Semiclassical theory of emission spectra of optical microcavities

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We developed a semiclassical theory for emission spectra of optical microcavities. The spontaneous emission is taken into consideration by adding a spontaneous-emission current term into the Maxwell's equations. The spontaneous-emission current is found to be equal to the element of the quantum current operator between the initial and final states of the spontaneous-emission transition. We showed that the optical-field distribution of the light emitted from optical cavities can be expressed as a superposition of eigenmodes, and the eigenvalue equation for eigenmodes is established. A detailed expression for emission spectra is also obtained. Numerical results for the emission spectra carried out for a microdisk laser match well with experimental data. By applying this theory in the calculation of the spectral linewidth of conventional lasers, we found the modified Schawlow-Townes spectral linewidth formula for gas lasers, and obtained an excellent agreement between the theoretical and experimental results in the case of semiconductor laser diodes. [S1050-2947(99)04109-8]

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I. INTRODUCTION

The emission spectra is an important topic in studies of optical microcavities. It is known that the spontaneous emission spectrum of optically active media can be strongly altered in optical microcavities, and a large fraction of spontaneous emission can be coupled into the lasing mode [1]. These properties of optical microcavities differ significantly from that of conventional laser cavities, thus the theory of spontaneous emission spectrum and lasing linewidth for conventional lasers may not be applicable for optical microcavities. To the best of our knowledge, there is not a theory for the emission spectrum of optical cavities that take into consideration both the effect of the cavity and the effect of the spontaneous emission at the same time. Theory for conventional laser cavities is frequently used in studies of emission spectra of optical microcavities [2,3].

In this paper, we introduce a semiclassical theory for the emission spectra of optical microcavities. In this theory the spontaneous emission is explicitly considered. From the point of view of electrodynamics, any optical emission is generated by a certain current at optical frequencies. Therefore, to include the spontaneous emission effect in this theory, we add a spontaneous emission current term into Maxwell's equations. This spontaneous emission current term is determined by using the quantum-transition theory.

An expression for the spontaneous emission current term is derived in Sec. II. We solve in Sec. III the wave equation with a spontaneous emission current term by expanding solutions in terms of eigenmodes. The formula for emission spectra of optical microcavities is established in Sec. IV. Numerical results for emission spectra of microdisk lasers and spectral linewidth of conventional lasers are carried out and compared with experimental data in Sec. V.

II. WAVE EQUATION WITH A SPONTANEOUS EMISSION CURRENT TERM

An optical cavity can be considered being formed from several homogeneous parts of continuous dielectric media without free electric charges. In each homogeneous dielectric medium the electromagnetic field satisfies the following Maxwell's equations:

$$\mathrm{rot}\,\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}+\vec{j},\quad \mathrm{rot}\,\vec{E}=-\mu_0\frac{\partial\vec{H}}{\partial t},$$

$$\mathrm{div}\,\vec{H}=0,\quad \mathrm{div}(\varepsilon\vec{E})=0, \tag{1}$$

in which μ_0 is the magnetic permeability of the vacuum and ε is the dielectric permittivity. At interfaces of different parts, the magnetic field $\vec{H}$, the tangential component of the electric field $\vec{E}$, and the normal component of the electric displacement vector $\varepsilon\vec{E}$ must be continuous.

It is convenient to present a real electromagnetic field in terms of its monochromatic components:

$$\vec{E}(t)=\int_0^\infty\vec{E}_\omega e^{-i\omega t}d\omega+\text{c.c.},\quad \vec{H}(t)=\int_0^\infty\vec{H}_\omega e^{-i\omega t}d\omega+\text{c.c.} \tag{2}$$

By applying this transformation, the first two equations in Eq. (1) become as

$$\mathrm{rot}\,\vec{H}_\omega=-i\omega\varepsilon\vec{E}_\omega+\vec{j}_\omega,\quad \mathrm{rot}\,\vec{E}_\omega=i\omega\mu_0\vec{H}_\omega. \tag{3}$$

In a dielectric medium with optical gain or absorption, the current density $\vec{j}_\omega$ should be proportional to the electric-field intensity $\vec{E}_\omega$. But, however, the existence of the spontaneous emission implies that there must also exist a free current source, which is independent of the electric-field intensity. We call this free current the spontaneous emission current and use $\vec{j}_\omega^{sp}$ to note it. The optical gain or absorption can be taken into consideration in Eqs. (3) by means of a complex refractive index.

Solutions of Eqs. (3) can be expressed in terms of the vector potential $\vec{A}_\omega$ and the scalar potential ϕ_ω as follows:

$$\vec{H}_{\omega}=\text{rot}\,\vec{A}_{\omega}, \quad \vec{E}_{\omega}=i\omega\mu_0\vec{A}_{\omega}-\text{grad}\,\phi_{\omega}. \tag{4}$$

The vector potential $\vec{A}_{\omega}$ satisfies the following wave equation:

$$\Delta\vec{A}_{\omega}+n^2k_0^2\vec{A}_{\omega}=-\vec{j}^{sp}_{\omega}, \tag{5}$$

where n is the complex refractive index of the medium, $k_0=\omega/c$, and c is the speed of light in the free space. There is a relation between the optical gain coefficient and the refractive index n:

$$g_{\omega}=-k_0\,\text{Im}(n^2). \tag{6}$$

The scalar potential ϕ_{ω} is related to the vector potential $\vec{A}_{\omega}$ by the following expression:

$$\phi_{\omega}=-i\frac{\mu_0 c}{n^2k_0}\text{div}\,\vec{A}_{\omega} \tag{7}$$

To solve Eq. (5) for $\vec{A}_{\omega}$, we need an explicit expression for the spontaneous emission current $\vec{j}^{sp}_{\omega}$. For obtaining such an expression, we consider a small optical cavity with volume V much smaller than the cube of wavelength. In this circumstance, we can write $\vec{j}^{sp}_{\omega}$ as $\vec{\mathcal{J}}_{\omega}\delta(\vec{r})$, where $\delta(\vec{r})$ is the Dirac's delta function, and

$$\vec{\mathcal{J}}_{\omega}=\int_V\vec{j}^{sp}_{\omega}dv. \tag{8}$$

We also have for this case $n^2=1$. The solution of Eq. (5) for this cavity is

$$\vec{A}_{\omega}=-\frac{\vec{\mathcal{J}}_{\omega}}{4\pi r}e^{ik_0 r}. \tag{9}$$

By using expression (9), we can calculate the total optical flux of spontaneous emission. We consider first a system with discrete spontaneous emission current spectrum:

$$\vec{\mathcal{J}}_{\omega}=\sum_m \vec{J}^m\delta(\omega-\omega^m). \tag{10}$$

The total optical flux at the circular frequency ω^m is then

$$P(\omega^m)=\frac{\mu_0(\omega^m)^2}{3\pi c}|\vec{J}^m|^2. \tag{11}$$

On the other hand, the total optical flux of spontaneous emission can also be obtained as the product of the spontaneous emission transition rate with the photon energy. By applying quantum-mechanic theory [4], we obtain

$$P(\omega^m)=\frac{\mu_0(\omega^m)^2}{3\pi c}|\langle f|\vec{j}|i\rangle|^2, \tag{12}$$

in which $\vec{j}$ is the electric current-density operator, and $|i\rangle$, $|f\rangle$ are initial and final quantum states of the spontaneous emission transition, respectively. The difference between the energy levels of the states $|i\rangle$ and $|f\rangle$ must be equal to $\omega^m\hbar$.

By comparing expressions (11) and (12), and neglecting an arbitrary phase factor, we obtain

$$\vec{J}^m=\int_V\psi_f^*\vec{j}\psi_i dv, \tag{13}$$

where ψ_i and ψ_f are time-independent wave functions of the quantum states $|i\rangle$ and $|fm\rangle$, respectively.

By substituting expression (13) into relation (10), we obtain

$$\vec{\mathcal{J}}_{\omega}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\sum_f\int_V\psi_f^*(t)\vec{j}\psi_i(t)dv, \tag{14}$$

in which $\psi_i(t)$ and $\psi_f(t)$ are time-dependent wave functions of the quantum state i and f, respectively. The summation is taken over all possible quantum states $|f\rangle$.

Expression (14) is obtained for systems with discrete spectra of a spontaneous emission current. Because a continuous spectrum can be considered as a discrete spectrum with infinitely closed spectral lines, expression (14) is also valid for systems with continuous spectra.

Because of the small volume of the considered optical cavity, from relation (14), we have

$$\vec{j}^{sp}_{\omega}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\sum_f\psi_f^*(t)\vec{j}\psi_i(t). \tag{15}$$

III. EIGENMODES OF OPTICAL CAVITIES

The wave equation, Eq. (5), can be solved by expanding the vector potential $\vec{A}_{\omega}$ into a linear combination of the eigenmodes of the optical cavity $\vec{u}^l_{\omega}$ that satisfies the following eigenvalue equation (see Appendix A):

$$(\Delta+n^2k_0^2)\vec{u}^l_{\omega}=-\rho n^2\Lambda^l_{\omega}\vec{u}^l_{\omega} \tag{16}$$

and the orthogonality condition:

$$k_0^3\int n^2\rho(\vec{u}^l_{\omega}\cdot\vec{u}^{l'}_{\omega})dv=\delta_{ll'}. \tag{17}$$

Λ^l_{ω} in Eq. (16) is the eigenvalue and $\rho(\vec{r})$ is a step function:

$$\rho(\vec{r})=\begin{cases}1 & \text{if } n(\vec{r})\neq 1 \quad \text{(in the cavity)}\\ 0 & \text{if } n(\vec{r})=1 \quad \text{(in free space)}.\end{cases} \tag{18}$$

We may write

$$\vec{A}_{\omega}=\sum_l\frac{a^l_{\omega}}{\Lambda^l_{\omega}}\vec{u}^l_{\omega}. \tag{19}$$

By substituting expression (19) into Eq. (5) and applying orthogonality relation (17) for $\vec{u}^l_{\omega}$ we find

$$a^l_{\omega}=k_0^3\int(\vec{j}^{sp}_{\omega}\cdot\vec{u}^l_{\omega})dv. \tag{20}$$

The eigenvalues Λ_ω^l are functions of the circle frequency ω. This function is determined by a certain algebraic equation derived from Eq. (16). We write this equation formally as

$$F(\Lambda_\omega^l,\omega,l)=0. \tag{21}$$

In the complex plan $\tilde{\omega}$, the equation

$$F(0,\tilde{\omega}_l,l)=0 \tag{22}$$

admits one solution $\tilde{\omega}_l$ for each mode l. For every $\tilde{\omega}_l$ obtained from Eq. (22), the eigenvalue $\Lambda_{\tilde{\omega}}^l$ determined by Eq. (21) is equal to zero. Let $\tilde{\omega}_l=\omega_l+i\Gamma_l$; then we may write

$$\Lambda_\omega^l=(\omega-\omega_l-i\Gamma_l)\,\eta_l\,. \tag{23}$$

We call ω_l the resonant frequency of the eigenmode l. η_l in the above expression is a function of ω. However, if we are concerned about only a limit wavelength range, we may take this function as a constant:

$$\eta_l=\left.\frac{d\Lambda_\omega^l}{d\omega}\right|_{\omega=\tilde{\omega}}. \tag{24}$$

By using relation (23), we now have

$$\vec{A}_\omega=\sum_l \frac{a_\omega^l}{\eta_l}\,\frac{\vec{u}_\omega^l}{\omega-\omega_l-i\Gamma_l}. \tag{25}$$

IV. EMISSION SPECTRA

The total output optical power from an optical cavity can be calculated by taking the average value, for a long time interval, of the surface integration of the Poynting's vector over a closed surface that contains the cavity. The surface integration can be transformed into a integration of the divergence of the Poynting's vector over the volume of the cavity. After performing this calculation, we obtain

$$P_{out}=\int_0^\infty P(\omega)\,d\omega, \tag{26}$$

where

$$P(\omega)=\lim_{T\to\infty}4\pi T^{-1}\int \mathrm{Re}\left(\frac{in^2k_0}{\mu_0 c}\vec{E}_\omega^*\cdot\vec{E}_{\omega,T}-\vec{E}_{\omega,T}\cdot\vec{j}_\omega^{sp*}\right)dv \tag{27}$$

and

$$\vec{E}_{\omega,T}=\frac{1}{2\pi}\int_{-T/2}^{T/2}\vec{E}(t)e^{i\omega t}dt. \tag{28}$$

The electric field $\vec{E}(t)$ can be calculated from the vector potential $\vec{A}_\omega$. By applying expression (25) for the vector potential, we find that the output optical power spectrum of an optical cavity is an overlap of contributions from all eigenmodes:

$$P(\omega)=\sum_l P_l(\omega), \tag{29}$$

with the power spectrum of an eigenmode given by the following expression (see Appendix B):

$$P_l(\omega)=\frac{\hbar\omega}{\pi}\left\{\frac{k_0^2}{|\eta_l|^2}\,\frac{g_\omega^{ll}(g_\omega^{ll}+\alpha_\omega^{ll})}{(\omega-\omega_l)^2+\Gamma_l^2}+\mathrm{Im}\left[\frac{k_0(\overline{g_\omega^l}+\overline{\alpha_\omega^l})}{\eta_l(\omega-\omega_l-i\Gamma_l)}\right]\right.$$
$$+\mathrm{Re}\left[\frac{2k_0^2}{\eta_l(\omega-\omega_l-i\Gamma_l)}\right.$$
$$\left.\left.\times\sum_{l'\neq l}\frac{g_\omega^{l'l}}{\eta_{l'}^*}\,\frac{g_\omega^{l'l}+\alpha_\omega^{l'l}}{\omega_l-\omega_{l'}+i(\Gamma_l+\Gamma_{l'})}\right]\right\}, \tag{30}$$

where

$$g_\omega^{ll'}=k_0^3\int g_\omega(\vec{u}_\omega^{l*}\cdot\vec{u}_\omega^{l'})dv,\quad \alpha_\omega^{ll'}=k_0^3\int \alpha_\omega(\vec{u}_\omega^{l*}\cdot\vec{u}_\omega^{l'})dv, \tag{31}$$

and

$$\overline{g_\omega^l}=k_0^3\int g_\omega(\vec{u}_\omega^l\cdot\vec{u}_\omega^l)dv,\quad \overline{\alpha_\omega^l}=k_0^3\int \alpha_\omega(\vec{u}_\omega^l\cdot\vec{u}_\omega^l)dv. \tag{32}$$

The quantities g_ω and α_ω in expressions (31) and (32) are the optical gain coefficient and optical absorption coefficient at the circle frequency ω, respectively.

The first term on the right side of Eq. (30) is proportional to the optical gain coefficient in optical cavities. This term describes the contribution from the stimulated emission process, or more exactly, it comes from the spontaneous emission amplified by the stimulated emission process. The second term can be regarded as the contribution from the spontaneous emission process. Although the last term is proportional to the optical gain coefficient, it also depends on characteristics of other eigenmodes. This term is a result of the coupling between different eigenmodes.

The stimulated emission term in Eq. (30) is a Lorentzian distribution modulated by the spectral function $g_\omega^{ll}(g_\omega^{ll}+\alpha_\omega^{ll})$. The Lorentzian linewidth $2\Gamma_l$ depends on the optical gain coefficient in cavities. For each eigenmode, there exists a threshold optical gain, which corresponds to a zero Lorentzian linewidth. Evidently, these threshold optical gain coefficients can be approached, but never reached, as the optical power must remain finite.

V. NUMERICAL RESULTS

Numerical calculation of the spontaneous emission spectrum was carried out for a disklike microcavity with a diameter of 2.2 μm and a thickness of 0.15 μm. A frequency-dependent refractive index $n(\omega)=[3.346+0.333(\hbar\omega-0.866\ \mathrm{eV})]$ is assumed [7]. The eigenvalue wave equation Eq. (16) for this system is solved by using the method presented in Refs. [5,6]. For the reason of simplicity, the following expression for the wavelength-dependent optical gain coefficient

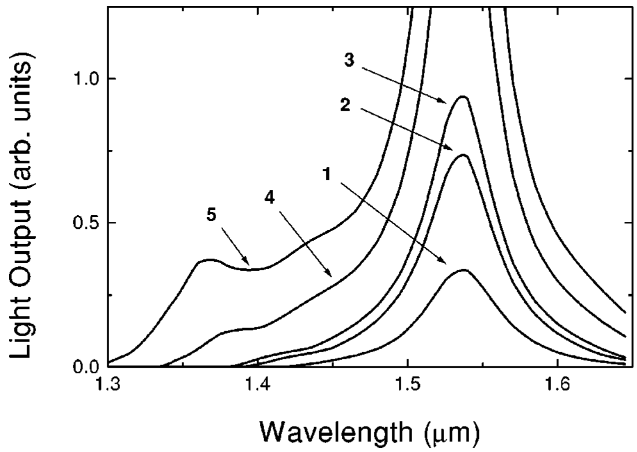

FIG. 1. Calculated emission spectra of a 2.2-μm diameter microdisk laser at different total output powers. The disk thickness is 150 nm. The total output powers are 7.5 mW (curve 1), 17 mW (curve 2), 22 mW (curve 3), 47 mW (curve 4), and 82 mW (curve 5).

$$g = g_1 - g_2(\lambda - \lambda_0)^2 \tag{33}$$

is used. Where g_1 is a pumping power-dependent parameter, $g_2 = 9.0\ \mu\mathrm{m}^{-3}$, λ is the wavelength, and $\lambda_0 = 1.55\ \mu$m. The calculated emission spectra of this microdisk cavity at different total output power ($P_{out} = 7.5$, 17, 22, 47, and 82 mW) are presented in Fig. 1. One may compare these results with the measured photoluminescence spectra of a microdisk laser with the same parameters at different pumping power [8], and find good agreement between these theoretical results and the experimental data.

To make a more precise comparison with experimental data, we consider the spectral linewidth of conventional single-mode lasers.

In the case of conventional lasers, an unidimensional model is appropriate. For an unidimensional cavity of length L, with front and rear mirrors at $z = \pm L/2$, the eigenvector $\vec{u}_\omega^l$ has the following form within the cavity:

$$\vec{u}_\omega^l = \vec{e}^l \sqrt{\frac{2}{k_0 L}} \cos[n(k_0^2 + \Lambda_\omega^l)^{1/2}(z - z_l) + \phi_l], \tag{34}$$

where $\vec{e}^l$ is an unit vector perpendicular to the z direction, $|z_l| < L/2$, and z_l and ϕ_l are determined by the boundary conditions at $z = \pm L/2$. In obtaining the normalization factor, the condition $k_0 L \gg 1$ is used. We emphasize the fact that the refractive index n is a complex number. In a lasing regime of a conventional laser, we have $\mathrm{Re}(n) \gg |\mathrm{Im}(n)|$. From the boundary conditions at $z = \pm L/2$, one can also determine the eigenvalue Λ_ω^l. In fact, what is determined directly from the boundary conditions is the quantity $k_0^2 + \Lambda_\omega^l$. For a given mode, this quantity varies slowly with the frequency, so we have

$$\eta_l = -2\frac{k_0}{c}. \tag{35}$$

In a single-mode conventional laser, only the lasing mode has significant contribution to the laser spectrum, contributions from nonlasing modes can be neglected. Let $l = 0$ for the lasing mode, then relation (29) is reduced to

$$P(\omega) \simeq P_0(\omega). \tag{36}$$

Expression (30) for $P_0(\omega)$ can also be simplified in this case, because we need to keep the stimulated emission term only. Conventional lasers have very narrow laser linewidth; therefore, we may replace g_ω^{00} and α_ω^{00} with $g_{\omega_0}^{00}$ and $\alpha_{\omega_0}^{00}$. When lasing action taking place, the optical gain coefficient of a conventional single-mode laser is very close the its threshold value of the lasing mode. Thus, in the following calculation, we can also replace the optical gain coefficient with its threshold value, which is equal to the total optical loss of the lasing mode. By applying relation (31), we obtain

$$g_{\omega_0}^{00}(g_{\omega_0}^{00} + \alpha_{\omega_0}^{00}) = n_{sp}\alpha_t^2 \left[\frac{2}{\alpha_t L} \sinh\left(\frac{\alpha_t L}{2}\right) \cosh(\alpha_t z_0)\right]^2, \tag{37}$$

where α_t is the total optical loss of the lasing mode and n_{sp} is the spontaneous emission factor defined as

$$n_{sp} = \frac{g_\omega + \alpha_\omega}{g_\omega}. \tag{38}$$

The condition $\mathrm{Re}(n) \gg |\mathrm{Im}(n)|$ is used in obtaining expression (37).

By applying these approximations, we find the following expression for the laser spectrum of a conventional single mode laser:

$$P(\omega) = \frac{\hbar\omega_0}{4\pi} \frac{n_{sp}\alpha_t^2 c^2}{(\omega - \omega_0)^2 + \Gamma_0^2} \left[\frac{2}{\alpha_t L} \sinh\left(\frac{\alpha_t L}{2}\right) \cosh(\alpha_t z_0)\right]^2. \tag{39}$$

The total output power can also be calculated:

$$P_{out} = \frac{\hbar\omega}{4\Gamma_0} n_{sp}\alpha_t^2 c^2 \left[\frac{2}{\alpha_t L} \sinh\left(\frac{\alpha_t L}{2}\right) \cosh(\alpha_t z_0)\right]^2. \tag{40}$$

The spectral linewidth is determined by the relation

$$\Delta\nu = \frac{2\Gamma_0}{2\pi}. \tag{41}$$

For gas lasers, the total optical loss is just the mirror loss α_m. Due to the high Q value of optical cavities of gas lasers, the quantity $\alpha_t L$ is very small; thus we have

$$\left[\frac{2}{\alpha_t L} \sinh\left(\frac{\alpha_t L}{2}\right) \cosh(\alpha_t z_0)\right]^2 \simeq 1. \tag{42}$$

From relations (41), (40), and (42), we obtain exactly the modified Schawlow-Townes relation for the laser linewidth that has been verified in detail for the He-Ne laser [9]:

$$\Delta\nu = \frac{\hbar\omega_0 n_{sp}\alpha_m^2 c^2}{4\pi P_{out}}. \tag{43}$$

In the case of semiconductor laser diodes with symmetric facets, we have $z_0 = 0$, and the total optical loss is the sum of the mirror loss α_m and the waveguide loss α_L. It is convenient to express the laser linewidth in terms of single facet output power P_0. P_0 is related to P_{out} by the following relation:

$$P_0 = \frac{\alpha_m}{2\alpha_t} P_{out}. \tag{44}$$

We obtain then the following relation for laser linewidth semiconductor laser diodes:

$$\Delta\nu = \frac{\hbar\omega_0 n_{sp}\alpha_m\alpha_t c^2}{8\pi P_0}\left[\frac{2}{\alpha_t L}\sinh\left(\frac{\alpha_t L}{2}\right)\right]^2. \tag{45}$$

By applying relation (45), we may calculate the laser linewidth of two GaAlAs single-mode laser diodes lasing at 817.5 nm and 832 nm, with the following parameters [10,11]: $L=280$ μm, $n_{sp}=2.6$, $\alpha_m=39$ cm^{-1}, $\alpha_L=45$ cm^{-1}, and $\alpha_t=\alpha_m+\alpha_L=84$ cm^{-1}. We obtain $\Delta\nu P_0=115$ MHz mW for the diode lasing at 817.5 nm, and $\Delta\nu P_0=113$ MHz mW for the other diode lasing at 832.0 nm. These values are in excellent agreement with the experimental results obtained by Fleming and Mooradian [10] for these two laser diodes: $\Delta\nu P_0=(114\pm5)$ MHz mW.

VI. CONCLUSION

In conformity with the classical electrodynamics, any emission of an optical wave is caused by a certain current source at optical frequencies. Thus, the spontaneous emission must be generated by certain spontaneous-emission current. Because the spontaneous emission is a quantum electrodynamic phenomenon, the correct form for the spontaneous-emission current cannot be obtained within the framework of the classical electrodynamics. We developed a semiclassical theory for emission spectra of lasers, in which the expression for the spontaneous-emission current is found by applying the criterion that this semiclassical theory must give the same spontaneous-emission power as the quantum-transition theory. We find that the spontaneous-emission current is equal to the element of quantum current operator between the initial and final states of the spontaneous-emission transition. The optical-field distribution of the light emitted from optical cavities, that is, the solution of the inhomogeneous wave equation with the spontaneous-emission current term, is obtained formally in terms of the eigenmodes. The eigenvalue equation for eigenmodes of optical cavities is established. The expression for emission spectra is derived. We calculated numerically the emission spectra of a microdisk laser at different output powers, and the numerical results match well with the measured photoluminescence spectra. We also applied this theory to calculate the laser linewidth of conventional lasers. We obtained the modified Schawlow-Townes formula for spectral linewidth of gas lasers, and in the case of semiconductor laser diodes, we got an excellent agreement between our theoretical results and experimental data reported in the literature.

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APPENDIX A

The wave equation, Eq. (5), can be rewritten formally as

$$\vec{A}_\omega = L_\omega[n^{-2}(\vec{r})\vec{j}^{sp}_\omega]. \tag{A1}$$

L_ω in Eq. (A1) is an operator defined by the following relation:

$$L_\omega[\vec{X}(\vec{r})] = -D_\omega^{-1}[n^2(\vec{r})\rho(\vec{r})\vec{X}(\vec{r})], \tag{A2}$$

in which $\vec{X}(\vec{r})$ is an arbitrary vector function and D_ω is the following operator:

$$D_\omega = \Delta + n^2 k_0^2. \tag{A3}$$

The step function $\rho(\vec{r})$ is introduced to ensure that $\vec{j}^{sp}_\omega=0$ where $n^2(\vec{r})\rho(\vec{r})\vec{X}(\vec{r})=\vec{0}$.

A way to solve Eq. (A1) is to expand the vector $\vec{j}^{sp}_\omega \cdot n^{-2}(\vec{r})$ into a linear combination of eigenvectors of the operator L_ω:

$$n^{-2}(\vec{r})\vec{j}^{sp}_\omega = \sum_l a^l_\omega \rho(\vec{r})\vec{u}^l_\omega, \tag{A4}$$

in which the eigenvector $\vec{u}^l_\omega$ satisfies the following equation:

$$L_\omega \vec{u}^l_\omega = \frac{\vec{u}^l_\omega}{\Lambda^l_\omega}. \tag{A5}$$

After this expansion, we obtain the following expression for the vector potential:

$$\vec{A}_\omega = \sum_l \frac{a^l_\omega}{\Lambda^l_\omega}\vec{u}^l_\omega. \tag{A6}$$

Eigenvalue equation (A5) can be written explicitly as

$$(\Delta + n^2 k_0^2)\vec{u}^l_\omega = -\rho n^2 \Lambda^l_\omega \vec{u}^l_\omega. \tag{A7}$$

The vector functions $\vec{u}^l_\omega$ satisfy the same boundary conditions as the vector potential $\vec{A}_\omega$. The term on the right side of Eq. (A7) should satisfy the condition of continuity for the spontaneous emission current $\vec{j}^{sp}_\omega$, so we also request that the divergence of $\vec{u}^l_\omega$ be null:

$$\mathrm{div}\,\vec{u}^l_\omega = 0. \tag{A8}$$

It is easy to obtain a solution of Eq. (A7), which satisfies condition (A8), from an arbitrary solution $\vec{u'}^l_\omega$ of the same equation:

$$\vec{u}^l_\omega = \vec{u'}^l_\omega + \frac{\mathrm{grad}(\mathrm{div}\,\vec{u'}^l_\omega)}{(\Lambda^l_\omega + k_0^2)n^2}. \tag{A9}$$

The vector potential $\vec{A}_\omega$ obtained in this way is in the Coulomb gauge. We call the solutions of Eq. (A7), which satisfy condition (A8), the eigenmodes of the cavity.

The orthogonality condition for $\vec{u}^l_\omega$ is

$$k_0^3\int n^2\rho(\vec{u}_\omega^l\cdot\vec{u}_\omega^{l'})dv=\delta_{ll'}, \tag{A10}$$

where $\delta_{ll'}$ is the Kronecker symbol.

APPENDIX B

The output optical power spectrum is given by expression (27):

$$P(\omega)=\lim_{T\to\infty}4\pi T^{-1}\int \mathrm{Re}\left(\frac{in^2k_0}{\mu_0 c}\vec{E}_\omega^*\cdot\vec{E}_{\omega,T}-\vec{E}_{\omega,T}\cdot\vec{j}_\omega^{sp\,*}\right)dv \tag{B1}$$

with $\vec{E}_{\omega,T}$ defined in Eq. (28).

Evidently,

$$\vec{E}_\omega=\lim_{T\to\infty}\vec{E}_{\omega,T}. \tag{B2}$$

In the limit $T\to\infty$, we have for the Coulomb gauge,

$$\vec{E}_{\omega,T}=i\mu_0\omega\vec{A}_{\omega,T} \tag{B3}$$

in which

$$\vec{A}_{\omega,T}=\frac{1}{2\pi}\int_{-T/2}^{T/2}dt\,e^{i\omega t}\int_0^\infty(\vec{A}_{\omega'}e^{-i\omega' t}+\text{c.c.})d\omega'; \tag{B4}$$

thus,

$$P(\omega)=\lim_{T\to\infty}4\pi T^{-1}\left[\mu_0 ck_0^3\int \mathrm{Im}(-n^2)|\vec{A}_{\omega,T}|^2dv+\mu_0\omega\int \mathrm{Im}(\vec{A}_{\omega,T}\cdot\vec{j}_\omega^{sp\,*})dv\right]. \tag{B5}$$

From Eqs. (25) and (20) we have

$$\lim_{T\to\infty}\vec{A}_{\omega,T}=\lim_{T\to\infty}\sum_l \vec{b}_\omega^l\int dv\int_{-T/2}^{T/2}[\vec{j}^{sp}(t)\cdot\vec{u}_\omega^l]e^{i\omega t}dt, \tag{B6}$$

where

$$\vec{b}_\omega^l=\frac{k_0^3}{2\pi\eta_l}\frac{\vec{u}_\omega^l}{\omega-\omega_l-i\Gamma_l} \tag{B7}$$

and

$$\vec{j}^{sp}(t)=\int_0^\infty\vec{j}_\omega^{sp}e^{-i\omega t}d\omega+\text{c.c.} \tag{B8}$$

We may write

$$|\vec{A}_{\omega,T}|^2=\sum_{\beta=x,y,z}|(\vec{A}_{\omega,T})_\beta|^2. \tag{B9}$$

According to the theory of quantum mechanics, the term $|(\vec{A}_{\omega,T})_\beta|^2/\hbar^2T$, with $(\vec{A}_{\omega,T})_\beta$ given by expression (B6), gives a total stimulated transition rate in the cavity in the presence of an electromagnetic field with an effective vector potential

$$\vec{A}_{\beta,eq}=\sum_l(\vec{b}_\omega^l)_\beta^*\frac{\vec{u}_\omega^{l*}}{\mu_0}e^{-i\omega t}. \tag{B10}$$

This transition rate must be equal to the sum of optical power emitted from and absorbed in the cavity divided by the photon energy when such an electromagnetic field is present. Thus, we have

$$\lim_{T\to\infty}T^{-1}|(\vec{A}_{\omega,T})_\beta|^2=\hbar k_0\mu_0\int(g_\omega+\alpha_\omega)|\vec{A}_{\beta,eq}|^2dv, \tag{B11}$$

where g_ω and α_ω are the optical gain coefficient and optical absorption coefficient at the circle frequency ω, respectively. Generally, g_ω and α_ω are also functions of coordinates.

By using relations (B9), (B11), and (6) we obtain

$$\lim_{T\to\infty}4\pi T^{-1}\mu_0ck_0^3\int\mathrm{Im}(-n^2)|\vec{A}_{\omega,T}|^2dv=\frac{\hbar\omega}{\pi}\sum_{l,l'}\frac{k_0^2g_\omega^{ll'}(g_\omega^{ll'}+\alpha_\omega^{ll'})}{\eta_l^*\eta_{l'}(\omega-\omega_l+i\Gamma_l)(\omega-\omega_{l'}-i\Gamma_{l'})}, \tag{B12}$$

in which

$$g_\omega^{ll'}=k_0^3\int g_\omega(\vec{u}_\omega^{l*}\cdot\vec{u}_\omega^{l'})dv,\quad \alpha_\omega^{ll'}=k_0^3\int\alpha_\omega(\vec{u}_\omega^{l*}\cdot\vec{u}_\omega^{l'})dv. \tag{B13}$$

According to relation (B6), we have

$$\lim_{T\to\infty}4\pi T^{-1}\mu_0\omega\int\mathrm{Im}(\vec{A}_{\omega,T}\cdot\vec{j}_\omega^{sp*})dv=\lim_{T\to\infty}\sum_l 4\pi T^{-1}\mu_0\omega k_0^3\eta_l^{-1}(\omega-\omega_l-i\Gamma_l)^{-1}\times\int(\vec{j}_{\omega,T}^{sp*}\cdot\vec{u}_\omega^l)dv\int(\vec{j}_{\omega,T}^{sp}\cdot\vec{u}_\omega^l)dv. \tag{B14}$$

In expression (B14), the following notation was used:

$$\vec{j}_{\omega,T}^{sp}=\frac{1}{2\pi}\int_{-T/2}^{T/2}\vec{j}^{sp}(t)dt. \tag{B15}$$

But

$$\int(\vec{j}_{\omega,T}^{sp*}\cdot\vec{u}_\omega^l)dv\int(\vec{j}_{\omega,T}^{sp}\cdot\vec{u}_\omega^l)dv=\left|\int\vec{j}_{\omega,T}^{sp}\cdot\vec{u}_{\omega,r}^ldv\right|^2-\left|\int\vec{j}_{\omega,T}^{sp}\cdot\vec{u}_{\omega,i}^ldv\right|^2+\frac{i}{2}\left|\int\vec{j}_{\omega,T}^{sp}\cdot(\vec{u}_{\omega,r}^l+\vec{u}_{\omega,i}^l)dv\right|^2$$

$$-\frac{i}{2}\left|\int \vec{j}^{sp}_{\omega T}\cdot(\vec{u}^{l}_{\omega,r}-\vec{u}^{l}_{\omega,i})dv\right|^2, \tag{B16}$$

where $\vec{u}^{l}_{\omega,r}=\mathrm{Re}(\vec{u}^{l}_{\omega})$ and $\vec{u}^{l}_{\omega,i}=\mathrm{Im}(\vec{u}^{l}_{\omega})$; so, in a similar way, we may obtain

$$\lim_{T\to\infty} T^{-1}\int(\vec{j}^{sp}_{\omega,T}\cdot\vec{u}^{l}_{\omega})dv\int(\vec{j}^{sp*}_{\omega,T}\cdot\vec{u}^{l}_{\omega})dv = \frac{\hbar k_0}{4\pi^2\mu_0}\int(g_\omega+\alpha_\omega)(\vec{u}^{l}_{\omega}\cdot\vec{u}^{l}_{\omega})dv \tag{B17}$$

and, consequently,

$$\lim_{T\to\infty} 4\pi T^{-1}\mu_0\omega\int \mathrm{Im}(\vec{A}_{\omega,T}\cdot\vec{j}^{sp*}_{\omega})dv = \sum_l \frac{\hbar\omega k_0(\overline{g^l_\omega}+\overline{\alpha^l_\omega})}{\pi\eta_l(\omega-\omega_l-i\Gamma_l)}. \tag{B18}$$

In the above expressions, the following notations were used:

$$\overline{g^l_\omega}=k_0^3\int g_\omega(\vec{u}^{l}_{\omega}\cdot\vec{u}^{l}_{\omega})dv,\quad \overline{\alpha^l_\omega}=k_0^3\int \alpha_\omega(\vec{u}^{l}_{\omega}\cdot\vec{u}^{l}_{\omega})dv, \tag{B19}$$

By substituting expressions (B12) and (B19) into Eq. (B5), we obtain the following expression for the power spectrum of an eigenmode:

$$P_l(\omega)=\frac{\hbar\omega}{\pi}\left\{\frac{k_0^2}{|\eta_l|^2}\frac{g^{ll}_\omega(g^{ll}_\omega+\alpha^{ll}_\omega)}{(\omega-\omega_l)^2+\Gamma_l^2}+\mathrm{Im}\left[\frac{k_0(\overline{g^l_\omega}+\overline{\alpha^l_\omega})}{\eta_l(\omega-\omega_l-i\Gamma_l)}\right]+\mathrm{Re}\left[\frac{2k_0^2}{\eta_l(\omega-\omega_l-i\Gamma_l)}\times\sum_{l'\neq l}\frac{g^{l'l}_\omega}{\eta^*_{l'}}\frac{g^{l'l}_\omega+\alpha^{l'l}_\omega}{\omega_l-\omega_{l'}+i(\Gamma_l+\Gamma_{l'})}\right]\right\}. \tag{B20}$$

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