Retardation effects on the Efimov states

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The retardation effect causes a small but finite change in the two-body potential, which in turn results in a large reduction in the scattering length A when the strength is near its critical value for forming a zero-energy bound state. This coupling to the vacuum fluctuation field thus affects the two-body dynamics such that the Efimov phenomenon is destroyed. Enhanced response of the scattering length ΔA for a small change in the two-body potential may be employed to gain efficiency and accuracy in experiments; for example, in testing body bound states in a cold atomic gas. The retardation effect can remove several Efimov bound states, depending on how large the change in scattering length is. Manipulation of the two-body potential by external fields is suggested to modify or eliminate some Efimov states. The optimal choice suggested from this analysis is to manipulate A such that $A/\Delta A \ll 1$. Also included is a discussion on the additivity correction to the two-body polarization potential due to the Casimir effect. [S1050-2947(99)04309-7]

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I. INTRODUCTION

It was shown nearly 30 years ago by Efimov [1] that, when a pair of particles interacting via a short-range potential of range R_0 forms a zero-energy bound state (ZEBS), with the corresponding scattering length $A=-\infty$, a third particle may then interact with the pair via an effective potential of the type $U \cong -1/R^2$, which can support an infinite number of three-body bound states, $N \propto \ln(|A|/R_0) \to \infty$ as $|A| \to \infty$. This surprising Efimov phenomenon (EP), of $N \to \infty$ with $|A| \to \infty$, has since been discussed by a number of people [2–4]. Evidently, the Efimov states are spatially extended, and thus it is reasonable to ask whether the retardation effect [5–10] associated with the finiteness of the velocity of the force-mediating particles would affect the EP.

The study of the Efimov states at large spatial separations among the constituent particles is of special interest for cold atomic gas of low densities [11], in which atoms generally interact at a distance of the order of 10^4 to $10^5 a_0$, for typical densities attainable at present. With very low relative kinetic energies ($k_B T < 10^{-3}$ K) associated with the heavy atomic cores, complications due to short-range interactions are minimized, while the long-range behavior of the interactions is magnified. Studies of the retardation effect on collisions among the cold atoms are difficult. On the other hand, loosely bound three-body Efimov states may be more adaptable for high-precision experimentation.

In Sec. II, we briefly review the Efimov result, and point out the crucial simplification needed in Sec. IV. The most detailed study of the retardation effect thus far has been carried out on the van der Waal's (vdW) and electric-dipole polarization (EDP) potentials for the two-body systems, involving two atoms or an atom and an ion. Therefore, focusing our discussion on these cases makes our argument more concrete and transparent, as we can then proceed to treat systems of three atoms or two atoms and an ion, provided the binding-energy condition for the Efimov effect is (approximately) satisfied. We summarize the existing result in Sec. III, adding several critical comments. The effect of retardation on the EP is then discussed in Sec. IV. Some comments concerning the additivity versus multiplicative modes of the retardation correction are given in the Appendix; it is based on the coupling to the vacuum fluctuation field.

II. THREE-PARTICLE EFIMOV STATES

For simplicity we consider three identical bosons interacting by short-range potentials. By short range, we mean any interaction that falls off at large distance faster than $1/r^2$. There are several different ways by which Efimov phenomenon has been studied. Thus, a confluence of the two-body and three-body threshold singularities [2] at $E_{\text{tot}} \cong E_b \cong 0$ in terms of the Faddeev interaction kernel $K = G_0 t_{12}$ provides the EP, where G_0 and t_{12} are the three-body free Green's function and the two-body scattering amplitude, respectively. Here, E_b denotes the binding energy of the two-body subsystem, while E_{tot} is the total energy of the full three particles. That is, $\text{tr}(K) \to \infty$ as the total energy $E \to 0$. Alternatively, an exactly soluble model solution in the adiabatic approximation [3] was studied to show that the effective two-body interaction in the presence of the third particle behaves as $-1/R^2$ at large R (but $R < |A|$). However, its leading nonadiabatic correction [4] was shown to behave spuriously as $1/R$, while the complete nonadiabatic contribution is of shorter range than $1/R^2$. Finally, the effective potential obtained by averaging the interaction between the pair and the third particle [4] gives the correct qualitative result, although the numerical coefficient of the resulting potential is not complete [4]. This is probably the simplest treatment of the EP that retains all the essential features. So we adopt it in the following discussion. We take the Jacobi coordinates $(\vec{r}, \vec{R})$ corresponding to the relative coordinates $\vec{r}$ for particles 1 and 2, and $\vec{R}$ for particle 3 relative to the center of mass of the pair $1+2$. For simplicity and without loss of generality, we neglect the symmetrization among the three particles.

The Efimov phonomenon is obtained in two steps: First, the pair $1+2$ interacts with a short-range attractive potential V_{12} of range of R_0, and is assumed to form a zero-energy bound state, with very large scattering length $|A|$. Then, in the region of $r<|A|$, the asymptotic wave function for the pair $\psi(r)$ takes the simple form

$$\lim_{k\to 0}\lim_{r\to 0}\psi(r)\simeq\lim_{k\to 0}[\sin(kr+\delta)/(kr)]\simeq(r-A)/r\to\mathrm{const}/r, \quad (1)$$

for $R_0\ll r<|A|$. For convenience of discussion, we assume that the two-body potential contains a strength parameter γ. Eventually, as the coupling parameter γ in V_{12} approaches the critical value γ_0 where a ZEBS is produced, we have $|A|\to\infty$. Thus, A is especially *sensitive* to the two-body interaction V_{12}; that is, in the vicinity of γ_0, the phase shift is close to $\pi/2$ and the binding energy $E_b\simeq 0$, such that a small change in V_{12} is reflected in a large change in A. Furthermore, the asymptotic behavior $\psi\to 1/r$ of Eq. (1) is crucial to the EP, but this follows from the general flux conservation over the asymptotic surface, as $dS=r^2 d\Omega$. It is independent of the particle interaction.

As the second step, we introduce a third particle to interact with the above pair. Then, the interaction between the pair with the third particle 3 is obtained in the lowest-order approximation by the average

$$U(R)\simeq U_{\mathrm{ave}}(R)\equiv(\psi|V_{13}+V_{23}|\psi)_r\simeq-\mathrm{const}/R^2 \quad (2)$$

for large R, but still $R<|A|$. The constant in Eq. (2) contains the polarizability of particle 3. Unlike in Eq. (1), the behavior of $U(R)$ is *insensitive* to the strength of V_{ij}, so long as it is attractive and relatively short-ranged, of typical range R_0. The higher-order corrections to the static picture (2) is to change the constant factor [4] in U, but the R dependence of Eq. (2) is unchanged. For the present purpose, V_{ij} are assumed to decay faster than $1/r^2$ at large r, including the inverse power potentials of the vdW and EDP types as well as a $\delta(\vec{r}/2\pm\vec{R})$ function form. The number of bound states generated by this potential U may be estimated, for example, by the Bargman formula (or Calogero form), as

$$N=\mathrm{tr}_R(G_0U)\simeq\int_{R_0}^{|A|}(RU)dR\simeq 1/\pi\,\ln(|A|/R_0)\to\infty, \quad (3)$$

as $|A|\to\infty$. This is the essence of the EP. The average potential approach presented here is, except for the multiplicative constant factor, consistent with the argument given originally by Efimov [1] and also that followed by Amado and Noble [2]. It not only avoids unnecessary complications involved in the other approaches, but also brings out clearly the parts that will be potentially affected by the retardation correction. A slightly more complete treatment of the EP in the adiabatic picture [3,4] gives essentially the same result, in the limit of the two-body state binding energy $E_b\to 0$.

For V attractive ($V<0$) and weak, the phase shift near $E=0$ is positive and A is negative. As the strength of V increases in magnitude, A approaches negative infinity. A slight further increase in the potential strength changes A abruptly from $-\infty$ to $+\infty$. Therefore, the sign of A at $\pm\infty$ has no dynamical meaning, since this is caused simply by the definition of A in terms of the phase shift, as a limit of $-\tan\delta/k$ as $k\to 0$. With $V<0$, we have $A>0$ for an odd number of bound states, including the ZEBS, and $A<0$ for an even number. In general, the sign of A does not represent the sign of V, except when V is too weak to support a bound state. In the Efimov case, however, the presence of one or more two-body bound states tends to make the three-body Efimov states unstable.

III. RETARDATION EFFECT ON THE TWO-BODY INTERACTION

The physical effects of the zero-point vacuum field fluctuation (VFF) have been studied extensively in recent years [5–10]. The Lamb shift, the Casimir effect, spontaneous radiative emission, etc are interpreted in terms of the coupling to the VFF. We summarize here the main results relevant to our discussion, especially their effects on the van der Waal's and electric dipole polarization potentials. Some clarifying comments on the ''additive'' vs ''multiplicative'' feature of the retardation corrections are given in the Appendix. Those quantities modified by the retardation effect are denoted by tilde in the following.

We consider a pair of neutral atoms with the dynamical electric-dipole polarizabilities $\alpha_1(w)$ and $\alpha_2(w)$, which depend on the frequency w of perturbing fields. In the VFF approach, the zero-point field is assumed to induce dipole moments in particles 1 and 2, which in turn interact with each other in their mutual dipole fields. Thus, the vacuum fields $\vec{E}_0^{1,2}$ at the positions of atoms 1 and 2 induce the dipole moments $\vec{p}_{1,2}=\Sigma_{\vec{k}}\alpha_{1,2}(w_k)\vec{E}_{0k}^{1,2}$, and their interaction potential is given, after the angular integration $d\Omega_{\vec{k}}$, by [7,8,10]

$$V^{VF}(\vec{r})=-(1/2)\sum_{\vec{k}\vec{k}'}\vec{p}_1\cdot\vec{G}_{\vec{k},\vec{k}'}(\vec{r})\cdot\vec{p}_2$$
$$\simeq-(\hbar/\pi c^6)\int_0^\infty dw\,w^6\alpha_1(w)\alpha_2(w)G(x), \quad (4)$$

where we set $|\vec{E}_{0\vec{k}}|^2\simeq\hbar w$, and where

$$G(x)=\sin(2x)/x^2+2\cos(2x)/x^3-5\sin(2x)/x^4$$
$$-6\cos(2x)/x^5+3\sin(2x)/x^6\equiv\sum_{i=1}^{5}G_i(x), \quad (5)$$

with $x=wr/c\equiv w/w_r$. A part of the argument $(2x)$ comes explicitly from the retardation correction $\delta(t-r/c)$.

Equation (4) with Eq. (5) is used to derive both the vdW and EDP potentials, with and without the retardation effect. To begin with, it is important to recognize the different frequency scales involved in Eq. (4). In addition to $w_r=c/r$, we also have the typical atomic excitation frequencies w_0 associated with atoms 1 and 2, where $w_r\simeq w_0f_0\ll w_0$ and $f_0=1/137$. Furthermore, the classical polarization of a charge involves the mass of the charge m, where $w_m=mc^2/\hbar\simeq w_0/f_0^2$.

The vdW interaction without retardation comes from the last G_5 term in Eq. (5) and the conventional definition of the polarizabilities $\alpha_{1,2}$. In the single excitation frequency approximation, we have $\alpha_{1,2}=e^2/m/(w_0^2-w^2)$, and

$$V^{\mathrm{vdW}}\simeq-(3\hbar/4)\alpha_1(0)\alpha_2(0)w_0/r^6. \quad (6)$$

We emphasize that the integral involved in the V^{vdW} is finite and well behaved. Next, when the full retardation effect is included, all the terms in G contribute and we have

$$\tilde{V}^{\mathrm{vdW}}(r) = -(23/4)(\hbar c/\pi)\alpha_1(0)\alpha_2(0)/r^7 = V^{\mathrm{vdW}}_{\mathrm{tot}}(r), \tag{7}$$

and thus

$$\Delta V^{\mathrm{vdW}}_{\mathrm{ret}} = V^{\mathrm{vdW}}_{\mathrm{tot}} - V^{\mathrm{vdW}} > 0 \tag{8}$$

for $r \gg a_0/f_0$. That is, $\tilde{V}^{\mathrm{vdW}}$ replaces V^{vdW} of Eq. (6) and the retardation correction is multiplicative at large r, with a factor $1/r$ multiplying Eq. (6). The full G behaves as $1/w^3$ or better at $w=0$, so that all the integrals involved in Eq. (8) are again finite.

Now we briefly turn to the EDP potential, for completeness. Following the procedure given above for the vdW case, we require that the nonretarded potential should also come from the G_5 term and the classical ''dipole polarizability'' associated with a simple charged particle, at $w=0$. If we take the limit $w_0=0$ in α_1, as $\alpha_1(w) \to e^2/m/(-w^2) \equiv \alpha_{fr}(w)$, then the integral for V^{EDP} with α_{fr} and G_5 becomes divergent, due to the strong singularity at $w=0$. Instead of α_{fr}, we introduce $\alpha_{f0} = e^2/m/w_f^2$, where $w_f \equiv w_r\sqrt{w_0/w_m}$. Now, the integral converges (see the Appendix) and we recover the usual result

$$V^{\mathrm{EDP}}(r) = -[\alpha_2(0)e^2/2]/r^4. \tag{9}$$

Next, for the retarded case, the full G contributes and, even with the singular α_{fr}, the integral in Eq. (4) is finite, and we obtain

$$\tilde{V}^{\mathrm{EDP}}(r) = +[11\hbar e^2\alpha_2(0)/(4\pi mc)]/r^5, \tag{10}$$

where the contribution from the $w \simeq w_f$ region is not important.

Apparently, there are two contributions in the case of EDP, one with α_{f0} near $w \simeq w_f \ll w_r$ and the other part with α_{fr} for $w \gg w_f$. This suggests that the two contributions are additive [9,10], as

$$V^{\mathrm{EDP}}_{\mathrm{tot}} = V^{\mathrm{EDP}} + \tilde{V}^{\mathrm{EDP}}, \tag{11}$$

and therefore for large r

$$\Delta V^{\mathrm{EDP}}_{\mathrm{ret}} = \tilde{V}^{\mathrm{EDP}} > 0. \tag{12}$$

Important for the discussion to be given in Sec. IV is the qualitative effect of the retardation in the large r region of the two-particle sector. The asymptotic forms of the potentials $V^{\mathrm{VF}}_{\mathrm{tot}}$ given by Eqs. (7) and (11) show that the change is positive; i.e., the potentials with the retardation corrections become less attractive at $r \gg R_0/f_0$. The essential point for our purpose is that the vacuum fluctuation effect in the presence of external fields is observable, and $\Delta V_{\mathrm{ret}} \neq 0$.

IV. RETARDATION EFFECT ON THE EFIMOV STATES

In Secs. II and III, the salient properties of the three-body Efimov states and the effect of retardation on the two-body interaction have been summarized. We now consider the effect of the latter on the former, EP. We show that the EP of $N \to \infty$ as $A \to -\infty$ is destroyed when the retardation effect is included. In principle this requires a detailed perturbation calculation of many connected diagrams that involve multiple photon exchanges among the three particles, such as $(1T2,2T3,T3) + (1T2,1I2,1T3) + \cdots$, etc., where at least one transverse photon (T) for each particle is required. I is for an instantaneous Coulomb interaction, and $1T2$, for example, denotes the T photon exchange between particles 1 and 2. However, such complications may be avoided if we recall that, as stressed in Sec. II, the EP is sensitive to the two-particle potential V, but U is insensitive to such interaction and depends mainly on the asymptotic behavior of the two-particle wave function as dictated by the particle flux conservation. Once this picture is adopted, the proof of the EP breakdown becomes almost trivial. The argument is given in two steps.

(i) First, due to Eqs. (8) and (12), $\tilde{A}$ that includes the retardation effect will be less than A in magnitude. Because, while $A \to -\infty$ by the original assumption of a ZEBS supported by $V, V^{\mathrm{VF}}_{\mathrm{tot}} = V + \tilde{V}$ with the retardation correction $\tilde{V} > 0$, for example, gives

$$0 > \tilde{A} > A \to -\infty. \tag{13}$$

As emphasized at the end of Sec. II above, the behavior of A associated with the phase shifts that are near odd multiples of $\pi/2$ is not smooth and should be treated with caution; the change in A can be abrupt and nonmonotonic even though both the corresponding phase shifts and the potentials change slightly in a continuous and monotonic way. Of course, for the phase shifts that are far from the critical values mentioned above, the changes in A and V are both smooth and monotonic [12].

Although $\tilde{V}$ is small, the change $\Delta A = \tilde{A} - A$ can be very large. The enhancement in A due to a small change in V in the near critical region can be estimated as follows: Denote the phase shifts with V and $\tilde{V}$ by δ_0 and δ', respectively, with $\delta_0 \simeq \pi/2$ and δ' small. We also let $\xi \equiv \tan\delta_0 \tan\delta'$ and η be the enhancement factor in $\Delta A = a\eta$, where a is the scattering length associated with $\tilde{V}$ alone. Then, for $\xi \simeq 1$, we have $\eta \cong \tan^2\delta_0/(1-\xi)$; for $\xi \gg 1$, $\eta \simeq \tan\delta_0/\tan\delta'$; and for $\xi \ll 1$, $\eta \simeq \tan^2\delta_0$. In all three cases, the enhancement can be very large when $\delta_0 \simeq \pi/2$. Thus the relative change in A due to ΔV may be easily observed in the three-particle states, as compared to the small changes in the two-particle sector. For example, with δ' from $\tilde{V}_{\mathrm{ret}}$ of the order of 10^{-4}, the cases with $\xi \gtrsim 1$ can give $\Delta A/A \lesssim 1$ and $\Delta N \lesssim 0.1$. On the other hand, the case with $\xi \simeq 1$ seems to produce the largest change $\Delta A/A \gtrsim 100$ and $\Delta N \gtrsim 1$.

With all these complications on A, however, the form of the wave function at large r is unchanged from that given by (1b), when A is replaced by $\tilde{A}$, and this is the crucial feature needed. This is discussed next.

(ii) The $1/R^2$ behavior of the three-body potential, $U(R)$ depends on the two-particle wave function $\psi(r)$ at large r, as seen in Eqs. (1) and (2). This form is not expected to be affected by the retardation effect, because the $1/r$ dependence comes strictly from the flux conservation over the large surface element that is proportional to r^2. This is basically a static property. The retarded Green's function $G_{\rm ret}$ in the tr$K$ has the same behavior $1/R$ at large R. Therefore,

$$\widetilde{U}(R) \to -\text{const}/R^2 \quad \text{for} \quad R < \widetilde{A}. \tag{14}$$

Since $\widetilde{A}$ is finite, the number of three-body bound states is also finite,

$$\widetilde{N} \propto (1/\pi)\ln(|\widetilde{A}|/R_0) < \infty. \tag{15}$$

This completes the proof of the breakdown of EP with the retardation effect. That is, the Efimov phenomenon of $N \to \infty$ is destroyed by the retardation effect, when V alone gives the EP. A small $\Delta V_{\rm ret} \neq 0$ changes N such that $\widetilde{N} < \infty$. The experimentally convenient quantity to measure the retardation effect is $\Delta N = \widetilde{N} - N$, which can be greater than 1, presumably only under the condition $\xi \simeq 1$. That is, one can simply count the number of Efimov states, with or without the retardation effect. Of course, the value N must be estimated accurately theoretically. By contrast, ΔA can vary wildly, and so is less controllable.

Evidently, the task of presenting a proof was made simple by the crucial observation that Eqs. (13) and (14) are distinctly separated, with the $\widetilde{U}$ being insensitive to the pair interactions V. Although the EP in the strict sense is broken, there may still be a large number of Efimov states in the critical region of V for experimentation. Furthermore, as a simple corollary, the EP is valid if it is defined in terms of $V_{\rm tot}^{\rm VF}$, rather than with V. That is, we may consider a different system with new V' that is slightly more attractive than V. Then, with the new $V_{{\rm tot}'}$, $\widetilde{A}' \to -\infty$, and the Efimov phenomenon is restored. Establishing the EP is delicate, but conditions for its failure are relatively easy to demonstrate. From the earlier discussion, the $\xi \simeq 1$ case seems to be the optimal choice. This condition may be achieved experimentally by manipulating the V such that the phase shift δ_0 corresponding to V gets as close to $\pi/2$. With $\delta_0 = \pi/2 - \Delta_0$, the condition $\xi \simeq 1$ means $\Delta_0 \simeq \delta'$, where δ' is from $\widetilde{V}_{\rm ret}$. For $(\Delta_0 - \delta')/\Delta_0 \simeq 10^{-2}$, we expect $\Delta N \simeq 1.5$.

Finally, we examine the consequences of the average potential approximation adopted in Eq. (2) and in this section. As noted earlier, Eq. (2) gives the correct R dependence but not the constant coefficient. On the other hand, the adiabatic potential represents the full strength [4], with the nonadiabatic corrections behaving as $1/R^3$ or better. The retardation correction makes the two-body $\widetilde{A}$ finite, and, for $R > R_b \simeq 1/\sqrt{-\widetilde{E}_b}$, the U_{ad} decays exponentially, where E_b is the binding energy of the pair $(1+2)$. Since particle 3 is held fixed in position at R during the calculation of $U_{\rm ad}$, the retardation correction to $U_{\rm ad}$ would be "additive," as in the EDP case, with the $1/R^3$ or stronger behavior. Therefore, the retardation effect again spoils the EP at the two-body level.

V. DISCUSSION

We have given a simple and direct argument that the retardation effect can destroy the Efimov phenomenon. The proof is semirigorous, only because the full perturbation treatment involving multiphoton exchanges among the three particles is not considered. Instead, our result is based on the crucial but simplifying observation that, insofar as the EP is concerned, the two-body potential in the critical region of ZEBS is sensitive to the retardation correction, as manifested in the large change in A. On the other hand, the resulting three-body potential U is not sensitive to retardation, but dictated by the overall flux conservation. Therefore, the original three-body retardation problem is reduced essentially to that of a two-body problem. In the perturbation-theory terms, what we have included in Sec. IV are effectively the diagrams $(1T2,1I2,1I3)+(1T2,1T2,2I3)+\cdots$, etc. The simplifying assumption we made there implies that the contributions from diagrams such as $(1T2,1T3,2T3)+(1T2,1I2,1T3)+\cdots$ are of the shorter range and do not contribute to the EP.

Apparently, the change in the two-body potential due to the retardation correction and the resulting change in the pair binding energies are small. But the effect is magnified through A and N associated with the three-body Efimov bound states. The following frequency regions have to be distinguished: $w_f \simeq w_r f_0 \ll w_r \simeq w_0 f_0 \ll w_0 \simeq w_m f_0^2 \ll w_m = mc^2/\hbar$, where $f_0 = 1/137$. This magnification of the small effect may be observed in a dilute cold trapped atomic system at extremely low temperature [11], provided the experimental conditions are such that the complex short-range part of the interactions may be minimized and the long-range behavior enhanced. Also the condition $\xi \simeq 1$ in Sec. IV may be experimentally desirable. Obviously, the optimal size of the Efimov bound states should be at least of the order of a_0/f_0 or larger, and the higher-lying states near the edge of A may be even of size a_0/f_0^2 with very small binding energies. This situation may correspond to gas densities on the order of $10^{12}\,{\rm cm}^{-3}$ or less for trapped cold atoms, which is attainable experimentally. Unlike in collision studies involving two particles with V, where small effects are often difficult to detect, the Efimov bound states involving three particles may be easier to analyze experimentally, with high accuracy.

There are several points that should be further examined.

(i) Insofar as the retardation effect on the three-body systems is concerned, our result shows that we can concentrate the treatment only in the two-body sector, as the third body is interacting with a long-range R^{-2} potential that is insensitive to retardation. This is obviously an important simplification, and, as discussed above, a full retardation treatment involving all three particles must be carried out.

(ii) Furthermore, while the tr$(G_0 t_{ij})$ proof of Ref. [2] is consistent with the approach we have taken, in that the two-body off-shell amplitude in the presence of the third particle t_{ij} is retardation corrected while G_0 is left unchanged, a more careful analysis in terms of the relativistic tr$K$ may be of interest.

(iii) Cluster formation in a cold gas of both bosons and fermions is of special interest, in sharp contrast to condensation of a Bose gas. In particular, an exotic case of interaction of two pairs of ZEBS should be examined, in light of a recent study by Ropke *et al.* [13], on the four-particle cluster formation in nuclear matter.

(iv) Furthermore, the two-body potential critical for the EP may be manipulated by external field perturbations. This in turn provides a handle on the diffuse Efimov states, perhaps through a change in α; a collective contribution to α from high Rydberg and continuum states may be easily altered. Because of strong enhancement, the change needed in V is small, while many upper Efimov states can be created or destroyed by the change in A.

(v) Recently, a question was raised by Mukhamedzhanov [14] concerning an additional r^{-5} correction to the EDP potential, due to a possible contribution to the dipole polarizability from the fully continuum three-particle states that are Coulomb distorted. Possible double counting of the continuum-state contribution must be addressed, especially in view of the complex asymptotic behavior of the three-body wave functions [15], which depends on the degrees of pair correlations. Although this problem is presently unresolved, it is an important one and warrants attention.

(vi) Finally, explicit calculations of the Efimov states, with and without the retardation effect in the more realistic systems, are needed for detailed quantitative comparison with experiments. Enhanced changes in the three-body bound system caused by a small change $\tilde{V}_{\rm ret}$ in the two-body sector may be useful in making the retardation effect observable. Work on this is in progress and the preliminary result will be reported on elsewhere.

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APPENDIX

We present here a coherent derivation of both the vdW and the EDP potentials, with and without the retardation effects, all from Eq. (4) with Eq. (5), and all within the vacuum fluctuation field approach [7,8,10]. This may provide possible reasons for the retardation corrections being either multiplicative, as in the vdW case, or additive, as in the EDP case. But, more importantly, the new α_{f0} introduced below for the charged particle in the context of the vacuum field approach provides insight into the role of a charged particle in a static picture. It is also somewhat troublesome that the complete EDP potential was not derived from Eq. (4), contrary to the vdW case. Of course, the perturbation treatment of Ref. [10] and a dispersion theoretic treatment in Ref. [9] provide the rigorous and complete answer. The present discussion therefore is not meant to be a substitute. Our treatment below is simply based on the observation that, for the nonretarded potentials, the α_1 in the conventional form and G_5 can give the singular integrand, and thus should result in an additive correction.

For the vdW potential in the nonretardation limit, we examine the dominant term G_5. The integrand is well-behaved for the classical dipole polarizability $\alpha_{1,2}(w)=e^2/m/(w_0^2-w^2)$ in the single-state approximation, and we obtain the known result

$$\begin{aligned} V^{\rm vdW}(r) &= -[3\hbar/(\pi r c^6)]\int_0^\infty dw\,\alpha_1(w)\alpha_2(w)\sin(2wr/c) \\ &= -[3\hbar/(\pi r^6)]\int_0^\infty du\,\alpha_1(iu)\alpha_2(iu)\exp(-2ur/c) \\ &\simeq -(3\hbar/4)\alpha(0)^2 w_0/r^6, \end{aligned} \tag{A1}$$

where the numerical coefficient is also correct. Note that the dw part in the integral is not scaled in terms of $x=wr/c$. Instead, the integration contour was rotated [8] to the imaginary axis u, and the exponential factor was then set equal to 1. The important point to note here is that the integrand is perfectly well-behaved in the region $w=0$.

The retarded potential is obtained by reevaluating the entire expression (4) by retaining all the terms in G of Eq. (5), plus a damping factor $\exp(-\beta w)$ in the integrals, and making the approximation $\alpha(w)\simeq\alpha(0)$. The result is

$$\tilde{V}^{\rm vdW}(r) = -(23/4)(\hbar c/\pi)\alpha_A(0)\alpha_B(0)/r^7 \equiv \tilde{V}^{\rm vdW}_{\rm tot}(r). \tag{A2}$$

Evidently, the retardation effect on the vdW potential is ''multiplicative,'' by a factor $1/r$ at larger r to the static potential $V^{\rm vdW}$.

We now turn to the electrie-dipole polarization potential between a charge e and a neutral atom, closely following the steps given above for the vdW case. That is, we have a charge e, in place of the polarizable particle 1, interacting with particle 2.

For the nonretarded case, we again focus on G_5, which is presumably the dominant term at small x. For atom 1 as a charged particle, the conventional treatment adopts the simple limit $\alpha_1(w)\simeq(e^2/m)(w_0^2-w^2)\to-(e^2/m)/w^2=\alpha_{fr}(w)$ and the integral for the EDP potential diverges, due to the singularity at $w=0$ coming from α_{fr}, as

$$\begin{aligned} V^{\rm EDP}(r) &= -[\hbar e^2/(m\pi c^6)] \\ &\quad\times\int_0^\infty dw\,w^6(-w^{-2})\alpha_2(w)G_5(wr/c) \\ &\to\infty. \end{aligned} \tag{A3}$$

In order to rectify this problem, we suggest that this singular behavior of the integrand in Eq. (4) with G_5 alone is the basic difference between the vdW and EDP cases and must be treated separately as an additive contribution coming from α_{1A} at $w=0$. [The $\sin(2x)$ term does not contribute in the present case, without retardation, when the integration contour is rotated so that $w\to iu$ and then the exponential is set equal to 1.] Thus, we modify the free particle polarizability by reexamining the limit $w\to0$. Within the context of the present VFF formulation, we have the frequency scales $w_r=c/r$ and w_0 associated with typical atomic excitation frequency, and also $w_m=mc^2/\hbar$. Roughly we have $w_r\ll w_0$, where $w_r\simeq f_0 w_0$ and where $f_0=1/137$. In addition, $w_0\simeq w_m f_0^2$, so that $q_w^2\equiv\hbar w_0/w_m\simeq f_0^2$ or $q_w\simeq f_0$. Now we define a new $\alpha_f\equiv e^2/m/(w_f^2-w^2)$, where $w_f=w_r q_w$. For the EDP potential without retardation, we set $\alpha_f\simeq e^2/m/w_f^2=\alpha_{f0}$, while for the retarded part of the potential with the full G we use the original $\alpha_{fr}(w)=e^2/m/(-w^2)$. The full G, with all the oscillating factors retained, behaves only as w^{-3} at small w, so that possible contributions from the region $w\simeq w_f$ are wiped out by the w^6 factor.

When α_{f0} is substituted in Eq. (4) and G is replaced by G_5, the integral no longer diverges, and with an additional adjustment of the (1/3) factor that reduces the three-dimensionality of atom 1, we obtain the usual EDP potential in the static limit

$$V^{\mathrm{EDP}}(r)=-[\alpha_2(0)e^2/2]/r^4. \tag{A4}$$

The above derivation indicates that the physical content of the ''polarizability'' of a charged particle in the static EDP potential is associated with the frequency region that is much smaller than w_r; that is, the relevant distance involved is two orders of magnitude farther than that for the retardation part.

The retarded part of the potential is obtained by evaluating the contribution from all the terms of G and using $\alpha_f \simeq e^2/m/(-w^2)$ and a convergent factor $\exp(-\beta x)$, as

$$\tilde{V}^{\mathrm{EDP}}(r)=+[11\hbar e^2\alpha_B(0)/(4\pi mc)]/r^5, \tag{A5}$$

and thus $V_{\mathrm{tot}}^{\mathrm{EDP}}(r)=V_0^{\mathrm{EDP}}(r)+\tilde{V}^{\mathrm{EDP}}(r)$.

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