ and $\Gamma\omega_m\ll\gamma_\perp$. Close to resonance, at around $\gamma\sim\pm 0.00031$, crossing curve *b* results in a threshold lowering, which marks the transition to the pulsed FM mode-locking regime. The dotted line in the figure, almost overlapped with curve *b*, shows the threshold condition for the fundamental Gaussian mode-locking mode as predicted by the perturbation analysis developed in the Appendix [see Eq. (A26)]. To better understand the origin of the transition, it is worth considering the behavior of the threshold curves for a few low-order Floquet modes in the neighbor of the transition region, which are shown in Fig. 3. The figure clearly indicates that the transition from FM oscillation to FM mode locking is due to the crossing of the threshold curves of two different Floquet modes, labeled by *a* and *b* in the figure. The intensity profile and corresponding spectra of these Floquet modes for a few values of the frequency detuning parameter are shown in Figs. 4–6. It should be noted that curve *b* actually describes two distinct Floquet modes that are degenerate in threshold. These modes correspond to two opposite values of $\mathrm{Im}(\mu)$, and their spectra are symmetric about the center of the gain curve (see Figs. 5,6). From these figures it can be seen that, far away from the transition region [Figs. 4–6(a)], the three modes correspond to almost ideal FM signals as discussed in Sec. III B, and we can therefore use the notations introduced in that section to label the modes. The mode corresponding to curve *a* both in Figs. 2 and 3 is the central Bessel mode $|0\rangle$ with the lowest threshold, whereas the other two degenerate modes (curve *b*) are two higher-order modes $|\pm n\rangle$ which corresponds to FM signals with a carrier frequency shifted from the center of the gain line by $\sim\pm n\omega_c$. [See Eqs. (29),(30).] For the parameter values chosen in Figs. 2,3, it turns out that $n=14$. As the detuning parameter is adiabatically moved closer to the transition region, strong distorsions of the mode spectra from ideal Bessel amplitudes appear [Figs. 4–6(b)], until very close to or across the transition region it becomes even inappropriate to speak of FM modes [see Figs. 4–6(c),6(d)]. In that region, in fact, the perturbative analysis of Sec. III B becomes inadequate and the identification of the three modes with FM modes of the phase-modulated empty cavity is somewhat arbitrary. However, we can continue to identify the three modes using the notations $|0\rangle$, $|n\rangle$, and $|-n\rangle$, having in mind that they correspond to the modes given by Eq. (29) solely when the detuning parameter $|\gamma|$ is adiabatically increased sufficiently away from the transition region. In correspondence of this region, the spectrum of mode $|0\rangle$ turns out to be composed mainly by two lobes, which are shifted away from the center of the gain line and that resemble to the lobes of the original FM signal; conversely, the spectra of modes $|\pm n\rangle$ are formed by a single lobe which moves close to the center of the gain line as the resonance modulation condition is attained. In the time domain, the two degenerate modes $|\pm n\rangle$ correspond to short pulses which pass through the modulator nearby the two different stationary points of the phase perturbation, i.e., in proximity of the maxima or minima of the phase modulation [see Figs. 5,6(d)]. This kind of degeneracy is well-known in the theory of FM mode locking [11] and indicates that FM mode-locked lasers can support two different sequences of pulses differing in the positioning of the pulses within the drive cycle and (slightly) in

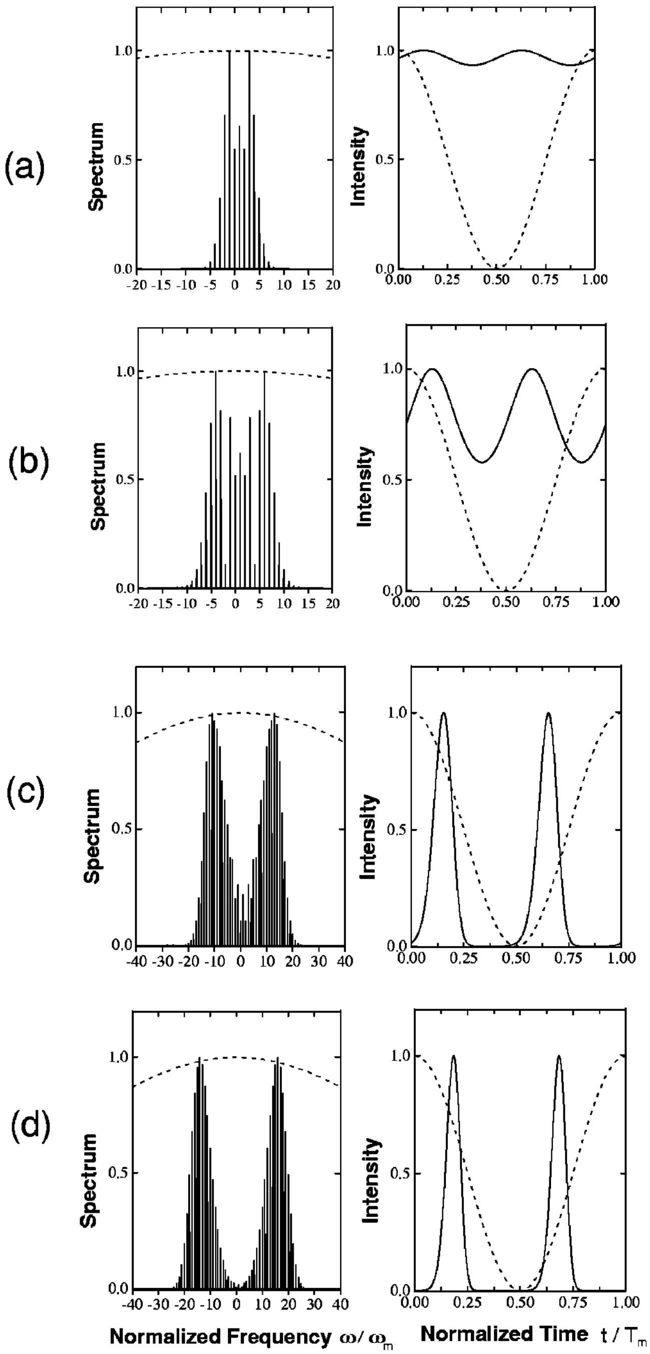

FIG. 4. Spectra (left columns) and corresponding intensity profiles (right columns) of the fundamental Floquet mode $|0\rangle$ (curve *a* in Figs. 2,3) for a few values of the detuning parameter γ. (a) $\gamma=0.002$, (b) $\gamma=0.001$, (c) $\gamma=0.00035$, and (d) $\gamma=0.000184$. The dashed curve in the figures on the left side is the spectral gain line of the filter, whereas the dashed curve in the figures on the right side shows the modulation signal impressed to the modulator.

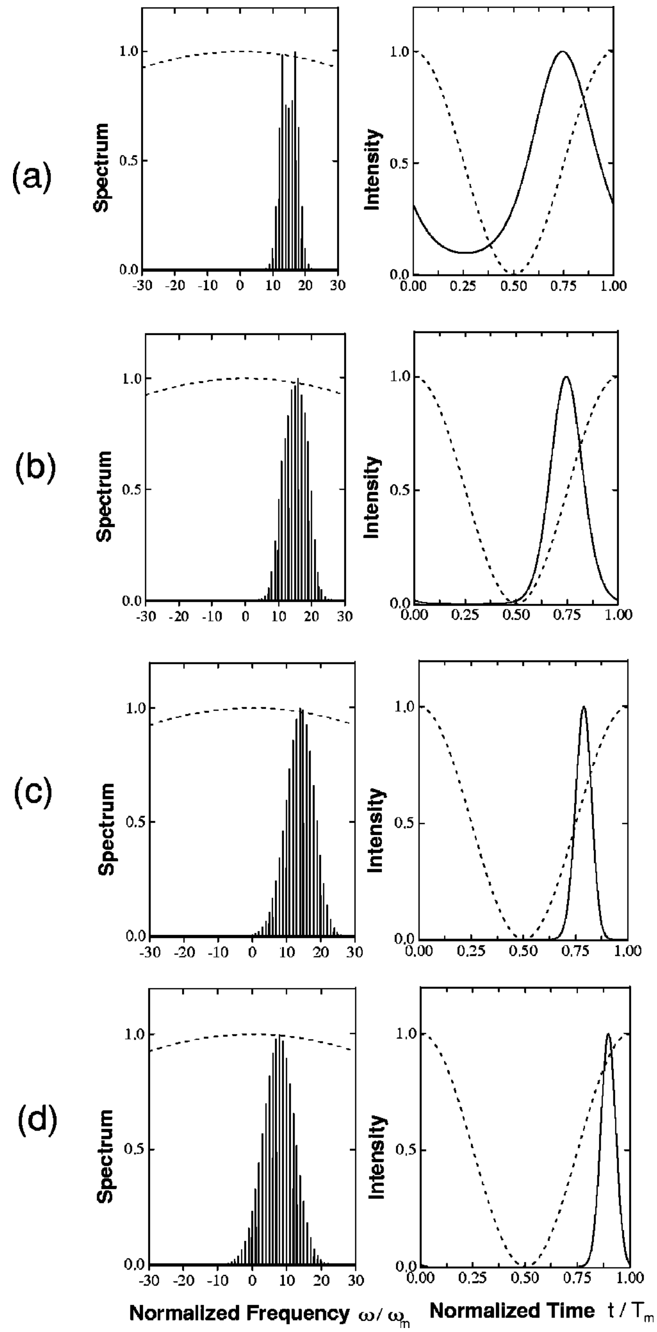

FIG. 5. Same as Fig. 4 but for the mode $|n\rangle$.

frequency due to the opposite values of $\mathrm{Im}(\mu)$ competing to the two modes. As the pulses in the two sequences differ by the chirp, they are usually called up-chirped and down-chirped pulses of the FM mode-locked laser.

The frequency detuning value γ_T at which the transition from FM oscillation to FM mode locking occurs is an important parameter which provides indicatively an upper limit for stable mode-locking operation. A typical dependence of this value on the modulation index Δ is shown in Fig. 7. An approximate dependence of the transition detuning γ_T on the dimensionless parameters Δ and $N=1/\omega_m\sqrt{D_g}$ can be obtained by imposing that the threshold values for the central Bessel mode $|0\rangle$ and the fundamental Gaussian mode-locking mode be equal. Using the approximate relations for these thresholds as given by Eqs. (45),(A26), we get

$$\gamma_T \sim \frac{1}{2\pi N^{3/2}}\sqrt{-\sqrt{\Delta}+\sqrt{\Delta+2\Delta^2 N^2}}. \qquad (47)$$

The above considerations are valid provided that cavity dispersion and finite gain bandwidth effects in the two-level gain medium are negligible. When these effects are considered in the analysis, there are some qualitative differences that can be briefly summarized as follows. The role of a

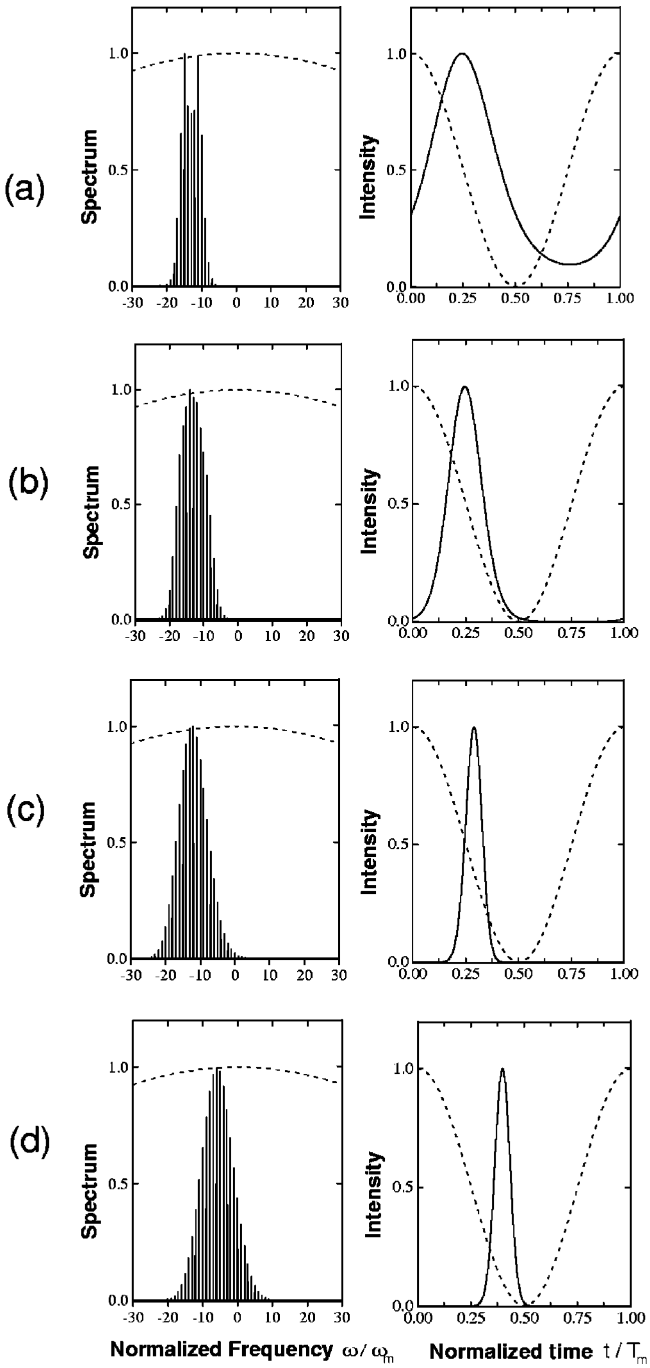

FIG. 6. Same as Fig. 4 but for the mode $|-n\rangle$.

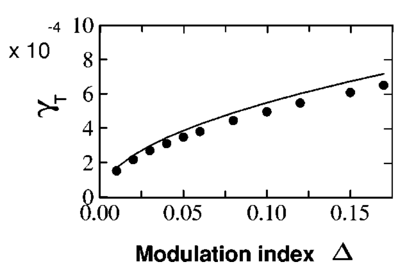

FIG. 7. Behavior of normalized frequency detuning γ_T at the transition as a function of the modulation index Δ. The points correspond to numerical simulations, whereas the continuous curve is the frequency detuning behavior as predicted by Eq. (47).

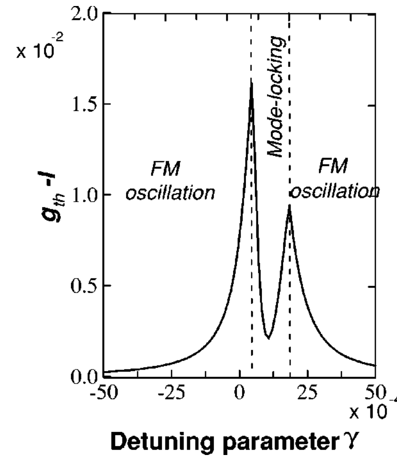

FIG. 8. Same as Fig. 2 for parameter values: $\Delta = 0.04$, $\gamma_\perp/\omega_m = 15$, $D_g = D_i = 0$, and $l = 10\%$.

finite gain line of the active medium is to introduce, besides a spectral filtering action analogous to that exploited by the filter, also a frequency-dependent refractive index near the resonance associated with the complex part of the Lorentzian function $\chi(\omega)$. Such a frequency-dependent refractive index is responsible for a slight shift of the exact synchronous modulation point and breaks the symmetric behavior around the resonance found in the previous analysis. As an example, Fig. 8 shows a typical behavior of the threshold curve g_{th}^{-2} of the laser as a function of the normalized detuning parameter γ as obtained by numerical solution of the eigenvalue equation (12) by assuming $D_f = 0$ and $\gamma_\perp = 15\omega_m$.

The role of cavity dispersion on FM mode locking has been already investigated in literature, with special emphasis payed to pulse chirp compensation and to dispersion-induced breakdown of threshold degeneracy of up-chirped and down-chirped pulses (see, for instance, Refs. [25]). Here we point out that cavity dispersion also changes the crossing point between FM mode locking and FM oscillation, pushing the frequency detuning γ_T of the transition away from the resonance point. This is clearly shown in Fig. 9, where a typical behavior of γ_T as a function of the normalized cavity dispersion D_i/D_g, as obtained by numerical analysis of Eq. (12), is reported. The threshold for laser oscillation at the transition point as a function of cavity dispersion is shown in Fig. 10. Note that the curves are symmetric around the zero dispersion point and that, as the cavity dispersion is increased, the transition occurs at larger detunings γ_T with a threshold lowering. This behavior can be understood from a physical viewpoint by observing that the mode-locking regime has a

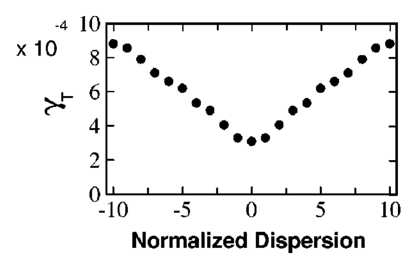

FIG. 9. Behavior of the transition detuning γ_T as a function of the normalized cavity dispersion D_i/D_g for the same parameter values as in Fig. 2.

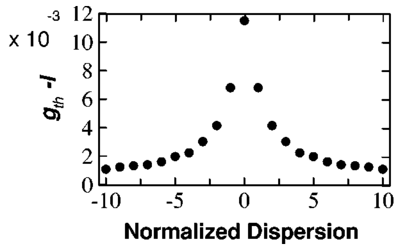

FIG. 10. Laser threshold at the transition point as a function of the normalized cavity dispersion for the same parameter values as in Fig. 9.

lower threshold than FM oscillation (see, for instance, Fig. 2), and the exact resonance regime is therefore the preferred regime of operation of the laser. In presence of a strong cavity dispersion and when the modulation frequency is tuned close to a resonance of the *dispersionless* cavity, the carrier frequency of the laser can adjust itself in such a way that exact synchronism is possible between the externally imposed phase perturbation and a cavity resonance of the *dispersive* cavity. In other words, we can say that the effect of cavity dispersion is to broaden the resonances of the laser cavity and to facilitate, as a consequence, the achievement of exact phase synchronism. This phenomenon has an important impact in enlarging the stability region of FM mode locking, and it has been successfully exploited to generate self-synchronized mode-locked pulses in erbium-doped fiber lasers under AM modulation [26].

IV. CONCLUSIONS

A detailed and comprehensive linear stability analysis of laser oscillation with an intracavity phase modulation of the optical field, based on a Floquet analysis of spatially extended laser equations, has been presented. We have shown that the Floquet solutions and corresponding Floquet exponents of the laser equations in the presence of a periodic (sinusoidal) phase perturbation can be derived as solutions of a nonlinear eigenvalue problem. The occurrence of resonances when the modulation frequency is close to an integer multiple of the cavity axial mode separation manifests itself in the strong dependence of Floquet modes on the frequency detuning parameter and explains the well-known cavity-enhanced effect of FM modulation near resonance and the transition to the pulsed FM mode locking regime when the modulation frequency is made closer to the synchronous condition. A perturbative analysis of Floquet modes for different scalings of parameters entering in the nonlinear eigenvalue problem allows us to recover, as particular cases, the well-known theories of FM laser oscillation [1] and FM mode locking [7] based on frequency or time domain methods. A noteworthy benefit of the Floquet approach is to clarify the transition from FM laser oscillation to pulsed FM mode locking near resonance, an issue that has been investigated with some detail in the paper. In particular, the behavior of threshold curves for low-order Floquet modes versus the frequency detuning parameter allows us to explain the transition from FM oscillation to FM mode locking as due to a crossing of two distinct modes. This fact allows us to answer to the question, not fully investigated in the literature yet, of whether the transition from FM oscillation to FM mode locking, which occurs when decreasing the detuning parameter close to resonance, is continuous or, as we have shown, abrupt. Finally, an approximate analytical expression for the transition frequency detuning has been derived, and the role of cavity dispersion on the transition has been clarified.

ACKNOWLEDGMENT

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APPENDIX: PERTURBATION THEORY OF FLOQUET MODES: THE RESONANT REGIME (FM MODE LOCKING)

In order to find approximate analytical expressions of Floquet modes and characteristics exponents in the parameter region corresponding to the pulsed operation of the FM laser, we introduce the following scaling for the various parameters entering in Eq. (12):

$$\Delta \sim \epsilon, \tag{A1}$$

$$\delta T \omega_m = 2\pi\gamma \sim \epsilon, \tag{A2}$$

$$\mathcal{B}(\mu) \sim \epsilon, \tag{A3}$$

$$\mu \sim \epsilon, \tag{A4}$$

where $\mathcal{B}(\mu)$ is given by Eq. (18) and ϵ is a smallness parameter. Using Eqs. (A1)–(A4), at leading order in ϵ the nonlinear eigenvalue equation (12) transforms into the linear eigenvalue problem

$$[i\Delta \cos(\omega_m t) + \mathcal{B}(0) + \delta T \partial_t]E = \frac{2\pi}{\omega_c}\mu E. \tag{A5}$$

If we restrict our analysis to the case where the finite bandwidth effects of the gain medium and cavity dispersion are negligible, we may assume in Eq. (A5)

$$\mathcal{B}(0) = g_0 \chi(\partial_t) + D_f \partial_t^2 - l \sim g_0 - l + D_g \partial_t^2 .$$

The spectral equation (A5) is defined in the space of functions that are periodic in the interval $[0,T_m]$. The problem of determining eigenvalues and eigenfunctions of Eq. (A5) is, in general, challenging and no exact analytical solutions seem available. However, as we are interested in determining pulsed solutions of the FM laser, we may focus our analysis by looking for a class of solutions of Eq. (A5) which assume a nonvanishing value solely in a narrow temporal window of the modulation period T_m. In this case, we may expand the sinusoidal phase perturbation at second order in time around the pulse position t_0

$$\cos(\omega_m t) \sim \cos(\omega_m t_0) - \sin(\omega_m t_0)(t-t_0) - \frac{1}{2}\cos(\omega_m t_0)(t-t_0)^2 \tag{A6}$$

and we may search for a solution of Eq. (A5) in the form of a generalized Gaussian function

$$E(t)=\left(\sum_n c_n(t-t_0)^n\right)\exp[-\sigma(t-t_0)^2+i\theta(t-t_0)]. \tag{A7}$$

The pulse parameters σ and θ, the pulse position t_0 and the coefficients c_n of the polynomial in Eq. (A7) have to be determined self-consistently by imposing the ansatz (A7) to be a solution to Eq. (A5). We note that the ansatz (A7), together with the parabolic approximation (A6), is basically an extension of the original Siegman-Kuizenga theory of detuned FM mode locking [12]. After substituting Eqs. (A6),(A7) into Eq. (A5) and collecting the terms in the equation of the same power in $(t-t_0)$, the following recurrence relation for the coefficients c_n is obtained:

$$\alpha_k c_{k+2}+\beta_k c_{k+1}+\delta_k c_k+\rho c_{k-1}+\omega c_{k-2}=0, \tag{A8}$$

where

$$\alpha_k=D_g(k+1)(k+2), \tag{A9}$$

$$\beta_k=\delta T(k+1)+2i\theta D_g(k+1), \tag{A10}$$

$$\delta_k=i\Delta\cos(\omega_m t_0)+i\delta T\theta+g_0-l-\frac{2\pi}{\omega_c}\mu-D_g[2\sigma(2k+1)+\theta^2], \tag{A11}$$

$$\rho=-2\delta T\sigma-i\Delta\omega_m\sin(\omega_m t_0)-4i\sigma\theta D_g, \tag{A12}$$

$$\omega=-\frac{i\Delta\omega_m^2}{2}\cos(\omega_m t_0)+4\sigma^2 D_g. \tag{A13}$$

The ansatz (A7) is meaningful provided that the series in Eq. (A7) reduces to a polynomial of order N, with $N=0,1,2,3,\ldots$. From Eq. (A8), it can be seen that the condition $c_N\neq 0$, $c_k=0$ for $k=N+1,N+2,\ldots$ is satisfied if and only if the following equations are valid:

$$\omega=\rho=0, \tag{A14}$$

$$\delta_N=0. \tag{A15}$$

For a fixed order N, the coefficients of the polynomial can be then calculated iteratively by the recurrence equation

$$c_N=1, \tag{A16}$$

$$c_{N-1}=-\frac{\beta_{N-1}}{\delta_{N-1}}, \tag{A17}$$

$$c_k=-\frac{\alpha_k c_{k+2}+\beta_k c_{k+1}}{\delta_k}\quad (k=0,1,2,\ldots,N-2). \tag{A18}$$

The condition given by Eq. (A14) determines the pulse parameters σ, θ and the pulse position t_0, which turn out to be independent of the mode order N and are given explicitly by

$$\sigma=(1\pm i)\xi, \tag{A19}$$

$$\theta=\pm\frac{\delta T}{2D_g}, \tag{A20}$$

where ξ and t_0 are the solutions of the equations

$$\cos(\omega_m t_0)=\pm\frac{16D_g\xi^2}{\Delta\omega_m^2}, \tag{A21}$$

$$\sin(\omega_m t_0)=\mp\frac{4\xi\delta T}{\Delta\omega_m}. \tag{A22}$$

Notice the double sign in Eqs. (A19),(A20), which indicates the existence of a mode degeneracy, and the complex nature of σ, which indicates that the pulses are chirped. On the other hand, condition (A15) allows us to calculate the characteristic exponent μ_N

$$\mu_N=\frac{\omega_c}{2\pi}[i\Delta\cos(\omega_m t_0)+i\theta\delta T+g_0-l-2D_g\sigma(2N+1)-D_g\theta^2]. \tag{A23}$$

The threshold condition for the mode of order N is then obtained by making $\mathrm{Re}(\mu_N)=0$, and reads

$$g_{th}(N)=l+D_g[2\xi(2N+1)+\theta^2]. \tag{A24}$$

From Eq. (A24) it follows that the fundamental Gaussian mode, corresponding to $N=0$, is the lowest threshold mode. At exact resonance, i.e., for $\delta T=0$, the solutions to Eqs. (A21),(A22) read

$$t_{0+}=0\quad t_{0-}=T_m/2\quad \xi=\sqrt{\frac{\Delta}{D_g}}\frac{\omega_m}{4} \tag{A25}$$

which correspond to the well-known down-chirped and up-chirped pulse sequences passing through the modulator in correspondence of either the maxima or minima of the phase modulation. For a small frequency detuning, such that $t_0\omega_m\ll\pi$, at leading order the solution to ξ remains unchanged, whereas the pulse position is shifted away from the extrema of the phase modulation by the small quantity $-4\xi\delta T/\Delta\omega_m^2$. Note also that, for small frequency detunings, the threshold for laser oscillation is an almost quadratic function of the time detuning δT and is explicitly given by

$$g_{th}=l+D_g\left[\frac{\omega_m}{2}\sqrt{\frac{\Delta}{D_g}}+\left(\frac{\delta T}{2D_g}\right)^2\right]. \tag{A26}$$

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