Casimir effect between two conducting plates

Reza Matloob
Department of Physics, University of Kerman, Kerman, Iran
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The Maxwell stress tensor is used to introduce the radiation pressure force of the electromagnetic field on a conducting surface. This expression is related to the imaginary part of the vector potential Green function for the fluctuating fields of the vacuum via the fluctuation dissipation theorem and Kubo's formula. The formalism allows one to evaluate the vacuum radiation pressure on a conducting surface without resorting to the process of field quantization. The latter formula is used to calculate the attractive and repulsive Casimir force between two conducting plates. What is more, in this formalism, there is no need to apply any regularization procedure to recover the final result. [S1050-2947(99)05210-5]

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I. INTRODUCTION

The Casimir force [1,2] on material bodies is interpreted either traditionally as a long range retarded phenomenon [3], or viewed from a modern perspective, as the force arising from a change of the zero point energy of the electromagnetic field in the presence of a boundary surface (or surfaces) [4,5]. In the modern theory the concept of the Casimir energy is introduced, first and necessarily, as the difference between the zero point energy in the presence and the absence of external constraints. In contrast to the vacuum energy itself, this energy which gives rise to the Casimir force is a finite quantity if an appropriate regularization procedure is used. In this sense, the Casimir force is the response of the vacuum against the presence of external bodies. The Casimir effect on an object, therefore, appears if the space surrounding it possesses an asymmetric nature. As a result this effect will manifest itself either as an attractive or repulsive mutual force between macroscopic objects. The repulsive effect is the notable difference between this phenomenon and the van der Waals–like forces.

The notion of the Casimir effect as the reaction of the vacuum against the existence of macroscopic bodies allows one to calculate the force acting on a given object by evaluating the vacuum radiation pressure on it [6–10]. The radiation pressure is an indication of the momentum inherent in an electromagnetic field even if the field is in its vacuum state. In this method one should necessarily go through the process of electromagnetic field quantization for the particular geometry in question, and thereafter calculate the Casimir force using the Maxwell stress tensor. Apart from a few simple configurations, considerable difficulty arises in most practical situations when the electromagnetic field is to be quantized. Any simplification in this formalism is therefore of value whenever more complicated geometries are involved. The aim of the present paper is to tackle the latter difficulty. This may be a step forward to make the theory closer to the experimental results [11–14].

In Sec. II we begin with the Maxwell stress tensor, the fluctuation dissipation theorem, and Kubo's formula to provide a general expression for the vacuum pressure force acting on a conducting surface. This expression is used in Secs. III and IV to calculate the attractive and repulsive forces between two conducting plates. Finally, the main points of the present work are summarized in the concluding section.

II. VACUUM RADIATION PRESSURE ON A CONDUCTING SURFACE

The conservation of linear momentum in the classical theory of electrodynamics leads to the introduction of the Maxwell stress tensor [15]

$$T_{\alpha\beta}=\epsilon_0 E_\alpha E_\beta+\frac{1}{\mu_0}B_\alpha B_\beta-\frac{1}{2}\left(\epsilon_0 E^2+\frac{1}{\mu_0}B^2\right)\delta_{\alpha\beta}, \tag{2.1}$$

where $\alpha,\beta=x,y,z$ denote the Cartesian coordinates, and ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively. The statement of the conservation of momentum indicates that the flow per unit area of momentum across a given surface S, or equivalently, the force per unit area acting on S is given by

$$\vec{F}=\int_S \mathbf{T}\cdot\hat{n}\,ds, \tag{2.2}$$

where $\mathbf{T}$ is the Maxwell stress tensor given by Eq. (2.1) and $\hat{n}$ is the unit outward normal vector at ds. Equation (2.2) can therefore be used to calculate the force acting on material objects due to the electromagnetic field.

Consider now a perfectly conducting medium with a single plane interface at $z=0$ which fills the half space $z\geqslant 0$. The x and y axes lie within the interface. Recalling the symmetry of the configuration, the radiation pressure force experienced by the plane interface of the conductor is in the z direction and it is given by the zz component of the stress tensor at $z=0$. The boundary conditions on the tangential **E** and normal **B** fields require that these two field components vanish on the interface. Therefore, T_{zz} has the following form on the conductor surface:

$$T_{zz}=\frac{1}{2}\epsilon_0 E_z^2-\frac{1}{2}\epsilon_0 c^2(B_x^2+B_y^2), \tag{2.3}$$

where $\mu_0\epsilon_0=c^{-2}$ has been used.

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The quantum expression of this force is obtained by converting the classical quantities into quantum mechanical counterparts. The expectation value of the latter expression evaluated for the vacuum state will then yield the force associated with the vacuum field. The vacuum radiation pressure force per unit area experienced by the plane interface of the conductor is then

$$F=-\frac{1}{2}\epsilon_0\langle 0|\hat{E}_z^2(\mathbf{r},t)|0\rangle+\epsilon_0 c^2\langle 0|\hat{B}_x^2(\mathbf{r},t)|0\rangle, \quad (2.4)$$

where account has been taken of the equal contributions of the second and third terms in Eq. (2.3) and $|0\rangle$ stands for the vacuum state of the electromagnetic field. To avoid any future ambiguity, the field argument $(\mathbf{r},t)$ is inserted in Eq. (2.4). The electric field operator has the form

$$\hat{\mathbf{E}}(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}d\omega[\hat{\mathbf{E}}^+(\mathbf{r},\omega)e^{-i\omega t}+\hat{\mathbf{E}}^-(\mathbf{r},\omega)e^{+i\omega t}], \quad (2.5)$$

where the positive and negative frequency parts in the integrand involve only the photon annihilation and creation operators, respectively. Using Eq. (2.5) and a similar decomposition for $\hat{\mathbf{B}}(\mathbf{r},t)$, Eq. (2.4) can be rewritten as

$$F=-\frac{\epsilon_0}{4\pi}\int_0^{+\infty}d\omega\int_0^{+\infty}d\omega'[\langle 0|\hat{E}_z^+(\mathbf{r},\omega)\hat{E}_z^-(\mathbf{r},\omega')|0\rangle -2c^2\langle 0|\hat{B}_x^+(\mathbf{r},\omega)\hat{B}_x^-(\mathbf{r},\omega')|0\rangle]e^{-i(\omega-\omega')t}, \quad (2.6)$$

where terms in which annihilation operators act directly on the vacuum state $|0\rangle$ have been set to zero.

The vector potential correlation function is related to the imaginary part of the vector potential Green function $G(\mathbf{r},\mathbf{r}',\omega)$ via the fluctuation dissipation theorem and Kubo's formula [16]

$$\langle 0|\hat{A}_\alpha^+(\mathbf{r},\omega)\hat{A}_\beta^-(\mathbf{r}',\omega')|0\rangle=2\hbar\,\mathrm{Im}\,G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\,\delta(\omega-\omega'), \quad (2.7)$$

where the same frequency decomposition is understood for $\hat{\mathbf{A}}(\mathbf{r},t)$. Taking advantage of the gauge in which the scalar potential vanishes, the electric and magnetic field operators are derived from the vector potential

$$\hat{\mathbf{E}}^+(\mathbf{r},\omega)=i\omega\hat{\mathbf{A}}^+(\mathbf{r},\omega),$$
$$\hat{\mathbf{B}}^+(\mathbf{r},\omega)=\nabla\times\hat{\mathbf{A}}^+(\mathbf{r},\omega). \quad (2.8)$$

The negative frequency part of each field operator is determined by taking the Hermitian conjugate of Eq. (2.8). Using Eqs. (2.7) and (2.8), the electric and magnetic field correlation functions are related to the imaginary part of the vector potential Green function according to

$$\langle 0|\hat{E}_\alpha^+(\mathbf{r},\omega)\hat{E}_\beta^-(\mathbf{r}',\omega')|0\rangle =2\hbar\omega^2\mathrm{Im}\,G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\,\delta(\omega-\omega') \quad (2.9)$$

and

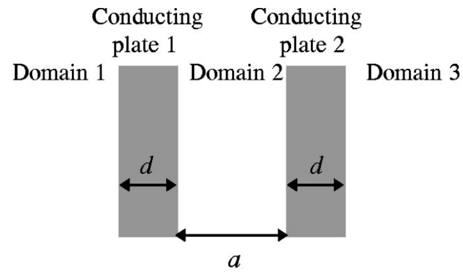

FIG. 1. Representation of the geometry of two conducting plates.

$$\langle 0|\hat{B}_\alpha^+(\mathbf{r},\omega)\hat{B}_\beta^-(\mathbf{r}',\omega')|0\rangle =2\hbar\,\epsilon_{\alpha\gamma\delta}\epsilon_{\beta\eta\nu}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x'_\eta}\mathrm{Im}\,G_{\delta\nu}(\mathbf{r},\mathbf{r}',\omega)\,\delta(\omega-\omega'), \quad (2.10)$$

where $\epsilon_{\alpha\gamma\delta}$ is the antisymmetric Levi-Civitá symbol and a summation over the repeated Cartesian indices is implied. Employing Eqs. (2.9) and (2.10), Eq. (2.6) can be simplified as

$$F=-\frac{\epsilon_0\hbar c^3}{2\pi}\mathrm{Im}\int_0^{+\infty}dq\left[\left(q^2\delta_{3\alpha}\delta_{3\beta}+2\epsilon_{1\gamma\alpha}\epsilon_{1\eta\beta}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x'_\eta}\right)G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\right]_{\substack{\mathbf{r}=\mathbf{r}'\\ z=0}}, \quad (2.11)$$

where 1,2,3 indicate the three Cartesian indices and $q=(\omega/c)$. The above expression is in fact the force per unit area acting on the interface of the conductor arising from the vacuum fluctuations of the field in the semi-infinite free space $z\leqslant 0$. The two terms in Eq. (2.11) provide the electric and magnetic field contributions, respectively. Note that the integral in Eq. (2.11) is divergent, but the net force per unit area evaluated for a conducting plate turns out to be a finite quantity.

III. ATTRACTIVE CASIMIR FORCE

We calculate in this section the Casimir force between two perfect conducting plates of thickness d located in empty space. To take advantage of the symmetry of the problem the z axis is chosen to be perpendicular to the interfaces with the origin at a distance $a/2$ from each plate as shown in Fig. 1. We label the left and right hand plates as well as the different domains of this geometry by 1 and 2 as well as 1, 2, and 3, respectively, as shown in Fig. 1. In order to use Eq. (2.11) to evaluate the Casimir force acting on the left hand plate we must find the appropriate response function. We need the explicit form of the coordinate space Green function for the case where both the source and observation points $\mathbf{r}$ and $\mathbf{r}'$ are within the gap between the two plates as well as for the case where both $\mathbf{r}$ and $\mathbf{r}'$ are in domain 1.

A. The Green function

The electromagnetic field operators satisfy Maxwell's equations, which in the frequency domain are of the form

$$\nabla\times\hat{\mathbf{E}}^{+}(\mathbf{r},\omega)=i\omega\hat{\mathbf{B}}^{+}(\mathbf{r},\omega), \tag{3.1}$$

$$\nabla\times\hat{\mathbf{B}}^{+}(\mathbf{r},\omega)=-i\omega\mu_0\hat{\mathbf{D}}^{+}(\mathbf{r},\omega)+\mu_0\hat{\mathbf{J}}^{+}(\mathbf{r},\omega), \tag{3.2}$$

where the monochromatic electric field and displacement operators are related by

$$\hat{\mathbf{D}}^{+}(\mathbf{r},\omega)=\epsilon_0\epsilon(\mathbf{r},\omega)\hat{\mathbf{E}}^{+}(\mathbf{r},\omega). \tag{3.3}$$

Combining Eqs. (3.2), (3.3), and (2.8), one can easily show that the spatial dependence of the positive frequency part of the vector potential operator is given by

$$\nabla\times[\nabla\times\hat{\mathbf{A}}^{+}(\mathbf{r},\omega)]-q^2\hat{\mathbf{A}}^{+}(\mathbf{r},\omega)=\frac{1}{\epsilon_0c^2}\hat{\mathbf{J}}^{+}(\mathbf{r},\omega), \tag{3.4}$$

where, as before, $q=(\omega/c)$. The Fourier time transformed Green function tensor is defined in the usual way as

$$\hat{A}^{+}_{\alpha}(\mathbf{r},\omega)=\sum_{\beta}\int d^3\mathbf{r}'G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\hat{J}^{+}_{\beta}(\mathbf{r}',\omega). \tag{3.5}$$

Substitution of Eq. (3.5) into Eq. (3.4) shows that $G_{\mu\nu}(\mathbf{r},\mathbf{r}',\omega)$ satisfies

$$\sum_{\mu}\left(q^2\delta_{\lambda\mu}-\frac{\partial^2}{\partial x_\lambda\partial x_\mu}+\delta_{\lambda\mu}\nabla^2\right)G_{\mu\nu}(\mathbf{r},\mathbf{r}',\omega)=-\frac{1}{\epsilon_0c^2}\delta_{\lambda\nu}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.6}$$

with the appropriate boundary conditions.

The homogeneous nature of the present configuration in the xy plane allows us to convert the Green function differential equations into a set of algebraic equations in the wave vector space $\mathbf{k}_{\|}=(k_x,k_y)$. Unfortunately, the homogeneity along the z axis is broken by the inhomogeneity of the medium in this direction. Therefore, one can conveniently express the components of the response function tensor in terms of their Fourier transform as

$$G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)=\frac{1}{2\pi}\int d^2\mathbf{k}_{\|}G_{\alpha\beta}(\mathbf{k}_{\|},\omega,z,z')e^{i\mathbf{k}_{\|}\cdot(\mathbf{x}_{\|}-\mathbf{x}'_{\|})}. \tag{3.7}$$

This converts the set of partial differential equations (3.6) into a set of ordinary differential equations in variable z.

The symmetry of the problem reduces the number of nonvanishing off-diagonal components of the Green function tensor. To exploit the remaining symmetry in the xy plane, it is sufficient to premultiply and postmultiply the new sets of differential equations by the matrices $O(\mathbf{k}_{\|})$ and $O^{-1}(\mathbf{k}_{\|})$, respectively, where

$$O(\mathbf{k}_{\|})=\frac{1}{k_{\|}}\begin{pmatrix}k_x & k_y & 0\\ -k_y & k_x & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{3.8}$$

It is convenient also to introduce the auxiliary tensor $g_{\alpha\beta}$ as

$$g=OGO^{-1}, \tag{3.9}$$

where the argument $(\mathbf{k}_{\|},\omega,z,z')$ has been omitted for simplicity. A detailed calculation shows that the four functions g_{yx}, g_{yz}, g_{xz}, and g_{zy} all vanish, while the other five components satisfy a set of five differential equations [17]. The two functions g_{xz} and g_{zz} are the solutions of the set of two coupled differential equations of the form

$$\left(\frac{d^2}{dz^2}+q^2\right)g_{xz}-ik_{\|}\frac{d}{dz}g_{zz}=0,$$
$$-ik_{\|}\frac{d}{dz}g_{xz}+(q^2-k_{\|}^2)g_{zz}=-\frac{1}{2\pi\epsilon_0c^2}\delta(z-z'). \tag{3.10}$$

The imposition of the boundary conditions makes it more convenient to find g_{xz} first and then to use Eq. (3.10) to evaluate g_{zz}. Combining the above differential equations, one can easily show that g_{xz} is given by the solution of

$$\left(\frac{d^2}{dz^2}+k^2\right)g_{xz}=-\frac{ik_{\|}}{2\pi\epsilon_0\omega^2}\frac{d}{dz}\delta(z-z'), \tag{3.11}$$

where

$$k=-(q^2-k_{\|}^2)^{1/2}. \tag{3.12}$$

The choice of the sign in Eq. (3.12) ensures that $\mathrm{Im}\,k>0$ when $k_{\|}>q$. The function g_{zz} is obtained by

$$g_{zz}=\frac{ik_{\|}}{k^2}\frac{d}{dz}g_{xz}-\frac{1}{2\pi\epsilon_0c^2k^2}\delta(z-z'). \tag{3.13}$$

A similar treatment shows that the function g_{xx} is determined by

$$\left(\frac{d^2}{dz^2}+k^2\right)g_{xx}=-\frac{k^2}{2\pi\epsilon_0\omega^2}\delta(z-z') \tag{3.14}$$

and g_{zx} can be derived using the explicit form of g_{xx} as

$$g_{zx}=\frac{ik_{\|}}{k^2}\frac{d}{dz}g_{xx}. \tag{3.15}$$

Finally, the yy component of the auxiliary tensor $g_{\mu\nu}$ satisfies

$$\left(\frac{d^2}{dz^2}+k^2\right)g_{yy}=-\frac{1}{2\pi\epsilon_0c^2}\delta(z-z'). \tag{3.16}$$

There are two types of boundary conditions on the functions $g_{\mu\nu}$. The first are the boundary conditions at $z=\pm\infty$. They are easily imposed by assuming outgoing or exponentially decaying waves at infinity, depending on the value of $k_\parallel$ and q. The second type of boundary conditions are those at the plane interfaces of the conductors. These are governed by the boundary conditions on the different components of the electromagnetic fields. It is not difficult to impose the continuity of the tangential components of $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ and the normal components of $\hat{\mathbf{D}}$ and $\hat{\mathbf{B}}$ to subsequently find the boundary conditions on the components of $G_{\mu\nu}$ and $g_{\mu\nu}$. The vanishing of the tangential electric field operator on the conductors entails that the functions of main concern, g_{xz}, g_{xx}, and g_{yy} should necessarily vanish when the observation point $\mathbf{r}$ is on the interfaces of the conducting plates.

Let us begin with calculating g_{xz} in domain 1. The general solution of Eq. (3.11) consists of a complementary solution and a particular solution of the form

$$g_{xz}=-\frac{ik_\parallel}{4\pi\epsilon_0\omega^2}e^{-ik|z-z'|}\mathrm{sgn}(z-z')+Re^{ikz}. \tag{3.17}$$

The coefficient R is determined by the imposition $g_{xz}|_{z=-(a/2)-d}=0$. It is straightforward, upon evaluating R and using Eq. (3.13), to show that the explicit forms of g_{xz} and g_{zz} are

$$g_{xz}(\mathbf{k}_\parallel,\omega,z,z')=-\frac{ik_\parallel}{4\pi\epsilon_0\omega^2}(e^{-ik|z-z'|}\mathrm{sgn}(z-z') -e^{ik[(z+z')+a+2d]}) \tag{3.18}$$

and

$$g_{zz}(\mathbf{k}_\parallel,\omega,z,z')=-\frac{ik_\parallel^2}{4\pi\epsilon_0\omega^2 k}(e^{-ik|z-z'|}+e^{ik[(z+z')+a+2d]}) -\frac{1}{2\pi\epsilon_0\omega^2}\delta(z-z'). \tag{3.19}$$

A similar calculation must be carried out in domain 2. In this case the particular solution of Eq. (3.11) has the same form as in Eq. (3.17), while the complementary solution consists of both the rightwards and leftwards propagating waves. The general solution of Eq. (3.11) is then

$$g_{xz}=-\frac{ik_\parallel}{4\pi\epsilon_0\omega^2}e^{-ik|z-z'|}\mathrm{sgn}(z-z')+C_Le^{ikz}+C_Re^{-ikz}. \tag{3.20}$$

The coefficients C_L and C_R are obtained by the boundary conditions $g_{xz}|_{z=-(a/2)}=0$ and $g_{xz}|_{z=(a/2)}=0$. It is not difficult to show that g_{xz} and g_{zz} take on the forms

$$\begin{aligned}g_{xz}(\mathbf{k}_\parallel,\omega,z,z')=&-\frac{ik_\parallel}{4\pi\epsilon_0\omega^2}\{e^{-ik|z-z'|}\mathrm{sgn}(z-z')\\&+(1-e^{-2ika})^{-1}[-e^{ik[(z+z')-a]}\\&-e^{ik[(z-z')-2a]}+e^{-ik[(z-z')+2a]}\\&+e^{-ik[(z+z')+a]}]\}\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}g_{zz}(\mathbf{k}_\parallel,\omega,z,z')=&-\frac{ik_\parallel^2}{4\pi\epsilon_0\omega^2 k}\{e^{-ik|z-z'|}\\&+(1-e^{-2ika})^{-1}[e^{ik[(z+z')-a]}\\&+e^{ik[(z-z')-2a]}+e^{-ik[(z-z')+2a]}\\&+e^{-ik[(z+z')+a]}]\}-\frac{1}{2\pi\epsilon_0\omega^2}\delta(z-z'),\end{aligned} \tag{3.22}$$

where Eq. (3.13) has been used to evaluate Eq. (3.22).

The coordinate space Green function needed for substitution into Eq. (2.11) is obtained with the help of Eq. (3.7) where due account must be taken of the rotation (3.9). Let us begin with the evaluation of $G_{zz}(\mathbf{r},\mathbf{r}',\omega)$. Recalling the fact that Eq. (3.9) is a rotation about the k_z axis in the wave vector space, it is clear that

$$G_{zz}(\mathbf{k}_\parallel,\omega,z,z')=g_{zz}(\mathbf{k}_\parallel,\omega,z,z'). \tag{3.23}$$

The latter statement may easily be checked by substituting Eq. (3.8) and its inverse into Eq. (3.9). Therefore, the zz component of the coordinate space Green function corresponding to Eq. (3.19) is

$$\begin{aligned}G_{zz}&(\mathbf{r},\mathbf{r}',\omega)\\&=-\frac{i}{4\pi\epsilon_0\omega^2}\left\{\int_0^{+\infty}dk_\parallel\frac{k_\parallel^3}{k}e^{-ik|z-z'|}J_0(k_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|)\right.\\&\quad\left.+\int_0^{+\infty}dk_\parallel\frac{k_\parallel^3}{k}e^{-ik|(z+z')+a+2d|}J_0(k_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|)\right\}\\&\quad-\frac{1}{\epsilon_0\omega^2}\delta(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{3.24}$$

where the integral representation of J_0, the zero order Bessel function of the first kind of the form

$$J_0(k_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|)=\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{ik_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|\cos\theta} \tag{3.25}$$

has been used for the integration over the azimuth angle in wave vector space. The integrands in the first and second terms have the same form and the integration over $k_\parallel$ can be performed easily. The details are omitted here, but the evaluation of this integral is given in the Appendix. With the explicit result provided in the Appendix, expression (3.24) can be rewritten as

$$G_{zz}(\mathbf{r},\mathbf{r}',\omega)=\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_0',\omega)+\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_1',\omega t) -\frac{1}{3\epsilon_0\omega^2}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.26}$$

where

$$\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_i',\omega)=\frac{q^3}{4\pi\epsilon_0\omega^2}\left\{\left(\frac{1}{(qr_{\rm rel})}+\frac{i}{(qr_{\rm rel})^2}-\frac{1}{(qr_{\rm rel})^3}\right)-\left(\frac{1}{(qr_{\rm rel})}+\frac{3i}{(qr_{\rm rel})^2}-\frac{3}{(qr_{\rm rel})^3}\right)\frac{(z_{\rm rel})^2}{(r_{\rm rel})^2}\right\}e^{iqr_{\rm rel}}, \tag{3.27}$$

in which

$$\mathbf{r}_{\rm rel}=\mathbf{r}-\mathbf{r}_i', \quad i=0,1,\ldots\,. \tag{3.28}$$

The vectors $\mathbf{r}_i'$ are defined as

$$\mathbf{r}_0'=\mathbf{r}', \tag{3.29}$$

$$\mathbf{r}_1'=x'\hat{i}+y'\hat{j}-(z'+a+2d)\hat{k}.$$

The zz component of the coordinate space Green function corresponding to Eq. (3.22) is calculated by employing a similar approach. Substitution of Eq. (3.22) into Eq. (3.7) yields

$$G_{zz}(\mathbf{r},\mathbf{r}',\omega)=-\frac{i}{4\pi\epsilon_0\omega^2}\int_0^{+\infty}dk_\parallel\frac{k_\parallel^3}{k}F(k_\parallel,\omega,z,z')\times J_0(k_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|)-\frac{1}{\epsilon_0\omega^2}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.30}$$

in which

$$F(k_\parallel,\omega,z,z')=e^{-ik|z-z'|}+\sum_{m=0}^{\infty}\{e^{-ik|(2m+1)a-(z+z')|}+e^{-ik|2(m+1)a-(z-z')|}+e^{-ik|2(m+1)a+(z-z')|}+e^{-ik|(2m+1)a+(z+z')|}\}, \tag{3.31}$$

where the prefactor of the square brackets in Eq. (3.22) has been expanded for later calculations and Eq. (3.25) has been used for integrating over θ. The integrands of all the five terms appearing in Eq. (3.30) have the same form and the integration over $k_\parallel$ can be performed using the integration given in the Appendix. The result is summarized as

$$G_{zz}(\mathbf{r},\mathbf{r}',\omega)=\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_0',\omega)+\sum_{m=0}^{\infty}\{\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_2',\omega)+\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_3',\omega)+\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_4',\omega)+\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_5',\omega)\}-\frac{1}{3\epsilon_0\omega^2}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.32}$$

where $\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_i',\omega)$ is given by Eq. (3.27) and $\mathbf{r}_i'$($i=2,3,\ldots$) have the same x and y coordinates as $\mathbf{r}'$ and their z coordinates are

$$z_2'=[-z'+(2m+1)a], \quad z_3'=[z'+2(m+1)a], \tag{3.33}$$

$$z_4'=[z'-2(m+1)a], \quad z_5'=[-z'-(2m+1)a].$$

The evaluations of the other components are similar and the details are omitted here for the sake of brevity. The explicit form of the coordinate space Green function with both $\mathbf{r}$ and $\mathbf{r}'$ in domain 1 can be written as

$$G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)=\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_0',\omega)\pm\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_1',\omega)-\frac{1}{3\epsilon_0\omega^2}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.34}$$

where the upper sign holds for $\alpha\beta=xz$, yz, zz and the lower sign stands for the other components. The tensor $\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_i',\omega)$ is given by

$$\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_i',\omega)=\frac{q^3}{4\pi\epsilon_0\omega^2}\left\{\left(\frac{1}{(qr_{\rm rel})}+\frac{i}{(qr_{\rm rel})^2}-\frac{1}{(qr_{\rm rel})^3}\right)\boldsymbol{\delta}_{\alpha\beta}-\left(\frac{1}{(qr_{\rm rel})}+\frac{3i}{(qr_{\rm rel})^2}-\frac{3}{(qr_{\rm rel})^3}\right)\frac{(\mathbf{r}_{\rm rel}\mathbf{r}_{\rm rel})_{\alpha\beta}}{(r_{\rm rel})^2}\right\}e^{iqr_{\rm rel}}, \tag{3.35}$$

where $\mathbf{r}_{\rm rel}\mathbf{r}_{\rm rel}$ is the normal Cartesian dyadic.

The structure of Eq. (3.34) is typical of a semi-infinite free space Green function. The surface part, the second term, corresponds to the communication between the points $\mathbf{r}$ and $\mathbf{r}'$ via reflection in the perfect conducting interface. The first term together with the Dirac δ function term, the bulk part, correspond to the direct communication between the two points. The latter term resembles the free space Green function with both the source and observation points $\mathbf{r}$ and $\mathbf{r}'$ positioned inside the region in which $G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)$ has been evaluated. The former term also has the same form as the free space response function, but its source point $\mathbf{r}_1'$ lies outside the region in question and thus the singular Dirac δ function term has disappeared. This term corresponds to the so-called image source.

The coordinate space response function with $\mathbf{r}$ and $\mathbf{r}'$ both in domain 2 is

$$G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)=\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_0',\omega)+\sum_{m=0}^{\infty}\{\pm\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_2',\omega)+\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_3',\omega)+\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_4',\omega)\pm\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_5',\omega)\}-\frac{1}{3\epsilon_0\omega^2}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.36}$$

with the same sign convention as Eq. (3.34).

The structure of Eq. (3.36) is typical of a cavity geometry made up of two perfectly conducting walls. The first term together with the Dirac δ function term correspond to the direct communication between the two points $\mathbf{r}$ and $\mathbf{r}'$. The surface parts, the other four terms, correspond to the communication between the two points $\mathbf{r}$ and $\mathbf{r}'$ via a series of infinite reflections in the cavity walls. Each of these terms has individually the same form as the free space response function, but their source points lie outside the cavity and thus their singular Dirac δ function terms have disappeared. The latter terms are associated with the infinite image sources produced in the cavity walls.

B. The Casimir force

The Casimir force acting on the left hand conducting plate is evaluated easily by taking into account the radiation pressures on both sides of this plate. Using Eq. (2.11) the net force per unit area on plate 1 is

$$F=-\frac{\epsilon_0\hbar c^3}{2\pi}\mathrm{Im}\int_0^{+\infty}dq\Bigg\{\Bigg[\Bigg(q^2\delta_{3\alpha}\delta_{3\beta}+2\epsilon_{1\gamma\alpha}\epsilon_{1\eta\beta}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x'_\eta}\Bigg)G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\Bigg]_{\substack{\mathbf{r}=\mathbf{r}'\\ z=-(a/2)-d}}-\Bigg[\Bigg(q^2\delta_{3\alpha}\delta_{3\beta}+2\epsilon_{1\gamma\alpha}\epsilon_{1\eta\beta}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x'_\eta}\Bigg)\times G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\Bigg]_{\substack{\mathbf{r}=\mathbf{r}'\\ z=-(a/2)}}\Bigg\}. \tag{3.37}$$

It is understood that a positive value of Eq. (3.37) indicates an attractive force between the two plates. The different components of the coordinate space Green function needed for substitution into the first and second terms of Eq. (3.37) are given by Eqs. (3.34) and (3.36), respectively. The bulk part contributions, which are identical in both Eqs. (3.34) and (3.36), cancel each other in Eq. (3.37). Furthermore, the fifth term in Eq. (3.36) with $m=0$ on the interior interface of plate 1 is identical with the reflection term in Eq. (3.34) on the exterior interface of the plate. These two terms cancel one another as well. Changing the subscript m in the fifth term as $m\to m+1$, along with taking into account that $\partial/\partial y=-\partial/\partial y'$ as well as $\partial/\partial z=\partial/\partial z'$ in the second and fifth terms of Eq. (3.36) while $\partial/\partial z=-\partial/\partial z'$ in the third and fourth terms, after some algebra one can easily show that Eq. (3.37) can be rewritten as

$$F=\frac{2\epsilon_0\hbar c^3}{\pi}\mathrm{Im}\sum_{m=0}^{\infty}\int_0^{+\infty}dq\Bigg\{\Bigg(q^2+2\frac{\partial^2}{\partial y^2}\Bigg)\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_2',\omega)+2\frac{\partial^2}{\partial z^2}\mathcal{G}_{yy}(\mathbf{r},\mathbf{r}_2',\omega)-4\frac{\partial^2}{\partial y\partial z}\mathcal{G}_{yz}(\mathbf{r},\mathbf{r}_2',\omega)\Bigg\}_{\substack{\mathbf{r}=\mathbf{r}'\\ z=-(a/2)}}. \tag{3.38}$$

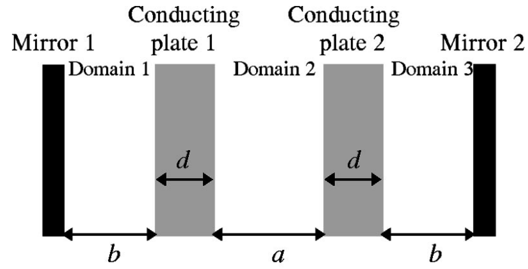

FIG. 2. Representation of the geometry of two conducting plates located in the Fabry-Perot cavity.

Note that in writing Eq. (3.38) the equal contribution of the zy and yz components of the response function allows us to avoid the explicit use of $\mathcal{G}_{yz}(\mathbf{r},\mathbf{r}_2',\omega)$. Furthermore, we have taken into account that the value of $\mathbf{r}_i'(i=2,3,\dots)$ on the interior interface of the conducting plate, that is, for $z=z'=-(a/2)$ and $\mathbf{x}_\parallel=\mathbf{x}_\parallel'$, are the same. Therefore, the four remaining terms have equal contributions in producing the Casimir force. The latter cancellation removes the need of any regularization procedure for the present formalism and, on the other hand, demonstrates how the asymmetric nature of the surroundings gives rise to the Casimir force on a macroscopic object. Using the explicit form of $\mathcal{G}_{zz}(\mathbf{r},\mathbf{r}_2',\omega)$, $\mathcal{G}_{yy}(\mathbf{r},\mathbf{r}_2',\omega)$, and $\mathcal{G}_{yz}(\mathbf{r},\mathbf{r}_2',\omega)$ in Eq. (3.38) one can show that

$$F=\frac{3\hbar c}{8\pi^2a^4}\sum_{m=0}^{\infty}\frac{1}{(m+1)^4}=\frac{\hbar c\pi^2}{240a^4}, \tag{3.39}$$

where in the last step the value of the ζ function has been used [18]. It is to be noted that the magnetic field contribution in Eq. (3.39) is twice as much as the electric field contribution. In other words, the nonvanishing field components have the same role in producing the final result.

IV. REPULSIVE CASIMIR FORCE

The repulsive Casimir force provides evidence for the difference between this phenomenon and the van der Waals–like forces. In the present formalism we can easily show that the sign of the Casimir force depends on the geometry of the boundaries.

To begin with, let us assume that the latter configuration is located inside a Fabry-Perot cavity made up of two perfect reflecting mirrors positioned at $z=\pm[(a/2)+d+b]$, see Fig. 2. The left and right hand plates as well as the different domains of this configuration are labeled, as before, by 1 and 2 as well as 1, 2, and 3, respectively. We need the explicit forms of the coordinate space Green function in domains 1 and 2 for the calculation of the Casimir force. The response function in domain 2 is given by Eq. (3.36). One may follow the same procedure and start with Eq. (3.20) along with the appropriate boundary conditions on the interfaces at $z=-a-b-d$ and $z=-a-d$ to find the explicit form of g_{xz} in domain 1. The coordinate space Green function is then obtained using Eqs. (3.9) and (3.7). To find the response function it is more convenient to use the following shortcut which avoids repeating a similar calculation. We may obtain the required Green's function by changing $a\to b$ along with

$z\to z+(a/2)+(b/2)+d$ and $z'\to z'+(a/2)+(b/2)+d$ in Eq. (3.36). Therefore, the explicit form of this response function is

$$G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)=\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_0',\omega)+\sum_{m=0}^{\infty}\{\pm\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_6',\omega)+\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_7',\omega)+\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_8',\omega)\pm\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_9',\omega)\}-\frac{1}{3\epsilon_0\omega^2}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'),\tag{4.1}$$

where $\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}_i',\omega)$ is given by Eq. (3.27) and $\mathbf{r}_i'(i=6,7,\dots)$ have the same x and y coordinates as $\mathbf{r}'$ and their z coordinates are

$$z_6'=-z'+2mb-a-2d,\quad z_7'=z'+2(m+1)b,$$
$$z_8'=z'-2(m+1)b,\quad z_9'=-z'-2(m+1)b-a-2d\tag{4.2}$$

Equation (4.1) also has the typical structure of the cavity geometry.

Having the coordinate space Green functions (3.36) and (4.1), and using Eq. (3.37), the evaluation of the Casimir force acting on the left hand plate is straightforward. It is clear that the positive (negative) value of Eq. (3.37) indicates an attractive (repulsive) force between the two plates. The bulk part contributions, which are identical in Eqs. (3.36) and (4.1), cancel each other in Eq. (3.37). Furthermore, the fifth term of Eq. (3.36) on the interior interface of plate 1 for $m=0$ is the same as the second term of Eq. (4.1) on the exterior interface of the plate with $m=0$. These two terms cancel one another as well. Changing the subscript m in the latter terms according to $m\to m+1$ and taking the similar structure of $G_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)$ on the interior and exterior interfaces of the left hand plate into account, one can easily show that

$$F=\frac{3\hbar c}{8\pi^2}\left(\frac{1}{a^4}-\frac{1}{b^4}\right)\sum_{m=0}^{\infty}\frac{1}{(m+1)^4}=\frac{\hbar c\pi^2}{240}\left(\frac{1}{a^4}-\frac{1}{b^4}\right),\tag{4.3}$$

where the value of the ζ function has been used. Therefore, the Casimir force between the two conducting plates is attractive when $b>a$ and repulsive when $b<a$.

V. CONCLUSIONS

The results derived in this paper amplify and extend the previously developed [10] Green function method for calculating the Casimir force. In Ref. [10] the method was applied to a one dimensional case, while the Casimir force calculation, in principle, does need the extension of the formalism to three dimensions. The final result of the attractive Casimir force between two conducting plates Eq. (3.39) is in complete agreement with those appearing in the literature [4] evaluated by the method of mode summation together with introducing an appropriate cutoff frequency.

Another field theoretical approach used for the evaluation of the Casimir force is based on the modified definition of the energy momentum tensor as the difference between that in the constrained field configuration and the one corresponding to the unconstrained field. The modified tensor is then related to the field propagator [4].

In this paper we have used the Maxwell stress tensor for the evaluation of the vacuum radiation pressure on a conducting surface. The relation of this expression with the imaginary part of the response function has been obtained using the fluctuation dissipation theorem and Kubo's formula (2.11). The attractive Casimir force between two conducting plates (3.39) is then calculated as the net vacuum radiation pressure acting on each plate. In a straightforward way the repulsive Casimir force has been calculated for two conducting plates located inside a Fabry-Perot cavity (4.3). For the sake of simplicity we have used the usual perfect conductor approximation with a frequency independent reflection coefficient. It is noteworthy that as in the one dimensional case this approximation should be improved, as in Ref. [10], to remedy the inconsistency of the theory.

The present formalism can be extended to the calculation of the Casimir force between two dielectric slabs exhibiting dissipation. The work on this problem is under way.

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APPENDIX

In Secs. III and IV the form of the following integral was required for the calculation of the coordinate space Green function:

$$L(\mathbf{r},\mathbf{r}',\omega)=\int_0^{+\infty}dk_\parallel\frac{k_\parallel^3}{k}e^{-ik|z-z'|}J_0(k_\parallel|\mathbf{x}_\parallel-\mathbf{x}_\parallel'|),\tag{A1}$$

where k is related to $k_\parallel$ by Eq. (3.9) and J_0 is the zero order Bessel function of the first kind. To simplify the later calculations let us assume $Z=z-z'$ and $\mathbf{X}_\parallel=\mathbf{x}_\parallel-\mathbf{x}_\parallel'$. Using the well-known recurrence relation

$$J_n(x)=\frac{(n+1)}{x}J_{n+1}(x)+J_{n+1}'(x)\tag{A2}$$

among the Bessel functions of the first kind in which the prime denotes derivative with respect to the argument, one can show that

$$k_\parallel J_0(k_\parallel X_\parallel)=\left(\frac{1}{X_\parallel}+\frac{d}{dX_\parallel}\right)J_1(k_\parallel X_\parallel).\tag{A3}$$

The successive applications of the latter recurrence relation allow one to write Eq. (A1) in the form

$$L(\mathbf{r},\mathbf{r}',\omega)=\left(\frac{1}{X_\parallel}+\frac{d}{dX_\parallel}\right)\left(\frac{2}{X_\parallel}+\frac{d}{dX_\parallel}\right)\left(\frac{3}{X_\parallel}+\frac{d}{dX_\parallel}\right)H(\mathbf{r},\mathbf{r}',\omega),\tag{A4}$$

where

$$H(\mathbf{r},\mathbf{r}',\omega)=\int_0^{+\infty}dk_\parallel \frac{1}{k}e^{-ik|Z|}J_3(k_\parallel X_\parallel). \qquad \text{(A5)}$$

This is a well-known integral and its value is [18]

$$H(\mathbf{r},\mathbf{r}',\omega)=iI_{3/2}\left(-\frac{iq}{2}(R-|Z|)\right)K_{3/2}\left(-\frac{iq}{2}(R+|Z|)\right), \qquad \text{(A6)}$$

where $I_{3/2}$ and $K_{3/2}$ are the modified Bessel functions of the first and second kind, respectively. Using the explicit forms of the latter Bessel functions,

$$I_{3/2}(x)=i\sqrt{\frac{2\pi}{x}}\left(\frac{\sin(ix)}{x^2}+\frac{\cos(ix)}{ix}\right), \qquad \text{(A7)}$$

$$K_{3/2}(x)=\sqrt{\frac{\pi x}{2}}e^{-x}\left(\frac{1}{x}+\frac{1}{x^2}\right),$$

after some algebra, we find that

$$H(\mathbf{r},\mathbf{r}',\omega)=-\frac{1}{qX_\parallel}\left\{\left(1+\frac{4iR}{qX_\parallel^2}-\frac{4}{q^2X_\parallel^2}\right)e^{iqR}+\left(1-\frac{4i|Z|}{qX_\parallel^2}+\frac{4}{q^2X_\parallel^2}\right)e^{iq|Z|}\right\}, \qquad \text{(A8)}$$

where $R=(X_\parallel^2+Z^2)^{1/2}$. Substituting Eq. (A8) into Eq. (A4) yields

$$L(\mathbf{r},\mathbf{r}',\omega)=iq^3\left\{\left(\frac{1}{(qR)}+\frac{i}{(qR)^2}-\frac{1}{(qR)^3}\right)+\left(-\frac{1}{(qR)}-\frac{3i}{(qR)^2}+\frac{3}{(qR)^3}\right)\frac{(Z)^2}{(R)^2}\right\}e^{iqR}-\frac{1}{3\epsilon_0\omega^2}\delta(R). \qquad \text{(A9)}$$

Note that the Dirac δ function term disappears if we require $R\neq 0$.

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