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Generalized coherent states for the *d*-dimensional Coulomb problem

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In this paper a set of generalized coherent states for the *d*-dimensional Coulomb problem in coordinate representation are constructed. A coordinate transformation in hyperspherical space is used that maps the *d*-dimensional Coulomb problem into the *D*-dimensional harmonic oscillator and the generalized coherent states for the *d*-dimensional Coulomb problem are then obtained. This exactly soluble model can provide an adequate means for a quantum coherency description of the Coulomb problem in arbitrary dimensions, specifically in the special case of the hydrogen atom, in many theoretical and applied related fields such as in coherent scattering. [S1050-2947(99)05008-8]

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I. INTRODUCTION

In quantum mechanics, the standard coherent states in the coordinate representation describe nonspreading wave packets for the harmonic oscillator which were considered by Schrödinger as early as 1926 [1]. Somewhat later von Neumann, in his famous monograph [2], studied an important subsystem of coherent states, related to the regular cell partition of the phase plane for a system with one degree of freedom. Among early works in this area, the important paper by Glauber [3] should be mentioned. There, the concept of the coherent state was introduced and it was shown that coherent states provide an adequate means for a quantum description of coherent laser light beams [4].

On the other hand, the problems associated with the Coulomb problem and harmonic oscillator [5,6], together with the connection between the two in arbitrary dimensions, which has been studied from various viewpoints [7–15], have been discussed in detail by many authors. The purpose of this paper is to take advantage of the above connection in order to construct the generalized coherent states for the Coulomb problem in arbitrary dimensions in the coordinate representation. As a result in a special three-dimensional case, these generalized coherent states would be the coherent states for the hydrogen atom, which is a very important result.

The paper is organized as follows. In Sec. II the coherent states for the harmonic oscillator are constructed and some properties of these states are studied. In Sec. III the Schrödinger equation for the *d*-dimensional Coulomb problem and the *D*-dimensional harmonic oscillator in hyperspherical coordinates are solved and their energy eigenvalues and eigenfunctions are obtained. In Sec. IV the Schrödinger equation for the *d*-dimensional Coulomb problem is mapped onto the *D*-dimensional harmonic oscillator by a coordinate transformation in hyperspherical space and the connection between energy eigenfunctions of these two systems is obtained. Finally, in Sec. V, by using the above connection, the generalized coherent states for the *d*-dimensional Coulomb problem are constructed in the coordinate representation.

II. COHERENT STATES FOR THE HARMONIC OSCILLATOR

The coherent states introduced by Schrödinger and Glauber were developed for the simple harmonic oscillator. The Hamiltonian of this system in one dimension being

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2, \tag{1}$$

it can be rewritten as

$$H=(a^\dagger a+\tfrac{1}{2})\hbar\omega \tag{2}$$

by defining annihilation and creation operators as

$$a=\left(x+\frac{i}{m\omega}p\right)\left(\frac{m\omega}{2}\right)^{1/2}, \tag{3}$$

$$a^\dagger=\left(x-\frac{i}{m\omega}p\right)\left(\frac{m\omega}{2}\right)^{1/2}, \tag{4}$$

respectively. The eigenstates of the Hamiltonian, $|n\rangle$, belonging to the energy eigenvalues $E_n=(n+1/2)\hbar\omega$, where n is a non-negative integer, may then be written as

$$a^\dagger a|n\rangle=n|n\rangle. \tag{5}$$

The coherent state is constructed as a superposition of the energy eigenstates of the harmonic oscillator

$$|\alpha\rangle=\exp\left(-\frac{1}{2}|a|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{6}$$

where α is a complex number. These states are normally defined in three equivalent ways [16].

(i) They minimize the uncertainty relation $\Delta x\Delta p=\hbar/2$, and have $\Delta x=(\hbar/2m\omega)^{1/2}$.

(ii) They are eigenstates of the annihilation operator a,

$$a|\alpha\rangle=\alpha|\alpha\rangle. \tag{7}$$

(iii) They are created from the ground state by a displacement,

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle, \tag{8}$$

where $D(\alpha)$ is unitary displacement operator. As is well known, all three definitions are equivalent and yield the same results, namely, Eq. (6). In fact, usually one of them is adopted as the definition of the harmonic-oscillator coherent states. The completeness relation for the coherent state is

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = 1, \tag{9}$$

where the integration is over the whole complex α plane. We also have

$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\alpha-\beta|^2), \tag{10}$$

which means that the coherent state is not a complete orthogonal state. It is over-complete. Since the coherent state Eq. (6) is a nonstationary state it develops with time in a rather simple manner and, taking $\alpha(t=0) = \lambda e^{-i\theta}$, it follows that

$$\langle\alpha,t|x|\alpha,t\rangle = \left[2\lambda\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right]\sin(\omega t+\theta). \tag{11}$$

Identifying the constant in square brackets with the amplitude, the expectation value of the displacement in the coherent state behaves like that of a classical oscillator. In this sense the coherent state is called a classical state [12]. By solving Eq. (7), we find an explicit expression for the coherent state in the coordinate representation,

$$\langle x|\alpha\rangle = \left(\frac{\pi\hbar}{m\omega}\right)^{-1/4}\exp\left[\frac{i}{\hbar}(2m\omega)^{1/2}\alpha'' x\right] \times \exp\left\{-\frac{m\omega}{2\hbar}\left[x-\left(\frac{2}{m\omega}\right)^{1/2}\alpha'\right]^2\right\}, \tag{12}$$

where α' and α'' are the real and imaginary parts of α, respectively. The probability distribution of the wave function Eq. (12) is a Gaussian function for all possible values of α. This leads to the coherent state is a minimum uncertainty state, namely,

$$\Delta x \Delta p = \hbar/2. \tag{13}$$

The wave function of the coherent state which is given in Eq. (12) becomes the ground state harmonic-oscillator wave function centered around the origin when $\alpha = 0$. When α is a real number, it is a displaced ground state harmonic-oscillator wave function whose maximum is at $x = (2/m\omega)^{1/2}\alpha$. When α is complex, it is a ground state wave function whose origin is displaced to a complex value.

III. SOLUTION OF THE SCHRÖDINGER EQUATION FOR COULOMB AND HARMONIC-OSCILLATOR PROBLEMS IN ARBITRARY DIMENSIONS

The Schrödinger equation for the d-dimensional Coulomb problem is

$$\left(-\frac{\hbar^2}{2m}\nabla_d^2 - \frac{e^2}{r}\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{14}$$

where $\mathbf{r}$ is a d-dimensional position vector having Cartesian components $x_1, x_2, \ldots, x_d$ with magnitude $r = (\Sigma_{j=1}^d x_j^2)^{1/2}$ and the Laplacian ∇_d^2 given by

$$\nabla_d^2 = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}. \tag{15}$$

Because of the spherical symmetry of the problem it is convenient to introduce the hyperspherical coordinates, which are defined as follows [17]:

$$\begin{aligned} x_1 &= r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1},\\ x_2 &= r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-1},\\ x_3 &= r\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{d-1},\\ &\vdots\\ x_j &= r\cos\theta_{j-1} \sin\theta_j \cdots \sin\theta_{d-1},\\ &\vdots\\ x_{d-1} &= r\cos\theta_{d-2} \sin\theta_{d-1},\\ x_d &= r\cos\theta_{d-1}, \end{aligned} \tag{16}$$

where $d = 2,3,\ldots,$ $0 \leqslant r \leqslant \infty$, $0 \leqslant \theta_1 \leqslant 2\pi$, $0 \leqslant \theta_j \leqslant \pi$, and $j = 1,2,\ldots,$ $d-1$. As in three dimensions, we substitute the following in Eq. (14):

$$\psi(\mathbf{r}) = \mathcal{R}_{nl}(r) Y_{l_1,l_2,\ldots,l_{d-1}}(\theta_1,\theta_2,\ldots,\theta_{d-1}), \tag{17}$$

where $\mathcal{R}_{nl}(r)$ is the radial wave function and $Y_{l_1,l_2,\ldots,l_{d-1}}(\theta_1,\theta_2,\ldots,\theta_{d-1})$ is the generalized spherical harmonics, in which $l_{d-1} = 0,1,2,\ldots;$ $l_{d-2} = 0,1,2,\ldots,$ $l_{d-1};\ldots;$ $l_2 = 0,1,2,\ldots,$ l_3; $l_1 = -l_2, -l_2+1, \ldots,$ l_2-1, l_2. We obtain the radial part of the Schrödinger equation as

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{l(l+d-2)}{r^2}\right] - \frac{e^2}{r}\right\}\mathcal{R}_{nl}(r) = E\mathcal{R}_{nl}(r). \tag{18}$$

Equation (18) can be written as [18]

$$\left[\frac{d^2}{du^2} + \frac{d-1}{u}\frac{d}{du} - \frac{l(l+d-2)}{u^2} + \frac{k}{u} - \frac{1}{4}\right]\phi(u,d,n,l) = 0, \tag{19}$$

where $u = r/kr_0$, $r_0 = \hbar^2/2me^2$, $k = n + \frac{1}{2}(d-3)$, $l = 0,1,2,\ldots,$ $n-1$, and $n \geqslant l+1$.

The energy eigenvalues ϵ_n and their corresponding eigenfunctions $\phi(u,d,n,l)$ are given by

$$\epsilon_n = -\frac{\epsilon_0}{\left[n + \frac{1}{2}(d-3)\right]^2}, \tag{20}$$

where $\epsilon_0 = me^4/2\hbar^2$, and principal quantum number $n = 1,2,3,\ldots,$ and

$$\phi(u,d,n,l)=c(d,n,l)e^{-u/2}u^l L^{(2l+d-2)}_{n-l-1}(u), \qquad (21)$$

with the normalization constant

$$c(d,n,l)=r_0^{-d/2}[n+\tfrac{1}{2}(d-3)]^{-(d+1)/2}[\Gamma(n-1)]^{1/2} \times[2\Gamma(n+l+d-2)]^{-1/2}. \qquad (22)$$

Note that the Laguerre polynomials $L_n^{(\alpha)}$ are those defined in handbooks on mathematical functions and are not the more limited $L^{\alpha}_{n+\alpha}$ often used in discussions of the hydrogen atom eigenfunctions.

In a similar way, the radial equation of the D-dimensional harmonic oscillator is given by [17]

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{d^2}{dR^2}+\frac{D-1}{R}\frac{d}{dR}-\frac{L(L+D-2)}{R^2}\right] + \frac{1}{2}m\omega^2R^2\right\}\mathcal{R}_{nl}(R)=E\mathcal{R}_{nl}(R), \qquad (23)$$

which can be written as [19]

$$\left[\frac{d^2}{dU^2}+\frac{D-1\frac{d}{du}}{U}-\frac{L(L+D-2)}{U^2}-U^2+K\right]\Phi(U,D,N,L) =0, \qquad (24)$$

where $U=R/R_0$, $R_0=(m\omega/\hbar)^2$, $K=2N+D$, and $N\geqslant L$.

The energy eigenvalues E_N and their corresponding eigenfunctions $\Phi(U,D,N,L)$ are given by

$$E_N=\tfrac{1}{2}\hbar\omega(2N+D), \qquad (25)$$

$$\Phi(U,D,N,L)=C(D,N,L)e^{-U^2/2}U^L L^{(L+D/2-1)}_{N/2-L/2}(U^2), \qquad (26)$$

with the normalization constant

$$C(D,N,L)=R_0^{-D/2}\left[2\Gamma\left(\frac{N}{2}-\frac{L}{2}+1\right)\right]^{1/2} \times\left[\Gamma\left(\frac{N}{2}+\frac{L}{2}+\frac{D}{2}\right)\right]^{-1/2}. \qquad (27)$$

Having obtained the eigenfunctions for Coulomb and harmonic-oscillator problems in arbitrary dimensions, Eqs. (21) and (26), we will, in Sec. IV, set out to link the two cases by writing the d-dimensional Coulomb problem eigenfunctions in terms of the D-dimensional harmonic-oscillator eigenfunctions.

IV. MAPPING OF THE COULOMB PROBLEM ONTO THE HARMONIC OSCILLATOR IN ARBITRARY DIMENSIONS

The connection between the Coulomb and harmonic-oscillator problems, which has been studied from various viewpoints, has been discussed in detail by many authors. The main point in this section is the mapping of the d-dimensional Coulomb problem onto the D-dimensional harmonic oscillator. The map taking Eq. (19) into Eq. (24) is $u=U^2$. The appropriate relation between solutions Eqs. (21) and (26) with restricting D, N, and L to integers is [13]

$$\phi(u,d,n,l)=\Lambda\Phi(U,2d-2,2n-2,2l), \qquad (28)$$

where

$$\Lambda=\{\tfrac{1}{2}R_0^{2d-2}/r_0^d[n+\tfrac{1}{2}(d-3)]^{d+1}\}^{1/2}. \qquad (29)$$

The d- and n-dependent constant Λ arises because $\phi(u,d,n,l)$ and $\Phi(U,D,N,L)$ are normalized to unity in d and D dimensions, respectively. The identification Eq. (28) yields the solution

$$D=2d-2, \quad N=2n-2, \quad L=2l. \qquad (30)$$

It is a general feature of this mapping that the spectrum of the d-dimensional Coulomb problem is related to half the spectrum of the D-dimensional harmonic oscillator for any even integer D. However, the quantities in Eq. (30) have parameter spaces that are further restricted by the properties chosen for this mapping. From Eq. (30), we find that all states of the d-dimensional Coulomb problem with $n\geqslant1$ and $l\geqslant0$ can be mapped onto the appropriate harmonic oscillator with $N\geqslant0$ and $L\geqslant0$, except for $d=1$.

Now by using coordinates (16) and ignoring the constant Λ, we can write Eq. (28) in Cartesian space as [20]

$$\phi_{N_1,\ldots,N_D}=\prod_{j=1}^{D}(\beta/\sqrt{\pi}2^{N_j}N_j!)^{1/2}e^{-(\beta^2/2)x_j^2}H_{N_j}(\beta x_j), \qquad (31)$$

where $H_N(\beta x)$ is the Hermite polynomials of order N, and $\beta=(m\omega/\hbar)^{1/2}$. Thus the d-dimensional Coulomb problem wave function is expanded as a linear combination of simple harmonic-oscillator wave functions in Hermite polynomials.

V. GENERALIZED COHERENT STATES FOR COULOMB PROBLEM IN ARBITRARY DIMENSIONS

In this section we use the result from the preceding section and constitute the coherent states for the d-dimensional Coulomb problem. By using Eqs. (12) and (31), we are able to construct the generalized coherent states for the Coulomb problem in arbitrary dimensions in the x coordinates:

$$\psi(x;\alpha)=\left(\frac{\pi\hbar}{m\omega}\right)^{-D/4}\prod_{j=1}^{D}\exp\left[\frac{i}{\hbar}(2m\omega)^{1/2}\alpha_j''x_j\right] \times\exp\left\{-\frac{m\omega}{2\hbar}\left[x_j-\left(\frac{2}{m\omega}\right)^{1/2}a_j'\right]^2\right\}, \qquad (32)$$

which is the desired result. In fact the wave function given in Eq. (32) is the wave function of coherent state for the d-dimensional Coulomb problem in coordinate representation. This Coulombic coherent state becomes the ground state wave function centered around the origin in the D-dimensional x space when $\alpha=0$. When α is a real number, it is a displaced ground state Coulombic wave function whose maximum is at $x=(2/m\omega)^{1/2}\alpha$. When α is complex, it is a ground state wave function whose origin is displaced by a complex value.

We can also construct this generalized coherent state as a superposition of the Coulombic eigenstates

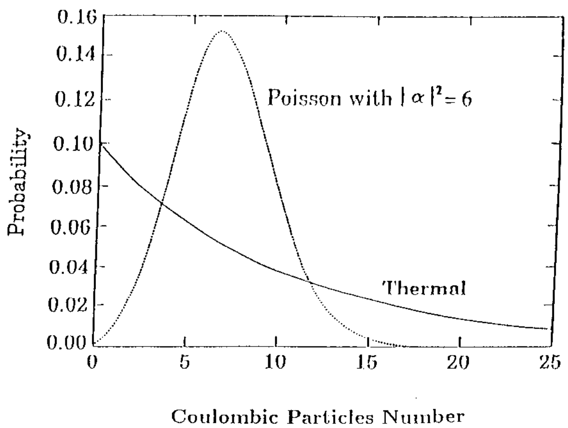

FIG. 1. Poisson and thermal distributions. It is expected that the Coulombic particle distribution in coherent scattering is Poissonian, whereas those without coherence are exponential distribution.

$$|[\alpha]\rangle = \exp\left(-\frac{1}{2}|[\alpha]|^2\right)\sum_{[n]=0}^{\infty}\frac{[\alpha]^{[n]}}{\sqrt{[n]!}}|[n]\rangle, \qquad (33)$$

where $[\alpha]=(\alpha_1,\alpha_2,\ldots,\alpha_D)$, $[n]=(n_1,n_2,\ldots,n_D)$, and $[n]!=n_1!n_2!\ldots n_D!$. The expression for the Coulombic coherent state given in Eq. (33) is normalized in the D-dimensional space. The probability of being in the $[n]$-Coulombic state is

$$p_{[n]}=\frac{|[\alpha]|^{2[n]}}{[n]!}\exp(-|[\alpha]^2). \qquad (34)$$

This means that the number of Coulombic particles in the coherent state has a Poisson distribution. This is the distribution expected from a coherent scattered beam of Coulombic particles. The cross section of this Poisson distribution and also thermal distribution is shown in Fig. 1. It is important to note that the three-dimensional representation of the above results is associated with the hydrogen atom itself. This in turn can provide an adequate means for a quantum coherency description of the hydrogen atom such as in the coherent scattering of the hydrogen atom. The coherent state for the hydrogen atom has already been obtained by the method of minimum uncertainty coherent states as the products of a radial wave function and an angular wave function in three dimensions [21,22]. But it is necessary to note that the generalized coherent state obtained in this paper first consists of both radial and angular coordinates in one equation, Eq. (32), and secondly it is valid for the Coulomb problem in arbitrary dimensions (except $d=1$), and thirdly it is represented in terms of the familiar harmonic-oscillator coherent states.

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