Creation of relativistic positronium. II. Photoproduction cross sections including Coulomb corrections

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Previous Born approximation calculations of the photoproduction of singlet (para) positronium are extended to give the cross section to all orders in the atomic number Z, i.e., to include Coulomb corrections. At the same time the cross section for triplet (ortho) positronium production—which does not exist in Born approximation—is obtained to all orders in Z. The calculations and results are given in a form closely related to the well-known high-energy pair production cross section including Coulomb corrections. The relations to pair production are discussed. [S1050-2947(99)06309-X]

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I. INTRODUCTION

The cross section for the production of singlet (para) positronium by photons in the field of an atom, $\gamma+Z=\mathrm{Ps}+Z$, was obtained some time ago in the Born approximation [1,2]. In the light of recent interest in the experimental production of relativistic positronium beams, briefly described below, the present paper is a presentation of the complete production cross section to all orders in Z, the atomic number. We thereby also obtain the corresponding triplet (ortho) positronium production cross section, which can only be obtained by the exchange of at least two virtual photons with the atom, i.e., in the second Born approximation. It follows simply from total angular momentum conservation that the amplitude for production of singlet positronium is odd in Z, while the corresponding amplitude for triplet positronium production is even in Z.

The first and only experimental observation of relativistic positronium production was made by Alekseev *et al.* [3] almost 15 years ago, when one of the photons from a π^0 decay was converted to a Dalitz pair which in turn (which happens very rarely) was converted into a bound-electron–positron state, positronium. Close to 200 positronium particles were recorded. This very important experimental work initiated interest in positronium production, first theoretical over many years [4,5], then a proposal of experiments; actually there is now a proposal to produce relativistic positronium beams at the planned REFER facility at Hiroshima University [5].

The paper is organized in the following way: In Sec. II the singlet and triplet positronium cross-sections, differential in the positronium polar and azimuth angles, are given, and in Sec. III the effects of photon linear and circular polarization are obtained. In Sec. IV the total singlet and triplet cross sections are given, and the results are presented in a way which demonstrates the relations to the pair production cross section. In Sec. V the effect of general screening is taken into account. A relation connecting the Coulomb correction functions for positronium and pair production is derived in Sec. VI and is of importance for obtaining simple and accurate formulas for singlet and triple positronium production cross sections. Comparison of production of pairs to production of positronium is discussed. Energies and momenta are given in units of m_ec^2 and m_ec, respectively.

II. SINGLET (PARA) AND TRIPLET (ORTHO) POSITRONIUM PRODUCTION CROSS SECTIONS

The positronium production cross section, including Coulomb corrections, is obtained in the same way as for the case of the Born approximation in Ref. [1]: the pair cross section $d^5\sigma_{\mathrm{pair}}$ for equal electron and positron momenta is multiplied with the appropriate ratio of phase space factors and by the inverse squared positronium normalization constant $N_{p_s}=\alpha^3 m_e^3/8\pi n^3$ for the nth positronium energy state. As in our previous paper (hereafter referred to as I), we discuss positronium s-states.

From the exact high-energy pair production process [6], as summarized in Appendix A one finds the positronium production amplitude (A5):

$$\vec{J}_\perp=\frac{8\pi}{\omega}(1-F(q))\frac{\vec{u}\xi}{q^2}V(x)/V(1)$$

$$J_z=\frac{4\pi i a}{\omega}(1-F(q))\xi^2(2\xi-1)W(x)/V(1). \qquad (1)$$

By choosing appropriate spinor polarization combinations for singlet and triplet positronium one obtains from Eqs. (A1) and (A2) the production cross sections for the nth energy level:

$$d^2\sigma_s^n=16\pi\frac{Z^2\alpha^6}{m_e^2n^3}[1-F(q)]^2 \times\frac{u^3du}{q^4}\xi^2(1-(\vec{e}\cdot\hat{u})^2)[V(x)/V(1)]^2\frac{d\varphi}{2\pi}, \qquad (2)$$

$$d^2\sigma_t^n = 4\pi \frac{Z^4\alpha^8}{m_e^2 n^3} u\,du\,\xi^4(2\xi-1)^2(1+\vec{s}\cdot\hat{p}_{p_s}\hat{k}\cdot\vec{\xi}) \times [W(x)/V(1)]^2 \frac{d\varphi}{2\pi}, \tag{3}$$

where $\vec{p}_{p_s} = 2\vec{p}_1$ is the positronium momentum and $\vec{s}$ the positronium polarization vector, $\vec{q} = \vec{k} - \vec{p}_{p_s}$, $\vec{u} = \vec{p}_{p_s}^{\perp}/2$, $\xi = (1+u^2)^{-1}$, $x = 1 - q^2\xi^2$, $F(q)$ is the screening function and $\vec{\xi} = i\vec{e}\times\vec{e^*}$ is the circular photon polarization unit vector. In addition to the Coulomb corrected singlet, Born approximation cross section as given in I, here we have obtained the triplet cross section which cannot occur in the Born approximation because of angular momentum conservation.

III. EFFECTS OF PHOTON POLARIZATIONS

From Eq. (2) follows that for a linearly polarized photon the singlet positronium is preferably emitted in a plane perpendicular to the polarization plane, with an azimuth angular distribution

$$d^2\sigma_s(\vec{e}\cdot\hat{u} = \cos\varphi) \sim \sin^2\varphi, \tag{4}$$

with φ the angle between the emission $(\vec{k},\vec{p}_{p_s})$ plane and the polarization $(\vec{k},\vec{e})$ plane.

A circular polarization of the photon has an effect on the production of triplet positronium. A circular polarization $\vec{P}_c = \vec{\xi}P_c$ is directly transferred to the longitudinal positronium polarization P_{p_s}:

$$P_{p_s} = \frac{d^2\sigma_t(\vec{s}\cdot\hat{p}_{p_s}=1) - d^2\sigma_t(\vec{s}\cdot\hat{p}_{p_s}=-1)}{d^2\sigma_t(\vec{s}\cdot\hat{p}_{p_s}=1) + d^2\sigma_t(\vec{s}\cdot\hat{p}_{p_s}=-1)} = \hat{k}\cdot\vec{\xi}P_c. \tag{5}$$

It should be noted that the transverse positronium spin polarization is of the order m_ec^2/E_1, and does not appear in this high-energy calculation, as expected.

IV. TOTAL CROSS SECTIONS

The integrations of the cross sections over φ and u are performed as for pair production [6], and the integrations are greatly simplified by the fact that Coulomb corrections and screening effects are separable, as explained in Appendix A. In fact while the singlet cross section obviously has contributions from $u\sim 1$, which implies Coulomb corrections and from $u\sim 1/\epsilon$, which implies screening corrections, the triplet cross section has only contributions from $u\sim 1$ and screening effects are absent. Therefore $F(q)=0$ for the triplet cross section [Eq. (3)]. As for pair production we use the high-energy Thomas-Fermi-Molière screening.

In the following we shall keep close to the integrations for pair production reference [6]. However as we shall see, the integrals cannot in the case of positronium be performed in such an elegant form as in the case of pair production in the work of Davies, Bethe, and Maximon [7].

For singlet positronium, we sort out the Born approximation term $\sigma_{\rm sing}^{\rm Born}$ which was given in I, and write the total cross section in the form [8]

$$\sigma_s^n = \sigma_s^{n,\rm Born} + \frac{\pi}{4}\frac{Z^2\alpha^6}{m_e^2n^3}\left[\int_0^{1-y_1}\frac{dx}{\sqrt{x}}\frac{1+x}{1-x}[V(x)/V(1)]^2 - 2(2\ln 2 - 1) + 2\ln y_1\right], \quad (y_1 \ll 1), \tag{6}$$

where $\sigma^{\rm Born}$ contains all screening effects, [1], and the Coulomb corrections are contained in the last term of Eq. (6). This demonstrates completely the separation of screening effects and Coulomb corrections for positronium.

For the triplet positronium cross section there is no Born approximation term, Eq. (3) integrated over φ, and u is therefore

$$\sigma_t^n = \frac{\pi}{2}\frac{Z^4\alpha^8}{m_e^2n^3}\int_0^1 dx\,\sqrt{x}(1+x)(W(x)/V(1))^2, \tag{7}$$

with screening effects absent.

Since the sum over energy levels will be of more use to the experimentalist, we sum over all energy levels, giving

$$\sum 1/n^3 = \zeta(3) = 1.20205,$$

where $\zeta(p)$ is the Riemann ζ function.

The total pair production cross section was given in Ref. [7], with

$$f(Z) = \sum_{n=1}^{\infty}\frac{a^2}{n(n^2+a^2)}, \quad a = \alpha Z,$$

$$\sigma = \frac{28}{9}\frac{Z^2\alpha^3}{m_e^2}\left[\ln\frac{2\omega}{m_e} - \frac{109}{42} - f(Z)\right] \quad \text{(no screening)}, \tag{8}$$

$$\sigma = \frac{28}{9}\frac{Z^2\alpha^3}{m_e^2}\left[\ln(183\,Z^{-1/3}) - \frac{1}{42} - f(Z)\right] \quad \text{(complete screening)}. \tag{9}$$

We write the positronium cross sections in the same style, from Eqs. (6) and (7):

$$\sigma_s = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\frac{\omega}{m_e} - 1 - f_s(Z)\right] \quad \text{(no screening)}, \tag{10}$$

$$\sigma_s = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)[\ln(242\,Z^{-1/3}) - 1 - f_s(Z)] \quad \text{(complete screening)}, \tag{11}$$

$$\sigma_t = \pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)f_t(Z) \quad \text{(irrespective of screening)}, \tag{12}$$

TABLE I. The Coulomb correction functions $f(Z)$ [9], $f_s(Z)$, and $f_t(Z)$.

	Z	$f(Z)$	$f_s(Z)$	$f_t(Z)$
C	6	0.0023	0.0057	0.0047
Al	13	0.0107	0.0265	0.0218
Fe	26	0.0420	0.100	0.0828
Kr	36	0.0784	0.185	0.1504
Sn	50	0.144	0.325	0.2632
Pt	78	0.303	0.655	0.4952
Pb	82	0.332	0.705	0.538
U	92	0.395	0.815	0.595

where the Born terms are taken from I, and

$$f_s(Z)=\frac{1}{4}\left[2(2\ln 2-1)-\ln y_1 - \int_0^{1-y_1}\frac{dx}{\sqrt{x}}\,\frac{1+x}{1-x}(V(x)/V(1))^2\right],\quad (y_1\ll 1), \tag{13}$$

$$f_t(Z)=\frac{1}{2}a^2\int_0^1 dx\,\sqrt{x}(1+x)[W(x)/V(1)]^2. \tag{14}$$

The numerical values of $f(Z)$, $f_s(Z)$, and $f_t(Z)$, for low values of Z, are $a=\alpha Z\ll 1$,

$$f(Z)=\zeta(3)a^2,$$

$$f_s(Z)=[-4(1-\ln 2)+\tfrac{7}{2}\zeta(3)]a^2,$$

$$f_t(Z)=8(1-\ln 2)a^2. \tag{15}$$

For selected values of Z, numerical values of $f(Z)$, $f_s(Z)$, and $f_t(Z)$ are given in Table I.

V. ARBITRARY SCREENING

We use the Thomas-Fermi-Molière model as in Ref. [6],

$$\frac{1-F(q)}{q^2}=\sum_{i=1}^{3}\frac{\alpha_i}{\beta_i^2+q^2},$$

with

$$\alpha_1=0.10,\quad \alpha_2=0.55,\quad \alpha_3=0.35,$$

$$\beta_i=(Z^{1/3}/121)b_i,\quad b_1=6.0,\quad b_2=1.20,\quad b_3=0.30.$$

The result is rather simple:

$$\sigma_s=\frac{28}{9}\,\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\frac{\omega}{m_e}-1-f_s(Z)+\mathcal{F}(2m_e/\omega)\right], \tag{16}$$

with

TABLE II. Thomas-Fermi-Molière screening functions with $\sqrt{B_1}=6(Z^{1/3}/121)(\omega/2m_e)$.

$\sqrt{B_1}$	0.5	2.0	8.0	20.0	30.0	60.0	80.0	100.0
$-\mathcal{F}(2m_e/\omega)$	0.014	0.140	0.676	1.37	1.73	2.39	2.68	2.90

$$\mathcal{F}(2m_e/\omega)=-\frac{1}{2}\sum_{i=1}^{3}\alpha_i^2\ln(1+B_i)+\sum_{i=1}^{3}\sum_{\substack{j=1\\ i\neq j}}^{3}\alpha_i\alpha_j \times\left[\frac{1+B_j}{B_i-B_j}\ln(1+B_j)+\frac{1}{2}\right], \tag{17}$$

where $B_i=\beta_i^2(2m_e/\omega)^2$. Values of $\mathcal{F}(2m_e/\omega)$ are given in Table II.

VI. RELATIONS TO PAIR PRODUCTION

In Appendix B the relation

$$2f_s(Z)-f_t(Z)=2f(Z)-g(Z) \tag{18}$$

is derived. This makes it possible to obtain numerical values avoiding the logarithmic singularity $\ln y_1$ in the integral in $f_s(Z)$, by calculating $f_t(Z)$ and $g(Z)$ which are both free of singularities, and deducing $f_s(Z)$ from Eq. (18).

For low values of Z, from Eq. (15) one obtains the relation

$$2f_s(Z)-f_t(Z)=2.915f(Z), \tag{19}$$

and for larger values of Z one obtains the approximate relation $g(Z)=-0.97f(Z)$, which gives

$$2f_s(Z)-f_t(Z)=2.97f(Z), \tag{20}$$

which does not differ much from Eq. (19). This indicates that the relations between the three $f(Z)$ functions are approximately constants. One finds

$$f_s(Z)=2.25f(Z),\quad f_t(Z)=1.75f(Z), \tag{21}$$

with an error in the cross section for singlet positronium which is of the order of 1% or smaller. For triplet positronium the error is somewhat larger.

These results may be used to relate the positronium production cross sections approximately directly to the pair production cross section by writing, for no-screening equations (10) and (11),

$$\sigma_s=\pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\frac{\omega}{m_e}-1-2.25\,f(Z)\right],$$

$$\sigma_t=\pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)1.75f(Z), \tag{22}$$

with $\ln(\omega/m_e)$ replaced by $\ln(242Z^{-1/3})$ for complete screening. The total cross section for positronium production,

$$\sigma_s+\sigma_t=\pi\frac{Z^2\alpha^6}{m_e^2}\zeta(3)\left[\ln\frac{\omega}{m_e}-1-0.50f(Z)\right], \quad (23)$$

shows that the Coulomb correction effect is smaller when a bound state, the neutral particle positronium, is produced than when the separate electron and positron are produced [Eqs. (8) and (9)]. This is what would be expected, producing an almost neutral particle would show almost no final state charge effects. The Coulomb correction effect in Eq. (23) gives a picture of the creation process of positronium, which is a ''large'' particle, twice as large as the hydrogen atom, and therefore represents a charge distribution of the size of the interaction volume. It is interesting to note that the observed magnitude of the Coulomb correction to positronium creation gives an insight into the creation process.

APPENDIX A

Here we give a brief summary of the main results for pair production [6]. The exact high-energy pair production cross section, including photon, electron, and positron polarizations, is formulated in a convenient way in Ref. [6],

$$d^5\sigma_{\text{pair}}=\frac{1}{(2\pi)^4}\frac{e^2}{m_ec^2}\frac{\hbar a^2}{m_ec}\frac{\epsilon_2^2}{\omega}|\vec{A}\cdot\vec{e}|^2p_1^2dp_1d\Omega_1d\Omega_2, \quad \text{(A1)}$$

with $a=\alpha Z$ and

$$|\vec{A}\cdot\vec{e}|=\frac{1}{2}\omega^2\left(|\vec{J}|^2-\frac{4\epsilon_1\epsilon_2}{\omega^2}|\vec{J}\cdot\vec{e}|^2\right)(1-\vec{\zeta}_1\cdot\vec{\zeta}_2)$$
$$+\text{Re}[\omega^2\vec{J}\cdot\vec{\zeta}_1\vec{J}^*\cdot\vec{\zeta}_2$$
$$+\omega(\epsilon_1\vec{\zeta}_2+\epsilon_2\vec{\zeta}_1)\cdot\vec{J}\vec{J}^*\cdot(i\vec{e}\times\overrightarrow{e^*}). \quad \text{(A2)}$$

$|\vec{A}\cdot\vec{e}|$ is the amplitude of the pair production process, with $\vec{e}$ the photon polarization vector. Here $(\epsilon_1,\vec{p}_1)$ and $(\epsilon_2,\vec{p}_2)$ are the electron and positron energies and momenta, respectively, $(\omega,\vec{k})$ the photon energy and momentum, and $\vec{\zeta}_1$ and $\vec{\zeta}_2$ the electron and positron polarization unit vectors, respectively, defined by

$$\vec{\sigma}\cdot\vec{\zeta}_{1,2}|\vec{\zeta}_{1,2}\rangle=\zeta_{1,2}|\vec{\zeta}_{1,2}\rangle. \quad \text{(A3)}$$

Energies and momenta are here given in units of m_ec^2 and m_ec, respectively. In $|\vec{A}\cdot\vec{e}|^2$ we have kept (Ref. [6]), only the terms which are relevant for our further calculations. The amplitude vector is given by

$$\vec{J}=\vec{J}_\perp+\hat{k}J_z, \quad \text{(A4)}$$

where the components perpendicular and parallel to $\vec{k}$, $\vec{J}_\perp$, and J_z, respectively, are

$$\vec{J}_\perp=\frac{4\pi}{\omega V(1)}(1-F(q))\left\{\frac{1}{q^2}(\vec{u}\xi+\vec{v}\eta)V(x)\right.$$
$$\left.+ia\xi\eta(\vec{u}\xi-\vec{v}\eta)W(x)\right\},$$

$$J_z=\frac{4\pi}{\omega V(1)}(1-F(q))\left\{\frac{1}{q^2}(\xi-\eta)V(x)\right.$$
$$\left.+ia\xi\eta(\xi+\eta-1)W(x)\right\}, \quad \text{(A5)}$$

with $V(x)$ and $W(x)$ given by hypergeometric functions:

$$V(x)=F(ia,-ia;1;x),$$
$$V(1)=|\Gamma(1+ia)|^{-2}=\sinh\pi a/\pi a, \quad \text{(A6)}$$
$$W(x)=(1/a^2)dV(x)/dx=F(1+ia,1-ia,2,x).$$

Here $a=\alpha Z\approx Z/137$, $\vec{u}$ and $\vec{v}$ are the components of $\vec{p}_1$ and $\vec{p}_2$ transverse to $\vec{k}$, $\xi=(1+u^2)^{-1}$, $\eta=(1+v^2)^{-1}$, and $\vec{q}$ is the momentum transfer to the atom,

$$\vec{q}=\vec{k}-\vec{p}_1-\vec{p}_2. \quad \text{(A7)}$$

The variable $x=1-\xi\eta q^2$ has the property that for small values of q, $q\sim q_{\min}=\omega/2\epsilon_1\epsilon_2\ll1$, $x=1$ and the cross section is given by the Born approximation, as can be seen from Eq. (2): the dependence on a drops out beyond the leading order for $x=1$. In this region, where screening may be important, Coulomb corrections are absent. Conversely, for larger values of q, $q\sim1$, where Coulomb corrections are important, screening effects are negligible. This shows that Coulomb corrections and screening effects are separable. Accordingly, we have included the screening effect in Eq. (2) by multiplying with the screening factor $1-F(q)$.

APPENDIX B

The differential equation for the hypergeometric function $V(x)=F(ia,-ia;1;x)$ [10],

$$(1-x)\frac{d}{dx}\left(x\frac{dV}{dx}\right)=a^2V,$$

gives, with $dV/dx=a^2W$,

$$\frac{1}{1-x}V^2(x)+a^2xW^2(x)=\frac{d}{dx}(xV(x)W(x)).$$

Multiplication with $(1+x)/\sqrt{x}$ and integration directly give the functions f_s and f_t:

$$-4f_s(Z)+2f_t(Z)=-V^2(1)\int_0^{1-y_1}\frac{dx}{x}\frac{1+x}{1-x}$$
$$+(1+x)\sqrt{x}V(x)W(x)_0^{1-y_1}$$
$$+\int_0^1dx\frac{1-x}{\sqrt{x}}V(x)W(x).$$

Using $W(1-y_1)=V(1)(\ln y_1+2f(Z))$, $y_1 \ll 1$ [7], one finds the desired relation

$$2f_s(Z)-f_t(Z)=2f(Z)-g(Z), \tag{B1}$$

with

$$g(Z)=1-2\ln 2+\frac{1}{4V^2(1)}\int_0^1 \frac{dx}{\sqrt{x}}(1-x)V(x)W(x), \tag{B2}$$

a function, like $f_t(Z)$, free of the logarithmic singularity $\ln y_1$, and therefore convenient for numerical work.

[1] H.A. Olsen, Phys. Rev. D **33**, 2033 (1986). In the following this is referred to as I.

[2] V.L. Lyuboshits, Yad. Fiz. **45**, 1099 (1987) [Sov. J. Nucl. Phys. **45**, 682 (1987)].

[3] G.D. Alekseev *et al.*, Yad. Fiz. **40**, 139 (1984) [Sov. J. Nucl. Phys. **40**, 87 (1984)].

[4] Yu. L. Pivovarov, Yu. P. Kunashenko, I. Endo, and T. Isshiki, Nucl. Instrum. Methods Phys. Res. B (to be published).

[5] E. Holvik and H.A. Olsen, Phys. Rev. D **35**, 2124 (1987) (electron-nucleus production); Yu.P. Kunashenko and Yu.L. Pivovarov, Yad. Fiz. **51**, 627 (1990) [Sov. J. Nucl. Phys. **51**, 397 (1990)]; Nucl. Instrum. Methods Phys. Res. B **114**, 237 (1996); **115**, 390 (1996); G.I. Sandnes and H.A. Olsen, Phys. Rev. A **48**, 3725 (1993) (production in crystals).

[6] H.A. Olsen and L.C. Maximon, Phys. Rev. **114**, 887 (1959).

[7] H. Davies, H.A. Bethe, and L.C. Maximon, Phys. Rev. **93**, 788 (1954).

[8] We agree in Eqs. (6) and (7) with the results obtained by S.R. Gevorkyan, E.A. Kuraev, A. Shiller, V.G. Serbo, and A.V. Tarasov, Phys. Rev. A **58**, 4556 (1998).

[9] $f(Z)$ values are taken from J.W. Motz, H.A. Olsen, and H.W. Koch, Rev. Mod. Phys. **41**, 581 (1969), p. 623, Table 6.05.

[10] See, e.g., W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. (Springer-Verlag, Berlin, 1966), Chap. II.