General teleportation channel, singlet fraction, and quasidistillation

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(Received 21 August 1998)

We prove a theorem on direct relation between the optimal fidelity f_{max} of teleportation and the maximal singlet fraction F_{max} attainable by means of trace-preserving local quantum and classical communication (LQCC) action. For a given bipartite state acting on $C^d \otimes C^d$ we have $f_{max} = (F_{max} d + 1)/(d+1)$. We assume completely general teleportation scheme (trace preserving LQCC action over the pair and the third particle in unknown state). The proof involves the isomorphism between quantum channels and a class of bipartite states. We also exploit the technique of $U \otimes U^*$ twirling states (random application of unitary transformation of the above form) and the introduced analogous twirling of channels. We illustrate the power of the theorem by showing that *any* bound entangled state does not provide better fidelity of teleportation than for the purely classical channel. Subsequently, we apply our tools to the problem of the so-called conclusive teleportation, then reduced to the question of optimal conclusive increasing of singlet fraction. We provide an example of state for which Alice and Bob have no chance to obtain perfect singlet by LQCC action, but still singlet fraction arbitrarily close to unity can be obtained with nonzero probability. We show that a slight modification of the state has a threshold for singlet fraction, which cannot be exceeded anymore. [S1050-2947(99)03707-5]

PACS number(s): 03.67.Hk, 03.65.Bz

I. INTRODUCTION

Consider the following problem. Alice and Bob are far from each other, and they share one pair of spin-*s* particles in entangled quantum state ϱ. All the manipulations Alice and Bob are allowed to perform are local quantum operations and classical communication [called local quantum and classical communication (LQCC) or bi-local operations]. It means that, in particular, they cannot exchange quantum bits or establish quantum interaction between their labs. Now, suppose that Alice wants to teleport an unknown state of some other spin-*s* particle, but only if she is sure that the fidelity *f* of the transfer is better than some given threshold f_{min} ($f_{min}<1$). Which states ϱ give Alice a nonzero chance that after some LQCC operations she can teleport being sure that her requirement is satisfied? The answer to this question will be one of the results of this paper.

As one knows, quantum teleportation [1] allows us to transfer the quantum information through quantum entangled states as quantum channels (supported by classical channels) with the fidelity better than by means of a classical channel itself [2]. For example, two spin-*s* particles in the maximally entangled state,

$$P_+ = |\Psi_+\rangle\langle\Psi_+|, \quad |\Psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d} |i\rangle|i\rangle, \quad d = 2s+1 \tag{1}$$

(we shall call it singlet state, despite it is, in fact, local transformation of true singlet) shared by a sender Alice and a receiver Bob allows us to transmit faithfully an unknown spin-*s* state, with additional use of classical bits describing one of $(2s+1)^2$ elementary messages. If the state shared by Alice and Bob is pure but not maximally entangled, then one can perform *conclusive* teleportation [3]. The main idea of the latter is that, given a particle in *nonmaximally* entangled state Ψ providing small transmission fidelity *f*, Alice and Bob can transform the state by some deliberate LQCC operations. As a result, with some probability the final state provides much greater transmission fidelity (usually the perfect one). In the proposed protocol [3] the pairs of particles were treated *noncollectively*, i.e., each pair was processed separately. The concept of *collective* operations that involve interactions among different pairs, has been implemented for pure states in the protocol of *concentration of entanglement* by LQCC operations [4] (see [5] for interesting consequences for entanglement measures and [6] for analysis of probability distributions). In this approach some number of nonmaximally entangled states are converted via LQCC operations into the less number of maximally entangled ones that can be used, for instance, for faithful teleportation process.

In realistic conditions, instead of pure entangled states, Alice and Bob usually share a mixed state that contains noisy entanglement. The latter case is more complex and it has longer history. Popescu first has pointed out [2] that mixed entangled states can allow for teleportation with significantly better fidelity than the one achieved by using only classical bits. He also showed [7] that some noncollective LQCC op-

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erations can transform the $d\times d$ ($d\geq 5$) Werner entangled mixed states satisfying local hidden variable model [8] into the two spin-$\frac{1}{2}$ states violating Bell inequalities. Subsequently, a similar effect by means of local filtering for mixed two spin-$\frac{1}{2}$ states has been found [9] (see [4] for pure state case). At the same time the important idea of *distillation* (or purification) of noisy entanglement has been worked out [10,11]. Here the aim is to convert some number of *mixed* inseparable states into less number of states close to a maximally entangled pure one. The distillation protocols are usually accomplished by operating on collections of pairs rather than on single pairs. However, the single-pair operations introduced in Refs. [9,4] have been shown to play an important role in the distillation protocol capable to distill *all* entangled two-quantum bit (qubit) states [12]. On the other hand, single-pair operations are much simpler to perform experimentally [9,13]. General results concerning the limits for those operations have been provided by Linden, Massar, and Popescu [14] and by Kent [15].

In this paper we would like to consider the question concerning the conclusive teleportation we asked at the beginning. We first reduce the question to the problem of single-pair distillation. To this end we provide a number of tools, which can be useful also in more general context.

In Sec. II we consider the problem of equivalence between bipartite states and quantum channels. The connections between states and channels were considered in Refs. [16,17]. It is clear that if we have a channel, then we can produce a bipartite state sending half of the singlet down the channel [16]. However, given a state, it is not clear whether there is a channel, which can produce it in the above way. A way of ascribing a channel to a given state is to perform teleportation via the state (creating the teleportation channel). One can now ask what channels can be produced by means of teleportation via a given mixed state. Another question is the following [16]: suppose that a mixed state was produced from a channel, by sending half of a singlet. Can we recover the channel by means of some (probably very sophisticated) teleportation scheme applied to the state? This is the question of reversibility of the operation of producing states from channels. In Sec. II we prove that the operation of sending half of a singlet down the channel produces an isomorphism between channels states having the reduced density matrix of one of the subsystems maximally chaotic (proportional to identity).

Section III is devoted to the problem of optimal fidelity of teleportation. We first consider families of states and channels connected via the above isomorphism. The states are singlets with an admixture of a completely mixed state [18] (generalization of two-qubit Werner states [8,2]), which are the only states invariant under the $U\otimes U^*$ twirling introduced in Ref. [18]. The channels are the generalized depolarizing ones. We show that the families are not only connected via the isomorphism, but are also physically equivalent, i.e., the teleportation via state reproduces the channel. This generalizes similar observation for two-qubit Werner states [16]. We also introduce an operation on channels, which is equivalent to twirling of the corresponding states. We show that average fidelity and the entanglement fidelity are invariants of the twirling operation.

These are the main tools that allow us to prove a strict connection between the optimal fidelity f_{max} of teleportation via a given state and the maximal singlet fraction F_{max} attainable by means of trace preserving LQCC operations on a single pair. Namely, we prove that there holds the equality $f_{max}=(F_{max}d+1)/(d+1)$. We emphasize here that we consider the most general teleportation scheme, which is possible. Then the problem of optimal teleportation fidelity is reduced to the much less complicated (but still nontrivial) task of optimal increasing singlet fraction by means of trace-preserving LQCC operations. We illustrate the power of the result in Sec. IV applying it to the problem of optimal teleportation fidelity via bound entangled states (the entangled ones, which cannot be distilled [19]). This problem has been quite recently risen by Linden and Popescu [20] who showed that some of the bound entangled states do not provide better fidelity by teleportation via purely classical channel (see [2] in this context). Here we prove that it is true for any bound entangled states and for the most general teleportation schemes.

In subsequent sections we apply the results to the problem of conclusive teleportation, which is now reduced to the problem of conclusive increasing singlet fraction. We consider two concepts: (i) noncollective distillation, where Alice and Bob have a chance to obtain a perfect singlet and (ii) noncollective quasidistillation (in short quasidistillation), where the perfect singlet cannot be obtained, but there is a nonzero chance to obtain arbitrarily high singlet fraction. From the results concerning noncollective distillation [14,15] we know that for a broad class of mixtures it is impossible even to increase the singlet fraction. As shown in Ref. [15] the noncollective distillation is impossible for mixed states of full rank.

Here we address, in particular, the following question: does there exist a state which is not noncollectively distillable, but still is quasidistillable? To answer this question we determine (in Sec. VI) the class of the states, which can be distilled (we generalize the consideration taking into account the case of the singlet of less dimension than the dimension of the system: this corresponds to fidelity teleportation with the average calculated over inputs restricted to less Hilbert space). In Sec. VII we provide example of state that does not belong to this class, but can be quasidistilled. Then we slightly modify the state so that the new one cannot be quasidistilled having a threshold for singlet fraction, which cannot be exceeded. Thus for sufficiently high required fidelity of teleportation, Alice and Bob have no chance to obtain the fidelity for this state.

II. STATES AND CHANNELS

In this section we prove formally that the set of channels Λ on the set of d-dimensional states is isomorphic to the set of density matrices ϱ acting on the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2=C^d\otimes C^d$ satisfying $\mathrm{Tr}_{\mathcal{H}_2}\varrho=I/d$ (the partial trace over the second system gives maximally mixed state). By channel we mean here completely positive, trace-preserving map [21]. Given a channel Λ one can ascribe to it a state ϱ_Λ, sending half of singlet state down the channel [16,17]

$$\varrho_\Lambda=(I\otimes\Lambda)P_+. \qquad (2)$$

Such a state must have the first reduction the same as the state P_+ (i.e., I/d) since, due to impossibility of action at a distance the local reduced density matrix of the remote half of P_+ cannot be changed by any local action performed on the other half (this is a ''physical'' version of the proof of this well-known fact [22]).

Now consider a given state ϱ, with first reduction equal to I/d. Following Ref. [18], consider the spectral decomposition of the state

$$\varrho=\sum_{i=1}^{d^2} p_k|\psi_k\rangle\langle\psi_k|. \quad (3)$$

Let, e.g., $\psi_1=\Sigma_{i,j=1}^{d}c_{ij}|i\rangle\otimes|j\rangle$. Then it can be represented as

$$\psi_1=I\otimes V_1\psi_+, \quad (4)$$

where $\langle i|V_1|j\rangle=\sqrt{d}c_{ij}$. Defining analogously V_k for $k=1,\ldots,d^2$ we obtain

$$\varrho=\sum_{k=1}^{d^2} p_k I\otimes V_k P_+ I\otimes V_k^\dagger=(I\otimes\Lambda)P_+, \quad (5)$$

where $\Lambda(\sigma)=\Sigma_k p_k V_k\sigma V_k^\dagger$. Of course the map Λ defined in this way is completely positive, since it is of the common Stinespring form [23]. It is also trace-preserving. To show it we only need to check whether $A\equiv\Sigma_k p_k V_k^\dagger V_k=I$ [24]. Since the first reduction of our state ϱ is I/d, we have for any operator B,

$$\begin{aligned}\mathrm{Tr}\,B&=d\mathrm{Tr}(\varrho B\otimes I)=d\sum_k p_k\mathrm{Tr}(P_+B\otimes V_k^\dagger V_k)\\&=d\,\mathrm{Tr}(P_+B\otimes A).\end{aligned} \quad (6)$$

Now using the property that $C\otimes IP_+=I\otimes C^TP_+$ for any operator C and the fact that the reduction of the singlet is I/d, we obtain

$$\mathrm{Tr}\,B=\mathrm{Tr}\,B^TA=\mathrm{Tr}\,A^TB \quad (7)$$

for any B. This implies that $A^T=I$; hence, of course, also $A=I$.

Finally one should know that the channel Λ is determined uniquely. Suppose that there are two maps Λ and Λ' that produce the same state ϱ so that we have

$$[I\otimes(\Lambda-\Lambda')]P_+=0. \quad (8)$$

Denote the difference $\Lambda-\Lambda'$ by Γ. We will now show that Γ must be equal to 0. Indeed, consider the operator basis constituted by the following operators $P_{ij}=|i\rangle\langle j|$. In that basis we have $P_+=1/d\Sigma_{ij}P_{ij}\otimes P_{ij}$. Substituting it into formula (8) we obtain

$$\sum_{ij} P_{ij}\otimes\Gamma(P_{ij})=\sum_{ijkl}\gamma_{ijkl}P_{ij}\otimes P_{kl}=0, \quad (9)$$

where γ_{ijkl} are matrix elements of Γ in the basis P_{ij}. Since the operators $P_{ij}\otimes P_{kl}$ also constitute basis then it follows that all γ_{ijkl} must vanish so that Γ must be 0 operator. Then, we have shown that if two maps give rise to the same state ϱ then they must be equal, so that formula (2) determines Λ uniquely. Thus it constitutes an affine *isomorphism* between channels and states of maximally mixed reduced density matrix on one of the subsystems.

Having proved the isomorphism, given a state ϱ we will denote by Λ_ϱ the unique channel satisfying Eq. (2). Conversely, given a channel Λ we ascribe to it a state ϱ_Λ also by means of the formula (2). Note that so far this isomorphism has been established here only between channels, i.e., completely positive trace-preserving maps and quantum states with one of subsystems being completely mixed. One can easily see that the isomorphism can be extended to all states if one abandons the condition of preserving the trace. Then we have the one-to-one correspondence between the set of all states and the set of all completely positive maps. But we would like to stress here that we refer to a completely positive map as to a channel only if it is trace-preserving.

Let us now discuss the physical sense of the considered useful mathematical equivalence. If Alice and Bob are connected via a channel Λ then they can create state ϱ_Λ by sending half of the singlet down the channel. However, if they initially share the state ϱ_Λ, then can they say they dispose a channel Λ? As one knows, applying teleportation protocol, they, in fact, obtain some quantum channel. It remains an open question whether there exists some teleportation procedure, which *reproduce* the channel Λ [16]. It is highly probable that, in general, sending half of the singlet down the channel causes some *irreversible* loss of the capacity. Thus, the mathematical equivalence would not imply the physical one, in general. One knows that in some cases there is also a physical equivalence. Namely, for the two-qubit Werner state the corresponding channel (quantum depolarizing channel) can be retrieved by applying the standard teleportation protocol [16]. In the next section we will show that the same reversibility holds for generalized depolarizing channel (associated with the family of the $U\otimes U^*$ invariant states).

At the end of this section we define two parameters describing channels and states. We will use the same notation for states and for channels, but the parameters will, of course, have different interpretation. We will denote them by f and F. The first one is defined for channels in the following way,

$$f(\Lambda)=\int d\,\phi\langle\phi|\Lambda(|\phi\rangle\langle\phi|)|\phi\rangle, \quad (10)$$

where the integral is performed with respect to the uniform distribution $d\phi$ over all input pure states. It has the following interpretation: it is the probability that the output state $\Lambda(|\phi\rangle\langle\phi|)$ passes the test of being the input state ϕ, averaged over all input states. We will call it fidelity of the channel.

For the states, the parameter $f(\varrho)$ will denote the fidelity of the channel constituted by the standard teleportation via the state ϱ. Here we adjust the standard teleportation scheme [1] so that it provides perfect transmission for the state of the form (1) rather than for the true singlet one (as in original scheme).

The parameter F for states will denote simply the fraction of the singlet state given by $F(\varrho)=\langle\Psi_+|\varrho|\Psi_+\rangle$. For the channels, we will denote by $F(\Lambda)$ the entanglement fidelity of Λ [17] given exactly by $F(\varrho_\Lambda)$. Then if one sent half of the singlet down the channel, the entanglement fidelity says how close is the output state to the input one. By definition, F is invariant under the isomorphism (2).

III. FIDELITY OF TELEPORTATION AND SINGLET FRACTION

In this section we relate the optimal fidelity of teleportation via a given mixed state to the maximal singlet fraction attainable by means of *trace-preserving* LQCC operations. Our basic tool will be the alternative kind of twirling technique introduced in [18], i.e., random application of $U\otimes U^*$ unitary transformations. We also introduce *twirling channels*, which is operation on channels analogous to twirling states.

A. Teleportation

Suppose that Alice and Bob share a pair of particles in a given state ϱ acting on the Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B=C^d\otimes C^d$ and Alice has a third particle in unknown state $\psi\in\mathcal{H}_3=C^d$ to be teleported. The standard teleportation scheme has been described in the Introduction. The most general teleportation scheme is that Bob and Alice, given the particles in states described above, apply some trace-preserving (hence without selection of ensemble) LQCC operation T to the particles they share and Alice's particle. After the operation the state of Bob's particle (from the pair) is to be close to the unknown state of the third particle. The final state of Bob's particle is given by the following formula,

$$\varrho^{\psi}_{Bob}=\mathrm{Tr}_{3,A}[T(|\psi\rangle\langle\psi|\otimes\varrho)]. \tag{11}$$

This establishes a quantum channel $\Lambda_{T,\varrho}$ that maps the input state (the state of the third particle) onto the output one — the final state of Bob particle

$$\Lambda_{T,\varrho}(|\psi\rangle\langle\psi|)=\varrho^{\psi}_{Bob}. \tag{12}$$

This is a different way of ascribing a channel to the given state than the isomorphism (2). It is determined by each established teleportation protocol T, and in contrast to the isomorphism, it is in general not a one-to-one association: two different states ϱ and ϱ' can give the same teleportation channel $\Lambda_{T,\varrho}=\Lambda_{T,\varrho'}$ (this was discussed in Ref. [16]). The fidelity of a teleportation protocol T (via a given state ϱ) is the fidelity of the arising channel $f(\Lambda_{T,\varrho})$. According to our definition of $f(\varrho)$, given in the previous section, we have $f(\varrho)=f(\Lambda_{T_0,\varrho})$, where T_0 is the standard teleportation protocol. We must stress here that, in general, we do not know whether $\Lambda_\varrho=\Lambda_{T,\varrho}$ for some protocol T, even if the protocol could *depend* on ϱ (thus we do not know whether the isomorphism implies also the physical equivalence, see previous section). A particular example where it is the case (depolarizing channel) will be discussed subsequently (see Sec. III C).

Finally, let us consider the situation with restricted input, i.e., if $\dim\mathcal{H}_1=m<d$. Then we must work with some established embedding of the space $\mathcal{H}_1$ into the Bob space $\mathcal{H}_2$ (of course the very form of the embedding is here irrelevant) so that the formula (10) for fidelity is well defined.

B. Noisy singlet

Consider the one-parameter family of states [18] given by

$$\varrho_p=pP_+ +(1-p)\frac{I\otimes I}{d^2},\quad 0\leqslant p\leqslant 1 \tag{13}$$

We will call them noisy singlets. They are the most natural generalization of the 2×2 Werner states [8,2].

Let us now calculate the two parameters f and F. To calculate f consider the standard teleportation scheme via the state. The scheme produces fidelity 1 for the singlet state. For the completely random noise represented by the state $I\otimes I/d^2$, the average final state of Bob's particle after the teleportation procedure is equal to I/d and does not depend on the unknown state to be teleported. Then, in this case the fidelity amounts to $1/d$. Thus for the noisy singlet we obtain

$$f=p+(1-p)\frac{1}{d},\quad \frac{1}{d}\leqslant f\leqslant 1 \tag{14}$$

The parameter F amounts to

$$F=p+(1-p)\frac{1}{d^2},\quad \frac{1}{d^2}\leqslant F\leqslant 1 \tag{15}$$

The two parameters are related in the following way,

$$f=\frac{Fd+1}{d+1}. \tag{16}$$

We see that the noisy singlet is uniquely determined by any of those parameters (so one can use notation ϱ_p, ϱ_F, or ϱ_f if we use one of these parametrizations). The separability of the state ϱ_p can be characterized in a very clear way. Namely, it is [18] separable if and only if $0\leqslant p\leqslant 1/(d+1)$. This is equivalent to $1/d^2\leqslant F\leqslant 1/d$ and $1/d\leqslant f\leqslant 2/(d+1)$.

Recall that the noisy singlets are the only states invariant under $U\otimes U^*$ transformations [18] (here the star denotes complex conjugation). Any state ϱ, if subjected to $U\otimes U^*$ twirling, produces noisy singlet:

$$\mathcal{T}(\varrho)\equiv\int d\,U U\otimes U^*\varrho U^\dagger\otimes U^{*\dagger}=\varrho_F, \tag{17}$$

with $F=\mathrm{Tr}(\varrho P_+)$. Thus the singlet fraction is invariant under the twirling procedure.

C. Depolarizing channel

The depolarizing channel [16,18] is defined as follows,

$$\Lambda^{dep}_p(\sigma)=p\sigma+(1-p)\frac{I}{d}, \tag{18}$$

where σ is the state acting on C^d. From the formula (18) it follows that with probability p the channel does not affect the input state, while with probability $1-p$ it completely randomizes the input state.

Now if we apply the considered channel to half of the singlet we obtain the noisy singlet with the same parameter p. Thus, we have the equivalence

$$\Lambda_p^{dep}=\Lambda_{\varrho_p}. \tag{19}$$

Then, it follows that $F(\Lambda_p^{dep})=F(\varrho_p)$. Even more, we have *full* physical equivalence between depolarizing channel and noisy singlet: the channel can be reproduced by standard teleportation applied to the noisy singlet (this is compatible with similar observation in Ref. [16] for two-qubit case). Thus we have the complete set of equivalences,

$$\Lambda_p^{dep}=\Lambda_{\varrho_p}=\Lambda(T_0,\varrho_p). \tag{20}$$

To prove the last equality, consider the standard teleportation scheme of an unknown state through the state ϱ_p. As described in the previous subsection, with probability p, Bob will obtain the input state undisturbed, while with probability $1-p$ he will end up with totally mixed state I/d.

Thus any given input state σ is in the process $\mathcal{T}_0$ transformed into the state $p\sigma+(1-p)I/d=\Lambda_p^{\rm dep}(\sigma)$. Then it follows that also the parameter f is the same for Λ_p^{dep} and ϱ_p, so that the formulas (14) and (15) hold also for the depolarizing channel, and any of the parameters F and f determines it uniquely.

D. Twirling channels

Here we will introduce an operation over the channels, which is equivalent to $U\otimes U^*$ twirling of states. Namely, for any channel Λ, one can consider one constructed in the following way. Given the incoming particle, Alice subjects it to a random unitary transformation U, then sends it through the channel and informs Bob, which unitary was applied. Subsequently, Bob, who received the particle, applies the inverse transformation $U^\dagger$.

Now, we will show that as expected, the following lemma is true.

Lemma 1 Any channel Λ subjected to the twirling procedure becomes a depolarizing channel with the same F, i.e., we have

$$\mathcal{T}(\Lambda)=\Lambda^{dep} \tag{21}$$

with $F(\Lambda^{dep})=F(\Lambda)$.

Proof Let us first show that

$$\varrho_{\mathcal{T}(\Lambda)}=\mathcal{T}(\varrho_\Lambda), \tag{22}$$

which can be illustrated by means of the following commutative diagram,

$$\begin{array}{ccc} \varrho & \leftrightarrow & \Lambda_\varrho \\ \mathcal{T}\downarrow & & \downarrow\mathcal{T}, \\ \varrho_p & \leftrightarrow & \Lambda_p^{dep} \end{array} \tag{23}$$

where the arrows $\leftrightarrow$ denote the isomorphism (2). That the diagram commutes can be verified directly,

$$\begin{aligned}\varrho_{\mathcal{T}(\Lambda)}&=[I\otimes\mathcal{T}(\Lambda)]P_+\\ &=\int dU[I\otimes U^\dagger I\otimes\Lambda(I\otimes UP_+I\otimes U^\dagger)I\otimes U]\\ &=\int dU[I\otimes U^\dagger I\otimes\Lambda(U^T\otimes IP_+U^*\otimes I)I\otimes U]\\ &=\int dU[U^{*\dagger}\otimes U^\dagger(I\otimes\Lambda)P_+U^*\otimes U]\\ &=\int dU U^*\otimes U\varrho_\Lambda U^{*\dagger}\otimes U^\dagger=\mathcal{T}(\varrho_\Lambda).\end{aligned} \tag{24}$$

Here we used the identity $I\otimes AP_+=A^T\otimes IP_+$ [24] and the invariance of the Haar measure under Hermitian conjugation.

Now, applying the isomorphism (2) we obtain that the channel $\mathcal{T}(\Lambda)$ is equal to the channel corresponding to the state $\mathcal{T}(\varrho_\Lambda)$. Since the latter is a noisy singlet, then the channel must be a depolarizing one. Let us compute entanglement fidelity of $\mathcal{T}(\Lambda)$. We obtain

$$F[\mathcal{T}(\Lambda)]\equiv F(\varrho_{\mathcal{T}(\Lambda)})=F[\mathcal{T}(\varrho_\Lambda)]=F(\varrho_\Lambda)\equiv F(\Lambda), \tag{25}$$

where we used definition of F, its invariance under twirling states, and the equality (22). Hence the entanglement fidelity is invariant under the twirling of the channel.

We have also the following lemma.

Lemma 2 The channel fidelity f is invariant under twirling.

$$f(\Lambda)=f[\mathcal{T}(\Lambda)]. \tag{26}$$

Proof This follows from direct calculation of f. Namely the formula (10) can be rewritten as follows,

$$f(\Lambda)=\int d\ \ U\,\mathrm{Tr}[U|\phi\rangle\langle\phi|U^\dagger\Lambda(U|\phi\rangle\langle\phi|U^\dagger)], \tag{27}$$

where ϕ is an arbitrarily established vector and the integral is performed over the uniform distribution on the group $U(d)$ (proportional to the Haar measure). Consequently, we have

$$\begin{aligned}f[\mathcal{T}(\Lambda)]&=\int dU\,\mathrm{Tr}\Big[U|\phi\rangle\langle\phi|U^\dagger\int dVV^\dagger\Lambda(VU|\phi\rangle\\ &\quad\times\langle\phi|U^\dagger V^\dagger)V\Big]\\ &=\int dV\int dU\,\mathrm{Tr}[VU|\phi\rangle\langle\phi|U^\dagger V^\dagger\Lambda(VU|\phi\rangle\\ &\quad\times\langle\phi|U^\dagger V^\dagger)]=\int dV\int dU\,\mathrm{Tr}[U|\phi\rangle\\ &\quad\times\langle\phi|U^\dagger\Lambda(U|\phi\rangle\langle\phi|U^\dagger)]=\int dVf(\Lambda)=f(\Lambda)\end{aligned} \tag{28}$$

with V unitary and dV representing the integration over Haar measure.

These two lemmas produce the following result.

Proposition 1 For any channel Λ, one has

$$f(\Lambda)=\frac{F(\Lambda)d+1}{d+1}. \tag{29}$$

Proof The above equality is true for depolarizing channel, but as shown above both f and F are invariants of twirling, so that it must be also true for any channel.

E. Teleportation and singlet fraction

Here we will prove the main result of this section. Namely, we will relate the maximal fidelity of general teleportation scheme to the maximal possible fraction of the singlet attainable by means of trace-preserving LQCC operations. This will reduce the problem of optimal teleportation scheme for a given mixed state to the less complicated problem of increasing singlet fraction.

Theorem Let F_{max} be the maximal possible fidelity (overlap with the state P_+) which can be obtained from a given state ϱ by means of trace-preserving LQCC operation. Then the maximal fidelity f_{max} of teleportation via the state ϱ attainable by means of trace-preserving LQCC operations is equal to

$$f_{max}=\frac{F_{max}d+1}{d+1}. \tag{30}$$

Proof First we will prove that $f_{max}\leqslant(F_{max}d+1)/(d+1)$. Suppose we have a teleportation channel of fidelity f_{max}. From Proposition 1 it follows that entanglement fidelity F of that channel satisfies $f_{max}=(Fd+1)/(d+1)$. Then, sending half of the singlet down the channel, one produces a state with F satisfying relation (30) and F_{max} is at least equal to F.

Conversely, suppose that by trace-preserving LQCC operations a state ϱ' of maximal F has been obtained. Apply twirling to this state. The resulting state is of the form (13). Then the fidelity f of standard teleportation via this state satisfies the relation (16). Thus the standard teleportation will achieve the required f, which ends the proof.

Remark It can be seen that we can assume that the final singlet is less dimensional than the space $\mathcal{H}_1\otimes\mathcal{H}_2$. For example, one can consider maximal attainable fraction F_m of $m\times m$ singlet $|\Psi_+^m\rangle=(1/\sqrt{m})\Sigma_{i=1}^m|i\rangle|i\rangle$ where $m<d$. In this case, the formula (30) describes the optimal fidelity of teleportation for restricted input (i.e., if the unknown state comes from the Hilbert space of dimension m [20]).

IV. OPTIMAL TELEPORTATION FIDELITY FOR BOUND ENTANGLED STATES

Here we will apply the results of the previous section to the question of optimal fidelity of teleportation via bound entangled (BE) states. These are the ones that are entangled (are not a mixture of product states) but cannot be distilled [19]. Linden and Popescu [20] asked the question whether the BE states allow for better fidelity than the one of purely classical teleportation (i.e., the one where Alice and Bob have no prior entanglement so that the quantum information is sent via classical bits themselves). Positive answer to the same question but in the context of states allowing local hidden variable model [8] allowed us to obtain nonclassical features of the states [2]. Now, for a class of BE states, those authors obtained a negative answer, taking into account a more general teleportation scheme than the original one (they allowed for arbitrary von Neumann measurement of Alice).

Here, we are able to obtain the fully general answer. Namely, we will show that *the optimal fidelity of teleportation via arbitrary* BE *state is equal to the classical teleportation fidelity*.

Let us first derive the expression for the classical teleportation fidelity. Due to the proved theorem, it suffices to find maximal singlet fraction attainable via classical communication. This, however, reduces to determining the maximal possible singlet fraction of separable states. Of course, it cannot be greater than $1/d$, since states with $F>1/d$ are entangled (even free entangled, i.e., distillable — explicit distillation protocol has been provided in Ref. [18]). On the other hand, as mentioned in the previous section, the noisy singlet with $F=1/d$ is separable. Hence, applying the formula (30) we obtain that the best fidelity of teleportation via classical channel is given by $f_{cl}=2/(d+1)$.

Consider now the BE states. Since the states with $F>1/d$ are free entangled, then the maximal possible F for BE states is also $F=1/d$. Then the maximal fidelity of teleportation via a given BE state is also less than or equal to the f_{cl}. In fact it is equal, as having any BE state one can simply get rid of it and perform classical teleportation attaining the fidelity f_{cl}.

V. CONCLUSIVE TELEPORTATION AND INCREASING SINGLET FRACTION

Here we will consider the problem of the conditional increasing of fidelity of teleportation, i.e., conclusive teleportation [3]. By the results of Sec. III this question will be directly related to the problem of conditional increasing of singlet fraction.

Suppose that Alice and Bob has a pair in state for which the optimal teleportation fidelity is f_0. Suppose further that the fidelity is too poor for some Alice and Bob purposes. What they can do to change the situation is to perform the so-called conclusive teleportation. Namely, they can perform some LQCC operation with two final outcomes 0 and 1. After obtaining outcome 0 they fail and decide to discard the pair. If the outcome is 1, they perform teleportation, and the fidelity is now much better that the initial f_0. Of course, the price they must pay is that the probability of the success (outcome 1) may be small. The scheme is illustrated in Fig. 1.

A simple example is the following. Suppose that Alice and Bob share a pair in pure state $\psi=a|00\rangle+b|11\rangle$, which is nearly product (i.e., a is close to 1). Then the standard teleportation scheme provides a rather poor fidelity $f=\frac{2}{3}(a^3-b^3)/(a-b)$ [25,26]. However, Alice can subject her particle to a filtering procedure [9,4] described by the operation

$$\Lambda=W(\cdot)W^{\dagger}+V(\cdot)V^{\dagger} \tag{31}$$

with $W=\mathrm{diag}(b,a), V=\mathrm{diag}(a,b)$. Here the outcome 1 (suc-

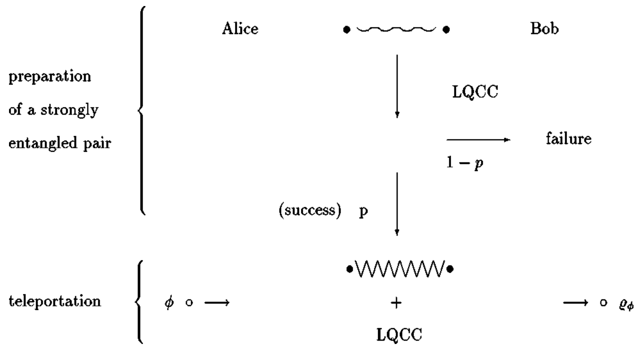

FIG. 1. Conclusive teleportation. Starting with a weakly entangled pair, Alice and Bob prepare with probability p a strongly entangled pair and then perform teleportation.

cess) corresponds to operator W. Indeed, if this outcome was obtained, the state collapses to the singlet one,

$$\tilde{\psi}=\frac{W\otimes I\psi}{||W\otimes I\psi||}=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle). \tag{32}$$

Then, in this case perfect teleportation can be performed. Thus, if Alice and Bob teleported directly via the initial state, they would obtain a very poor performance. Now, they have a small, but nonzero chance of performing perfect teleportation.

The main questions concerning the above scheme of conclusive teleportation are the following. Which states can provide perfect conclusive teleportation? More precisely, given a state ϱ, does there exist a nonzero probability p of success, for which Alice and Bob end up with pure singlet? Confining now to the class of states, which cannot be converted into pure singlets, one could ask: how large fidelity can be obtained? As we will see, *the fact that perfect conclusive teleportation is impossible does not, in general, mean that there is some fidelity threshold* $C<1$, *which cannot be exceeded*.

To analyze the above questions, we will apply the tools worked out in previous sections. Namely, there we have reduced the problem of optimal fidelity of teleportation to the problem of optimal increasing of singlet fraction. Let us now apply this result to the present situation. Namely, for a given probability p let f_p denote the maximal fidelity of conclusive teleportation with this probability of success. From the theorem it follows that $f_p=(F_d d+1)/(d+1)$ where F_d is the maximal singlet fraction attainable with probability p of success. So to obtain results concerning fidelity of teleportation we do not need to consider the conclusive teleportation scheme but the much simpler scheme of conclusive increasing singlet fraction. The scheme is illustrated in Fig. 2.

Again, Alice and Bob perform some LQCC operation with two outcomes. The outcome 0 denotes failure, while obtaining the other outcome, Alice and Bob have the final state of higher fraction of singlet than the initial one. Now the question concerning the fidelity of teleportation can be reformulated in the following way. For which states can the perfect singlet be obtained? For which states can arbitrarily high fraction of the singlet be obtained? Finally, for which states is there a threshold for singlet fraction that cannot be exceeded?

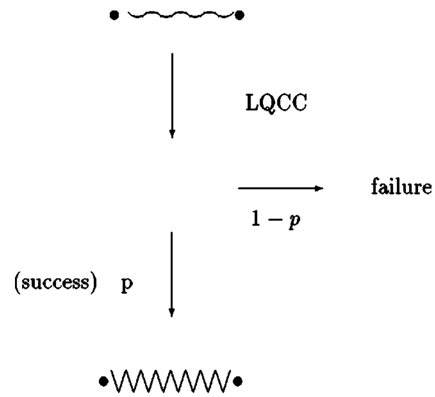

FIG. 2. Conclusive increasing singlet fraction. Alice and Bob with probability p of success obtain a state with higher singlet fraction than the one of the initial state.

Applying now the known results concerning increasing singlet fraction [15,14] we obtain that there is such a threshold for the states of full rank (i.e., with eigenvalues nonvanishing). However, we will provide the class of states of low rank, which do not allow for perfect conclusive teleportation, but still *arbitrarily high fidelity* can be obtained with nonzero probability (the latter depends on how high fidelity we would like to have). We will also provide a class of states of low rank for which we prove that the threshold exists. As we will see the proof is surprisingly complicated. Then the problem of determining whether a given state has an ultimate threshold for conclusive teleportation becomes highly nontrivial.

The above problems are closely related to the problem of distillation [10] by means of noncollective operations [14]. Namely, if for some state it is possible to obtain conclusively perfect singlets, then we have, in fact, a protocol of distillation, because we obtain a nonzero rate of produced singlets. Then such a state is noncollectively distillable. In the case where pure singlets cannot be produced, the noncollective operations cannot produce a nonzero asymptotic yield. If still an arbitrary high singlet fraction can be obtained, we will call the state noncollectively *quasidistillable* (as in this paper we deal only with noncollective protocols, so we will say briefly quasidistillable). The states that have the ultimate threshold of a fraction of the singlet we call nonquasidistillable. In subsequent sections we will define the notions more precisely, and we will consider the relevant examples.

VI. NONCOLLECTIVE $m\times m$ DISTILLATION

Let $P_+^m=|\Psi_+^m\rangle\langle\Psi_+^m|$. As mentioned in the Introduction, following the ideas presented in the papers [14,15] we use the following definition of noncollective distillation.

Definition One says that the $N\times M$ state ϱ can be $m\times m$ noncollectively distilled if there exist operators A,B such that

$$\frac{A\otimes B\varrho A^\dagger\otimes B^\dagger}{\mathrm{Tr}(A\otimes B\varrho A^\dagger\otimes B^\dagger)}=P_+^m. \tag{33}$$

We shall need other notions yet.

Definitions (i) If the state has the Schmidt decomposition

$$\Psi=\sum_{i=0}^{m-1} a_i|f_i'\rangle|f_i''\rangle, \quad a_i\neq 0 \tag{34}$$

then we shall call the number *m the Schmidt rank of state* Ψ and denote it by $r_s(\Psi)$.

(ii) We also shall call *product* $n\times m$ *projection* the product projection $P\otimes Q$ where the ranks of the projections P,Q are n,m, respectively. Hilbert subspace of the space $\mathcal{H}$ corresponding to any such projection we shall call *product* $n\times m$ *subspace*.

Here we simply characterize the states, which can be noncollectively distilled.

Proposition A given $N\times M$ state ϱ is $m\times m$ noncollectively distillable if there exists $m\times m$ product projection $P\otimes Q$ such that $P\otimes Q\varrho P\otimes Q$ is some pure (possibly unnormalized) projector of Schmidt rank m.

Proof Consider the given state ϱ that is noncollectively distillable. It means that there exists some $A\otimes B$ such that

$$A\otimes B\varrho A^\dagger\otimes B^\dagger=|\phi\rangle\langle\phi|, \tag{35}$$

and $|\phi\rangle$ is (possibly unnormalized) a maximally entangled vector of rank m. Note, that one can restrict to Hermitian A, B. It follows from two simple facts: (i) for any A,B there exist $\tilde{A},\tilde{B}$, and unitary U_A,U_B such that $A\otimes B=\tilde{A}U_A\otimes\tilde{B}U_B$ and (ii) product unitary transformation $U_A\otimes U_B$ does not change Schmidt rank.

Consider now Hermitian A,B satisfying Eq. (35). One can invert them on their supports: $A^{-1}A=P_A, B^{-1}B=P_B$ where P_A,P_B are projections onto the supports of A,B. Consider a vector given by $|\psi\rangle=A^{-1}\otimes B^{-1}|\phi\rangle$. As no product operator can increase the Schmidt rank [6] we have $r_s(\psi)\leq r_s(\phi)$. Since $|\phi\rangle=A\otimes B|\psi\rangle$, we obtain that, in fact, $r_s(\psi)=r_s(\phi)=m$. Also, by definition of P_A, P_B, A^{-1} and B^{-1} one has

$$|\psi\rangle\langle\psi|=P_A\otimes P_B\varrho P_A\otimes P_B. \tag{36}$$

The projector $P_A\otimes P_B$ must have at least rank $m\times m$ [otherwise $r_s(\psi)$ would have to be less than m]. If it has greater rank, then one can easily find (via Schmidt decomposition of ψ) the projectors $P_A'\otimes P_B'$ of rank $m\times m$ that still convert ϱ into $|\psi\rangle\langle\psi|$. Thus, if $P_A\otimes P_B$ has rank $m\times m$ then we take $P=P_A, Q=P_B$; otherwise, $P=P_A', Q=P_B'$.

Suppose now, conversely, that there exists $P\otimes Q$ of rank $m\times m$ such that $P\otimes Q\varrho P\otimes Q=|\psi\rangle\langle\psi|$ with $r_s(\psi)=m$. Then ψ is of the form (34). Now, taking $A=P, B=VQ$, with $\langle f_i''|V|f_j''\rangle=(1/a_i)\delta_{ij}$ [see Eq. (34)], one obtains that $A\otimes B\varrho A^\dagger\otimes B^\dagger$ is the maximally entangled state (of Schmidt rank m, of course). Note that the operator V plays the role of the suitable local filter [4,9]. Now, applying suitable product unitary transformation, we obtain (after normalization) the desired state P_+^m. This ends the proof.

Note that the above proposition provides the necessary and sufficient condition for noncollective distillation, which obviously does not automatically provide the best way to distill the state. From the proposition it follows directly that *no mixed state of* $d\times d$ *system can be converted into the maximally entangled state of the system.*

This is possible, however, for many states of the system $N\times M, M>N$. Simple examples of such states are the states of the form $p|\Psi_+\rangle\langle\Psi_+|+(1-p)\varrho'$ with reduced density matrix ϱ_2' of the matrix ϱ' orthogonal to the projector $P=\Sigma_{i=0}^{N-1}|i\rangle\langle i|$. The corresponding operators A,B turning such states into maximally entangled state are $A=I$ (identity operator on the first subsystem) and $B=P$.

VII. NONCOLLECTIVE QUASIDISTILLATION

One can ask whether it is possible by means of LQCC operations to make F_m arbitrary close to 1 with nonzero probability even if the noncollective distillation (33) is impossible. In fact, one can imagine a sequence of LQCC operations producing better and better F but with the probability tending to zero. Then the corresponding denominators of the expression (33) converge to zero, so that the hypothetical limiting operation does not exist. It corresponds to the existence of A_n, B_n such that

$$\frac{A_n\otimes B_n\varrho A_n^\dagger\otimes B_n^\dagger}{\mathrm{Tr}(A_n\otimes B_n\varrho A_n^\dagger\otimes B_n^\dagger)}\xrightarrow{n\to\infty} P_+^m. \tag{37}$$

The existence of such operators we shall call the noncollective *quasidistillation* as we allow corresponding sequence of probabilities $p_n=\mathrm{Tr}(A_n\otimes B_n\varrho A_n^\dagger\otimes B_n^\dagger)$ to decrease to zero. It means that if such noncollective operations were performed on many pairs of particles, then, unlike in the original distillation scheme [10], one would obtain zero rate [10,16] of pure singlet states P_+^m.

Now we are in position to present an example of the quasidistillation process. In this section we shall focus on the quasidistillation of the $d\times d$ system to the maximally entangled state $P_+=P_+^d$ (not to P_+^m with $m<d$). To be specific, we will deal with the case $d=3$.

The mixed state, which can exhibit arbitrary high fidelity F after noncollective local filtering, is the following [27]:

$$\sigma_F=FP_++(1-F)|01\rangle\langle 01|, \quad 0<F<1 \tag{38}$$

Following our remarks from previous section we know that the state, as a mixed one, cannot be distilled to the maximally entangled state P_+. According to the formula (30) this means that it is impossible to teleport with the fidelity $f=1$ via the state σ_F.

However, it can be easily seen that the operations:

$$A_n\equiv\mathrm{diag}\left[\frac{1}{n},1,1\right], \quad B_n\equiv\mathrm{diag}\left[1,\frac{1}{n},\frac{1}{n}\right], \tag{39}$$

allow for quasidistillation process (37). Indeed, then

$$A_n\otimes B_n\sigma A_n^\dagger\otimes B_n^\dagger=\frac{1}{n}\left[FP_++\frac{1-F}{n}\left|01\rangle\langle 01\right|\right], \tag{40}$$

which after suitable normalization leads to the desired result. The key point is that in the letter example the normalizing factor of the state converges to zero, i.e., we have an increase of the fidelity of the output state but, at the same time, the probability of obtaining this output state decreases to zero. Now, according to Eq. (30), it means that it is possible to teleport with the fidelity f arbitarily close to unity, although the value $f=1$ cannot be achieved.

Now we shall see how the above situation dramatically changes under seemingly not strong modifications of the input state (38). For this purpose consider the following state [27],

$$\varrho_F = FP_+ + \frac{(1-F)}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|),$$

$$0<F<1 \tag{41}$$

For convenience let us introduce the notation $\Theta_n(\sigma) = A_n \otimes B_n \sigma A_n^\dagger \otimes B_n^\dagger$, $\Theta(\sigma) = A \otimes B \sigma A^\dagger \otimes B^\dagger$, and $\langle \omega \rangle = \mathrm{Tr}(\omega)$. The same arguments as before lead to the conclusion that there are no operators A,B such that $\Theta(\varrho_F)/\langle \Theta(\varrho_F)\rangle = P_+$. We will show that for the considered state, unlike for σ_F, even the second, weaker form of distillation of entanglement is impossible. Let us assume, on the contrary, that Eq. (37) is possible. Then the output states can be written as convex combinations of two states, the second one of them being certainly *separable*,

$$\frac{\Theta_n(\varrho_F)}{\langle \Theta_n(\varrho_F)\rangle} = \left[F\frac{\langle \Theta_n(P_+)\rangle}{\langle \Theta_n(\varrho_F)\rangle}\right] \frac{\Theta_n(P_+)}{\langle \Theta_n(P_+)\rangle} + \left[(1-F)\frac{\langle \Theta_n(\sigma_+)\rangle}{\langle \Theta_n(\varrho_F)\rangle}\right]\frac{\Theta_n(\sigma_+)}{\langle \Theta_n(\sigma_+)\rangle}, \tag{42}$$

where $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$; the weights at both states have been put into square brackets [assume, for a while, that for all n, we have $\Theta_n(P_+) \neq 0$ and $\Theta_n(\sigma_+) \neq 0$]. Since the limit state must be a pure entangled state, the second weight must converge to zero. Otherwise, some of its subsequences would converge to the weight $w_2 > 0$ (recall that any bounded sequence has a convergent subsequence). As both the set of states and the set of separable states are compact it would lead to the conclusion that the limit state of the sequence (42) includes some separable state with the nonzero weight w_2. Such a state certainly could not be the pure entangled one. Thus the weight $(1-F)\langle \Theta_n(\sigma_+)\rangle / \langle \Theta_n(\varrho_F)\rangle$ must converge to zero. Together with the normalization condition it implies immediately that the weight at the state $\Theta_n(P_+)/\langle \Theta_n(P_+)\rangle$ must converge to unity. Hence, we have

$$\frac{\Theta_n(P_+)}{\langle \Theta_n(P_+)\rangle} \overset{n\to\infty}{\to} P_+ . \tag{43}$$

We also obtain that the ratio of the second weight to the first one must vanish in the limit of large n. This leads to the condition

$$\frac{\langle \Theta_n(\sigma_+)\rangle}{\langle \Theta_n(P_+)\rangle} \overset{n\to\infty}{\to} 0. \tag{44}$$

Subsequently, we shall show that satisfaction of the condition (43) is impossible if only Eq. (44) is satisfied. Let us introduce the notation $|a_i^n\rangle = A_n|i\rangle / \sqrt[4]{\langle \Theta_n(P_+)\rangle}$, $|b_i^n\rangle = B_n|i\rangle / \sqrt[4]{\langle \Theta_n(P_+)\rangle}$, $i=1,2,3$. Then the requirement (43) can be rewritten as

$$\Psi_n = \frac{1}{\sqrt{3}}(|a_0^n b_0^n\rangle + |a_1^n b_1^n\rangle + |a_2^n b_2^n\rangle) \overset{n\to\infty}{\to} \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{45}$$

But, at the same time, calculating that $\langle \Theta_n(\sigma_+)\rangle = \mathrm{Tr}[\Theta_n(\sigma_+)] = \frac{1}{3}\Sigma_{i=0}^2 ||A_n|i\rangle||^2 ||B_n|i\oplus 1\rangle||^2$ [here $x \oplus y = (x+y) \bmod 3$] we obtain via Eq. (44) that $\lim_{n\to\infty}\Sigma_{i=0}^2 |||a_i^n\rangle||^2 ||b_{i\oplus 1}^n\rangle||^2 = 0$. The latter is the sum of three non-negative sequences, so that any of them must converge to zero independently. Taking their square roots and multiplying them by each other we obtain, after suitable reordering, that

$$\lim_{n\to\infty}(||a_0^n||||b_0^n||)(||a_1^n||||b_1^n||)(||a_2^n||||b_2^n||) = 0. \tag{46}$$

Vanishing this limit, which is a product of three positive sequences, implies that at least one of them, say, $||a_0^n||||b_0^n||$, must converge to zero. But it means, turning back to Eq. (45), that $\lim_{n\to\infty}\Psi_n = \lim_{n\to\infty} 1/\sqrt{3}(|a_1^n b_1^n\rangle + |a_2^n b_2^n\rangle)$. This limit vector obviously cannot be singlet state [as Eq. (42) requires] because its Schmidt decomposition can consist of at most two terms (it can be easily seen by looking at the spectrum of the corresponding reduced density matrix). In this way we have obtained the required contradiction.

Finally, note that if the condition of the nonvanishing of $\Theta_n(P_+)$ and $\Theta_n(\sigma_+)$ for all n is not satisfied, the result is still valid. Indeed, if for all but finite number of components of the sequence $\Theta_n(P_+)$ vanish, then the limit state is separable (hence certainly cannot be quasidistilled). If the same holds for the sequence $\Theta_n(\sigma_+)$, then the state (42) consists only of the first term and the proof still applies [the limits containing $\Theta_n(\sigma_+)$ can be replaced by zeros]. If none of the above conditions is fulfilled, one can take a subsequence $\Theta_{n_k}(P_+)$ and $\Theta_{n_k}(\sigma_+)$ with all components nonvanishing and apply the proof to the subsequence.

Thus we have proved that for the considered state the process (37) is impossible. In other words, no LQCC operations performed on Eq. (41) state can increase the fidelity $F(\varrho_F)$ upon some $C<1$. Following the results of Sec. III, the conclusive teleportation of the spin-1 state through the state ϱ_F can produce the fidelity of transmission at most equal to $f_{max} = (Cd+1)/(d+1)$.

VIII. SUMMARY AND CONCLUSION

We have developed the correspondence between states and channels. In particular, we have exploited the equivalence between $U \otimes U^*$ invariant states and the generalized depolarizing channel to provide a relation between the optimal fidelity of teleportation and the maximal attainable singlet fraction. If the maximal fraction F of singlet obtained from the initial two spin-s state ϱ by means of trace-preserving LQCC operations is equal to F_{max}, then the best possible transmission fidelity f of teleportation via state ϱ is $f_{max} = (F_{max}d+1)/(d+1)$. This result was applied to the case of conclusive teleportation. It gives the answer to the question announced at the beginning of this paper. Namely, if Alice wants to teleport only if she knows that the transfer fidelity is better than f_0, then the state *must* admit an LQCC operation converting it (possibly with some probability) into the state with singlet fraction greater than $F_0=(1+1/d)f_0-1/d$.

It is interesting that the result does not depend on the kind of teleportation scheme and at the same time it involves a quantity F that measures the degree of overlap of the channel state with the singlet state. The quantity was originally associated with the standard teleportation scheme [26]. Moreover, in this scheme Alice performs her complete measurement (required as a step of the scheme) in a maximally entangled basis. It suggests that the standard teleportation scheme might be optimal.

As the concept of conclusive teleportation with good fidelity appeared to be connected with the possibility of increasing of the fidelity F, we have considered the problem of noncollective distillation of the mixed state of a two component system. It involves a conversion (by means of noncollective LQCC operations) of some $d\times n$ $(d\geqslant n)$ channel state into the maximally entangled state P_+ or its $m\times m$ counterparts $(m<d)$. We have shown that the first kind of conversion is suppressed for mixed states (this generalizes the result for 2×2 singlets [15]). Thus there is an important difference between $d\times d$ mixed and pure states as there are some pure states that can be converted in such a way (see [18]). The states for which the second kind of conversion is possible have been characterized and the possibility of conversion of some $d\times n$ $(d<n)$ states into the singlet state has been pointed out.

Then we have introduced the concept of quasidistillation which means, by definition, the possibility of making via LQCC process the quantity F arbitrary close to unity with nonzero probability but with the latter allowed to depend on the desired F of output state. We focused on the *noncollective* quasidistillation, which is the possibility of making the fidelity F of the state arbitrary close to unity in *noncollective LQCC process*. It turned out that sometimes, despite that a state cannot be noncollectively distilled, it allows for noncollective quasidistillation. The key point is that the probability of achieving the required fidelity is less the higher the fidelity is. The examples of states which are quasidistillable but not distillable via noncollective processes have been provided. They show that impossiblity of perfect teleportation sometimes does not imply the threshold for fidelity of the conclusive teleportation, i.e., sometimes it is still possible to teleport with f arbitrarily close to unity.

Subsequently, some modifications of those states have been considered that do not fall into the classes considered so far in Refs. [14,15] and for which the approach proposed in Ref. [15] cannot be applied due to the low rank of the matrix of the state. Nevertheless, we have shown by different techniques that those states are not even quasidistillable [28]. One of the main results of this paper is the conclusion that, in the noncollective LQCC operations regime, any state which cannot be quasidistilled *never* allows for conclusive teleportation with the fidelity better than some boundary value f_{max}. This result has been achieved by means of general approach including *all* possible teleportation schemes. It was possible due to the application of the isomorphism between states and channels, which seems to be a promising technique in quantum information theory.

ACKNOWLEDGMENTS

We thank Noah Linden and Sandu Popescu for sending us their manuscript on bound entanglement and teleportation [20] and for useful comments. The paper is supported by the Polish Committee for Scientific Research, under Contract No. 2 P03B 103 16. M. H. and P. H. kindly acknowledge the support from the Foundation for Polish Science.

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