Fluctuations, time-correlation functions, and geometric phase

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The adiabatic approximation usually applies to a collection of lighter particles (the ''fast'' system) and heavier particles (the ''slow'' system), where the geometric phase naturally appears. With the help of the recently introduced gauge potential in the context of adiabatic Berry phase for open paths, we establish a fluctuation-correlation theorem by relating the quantum fluctuations in the generator of the parameter change (of the ''slow'' system) to the time integral of the quantum correlation function between the projection operator and force operator acting on the ''fast'' system. By taking a cue from linear response theory we relate the quantum fluctuation in the generator to the generalized susceptibility. The relation between the open-path geometric phase, diagonal elements of the quantum metric tensor, and the force-force correlation function are provided and the classical limit of the fluctuation-correlation theorem is discussed. [S1050-2947(99)08107-X]

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Fluctuations in a generic observable are inherent to all quantum systems that cannot be brought to zero, even in principle, unless the system is prepared in an eigenstate of the observable. In many-body systems the nature and the interrelation of quantum fluctuation and statistical fluctuation (which comes from time-correlation functions) is a topic of great interest. Given a composite system, the quantum fluctuation in an observable of a subsystem may drive the other subsystem towards equilibrium, and this information can be obtained by studying the time-correlation function of the later subsystem. One can describe the effect of coupling between the ''slow'' and the ''fast'' system by studying the time-correlation function between different operators pertinent to subsystems. Here, we use the concept of adiabatic geometric phase to bring out an important connection between the quantum fluctuation and time-correlation function of some observables of the ''fast'' system.

In the context of adiabatic theorem, Berry discovered the geometric phase [1] as an extra phase shift acquired by the wave function during cyclic variations of external parameters. Realizing its importance, this concept was further generalized to nonadiabatic situations by Aharonov and Anandan [2]. The Berry phase concept for noncyclic and nonunitary situations was generalized by Samuel and Bhandari [3] using geodesic closure rules. The present author has given a connection-form for noncyclic evolution of an arbitrary quantum system [4] without explicitly closing the open path by a geodesic. Further, we have generalized the geometric phase to the case of noncyclic, nonunitary, and non-Schrödinger evolutions [5] of quantum systems. Although the geometric phase can appear in a quite general context [6], purely related to the geometry of the Hilbert space [7], most of the application of the geometric phase theory deals with systems undergoing adiabatic evolutions. In the literature, the adiabatic Berry phase and its applications have been restricted to closed path evolutions. Therefore, we [8] have developed a theory of adiabatic Berry phase and Hannay angle for open paths and studied its semiclassical and classical limits. Most important and natural context, where adiabaticity holds is the system comprising of collection of electrons (the ''fast'' system) and nuclei (the ''slow'' system). Here, one applies the Born-Oppenheimer (BO) approximation [9] to solve the slow motion by integrating out the fast degrees of the system. Incidentally, the gauge potential (now called the Berry potential) was first highlighted by Mead [10] as a leading-order correction to usual BO approximation prior to Berry's observation. Recently, it has been shown by Berry and Robbins [11] that there are higher order corrections to the usual BO force called geometric magnetism and deterministic friction in a classical setting and half-classical setting. In a fully quantum mechanical setting, (when the underlying system is chaotic) the origin of damping of collective excitations in a finite Fermi system has been studied by Jain and Pati [12]. The interest in adiabatic geometric phase is still continuing, can be seen from recent papers by de Polavieja and Sjöqvist [13] and the generalization of adiabatic approximation to relativistic situations by Mostafazadeh [14].

In this paper, we show that the quantum fluctuation in the generator of the unitary operator (which induces the parameter change) is directly related to the time integral of the quantum correlation function between the projection operator and the force operator of the ''fast'' system. Further, invoking the ideas of linear response theory one can show that this quantum fluctuation is related to the generalized susceptibility. This fluctuation can be represented in terms of the diagonal elements of the quantum metric tensor, which in turn is related to the force-force correlation function. We provide a new expression for the adiabatic geometric phase when the slow coordinates undergo a noncyclic change. Also, we discuss the classical limit of the fluctuation-correlation theorem when the classical counterpart of the fast motion is both chaotic and integrable. The formal developments presented in this paper will have important applications in the areas such as nuclear physics and condensed matter physics and open up new avenues for studying classical limit of generalized quantum one-form for chaotic systems.

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Let us consider a composite, many-body system (''slow'' + ''fast'') and denote the ''slow'' and ''fast'' variables by $(\mathbf{R},\mathbf{P})$ and $(\mathbf{r},\mathbf{p})$, respectively. The Hamiltonian of the total system can be written as

$$H(\mathbf{r},\mathbf{p},\mathbf{R},\mathbf{P})=\frac{\mathbf{P}^2}{2M}+\frac{\mathbf{p}^2}{2m}+V(\mathbf{r},\mathbf{R})=h(\mathbf{r},\mathbf{p},\mathbf{R})+\frac{\mathbf{P}^2}{2M}, \tag{1}$$

where $h(\mathbf{r},\mathbf{p},\mathbf{R})=\mathbf{P}^2/2M+\mathbf{p}^2/2m$. Usually, one first solves for the fast Hamiltonian $h(\mathbf{r},\mathbf{p},\mathbf{R})$ for a fixed coordinate of the slow variable $\mathbf{R}$. The eigenvalue equation reads as

$$h(\mathbf{R})|n(\mathbf{R})\rangle=\epsilon_n(\mathbf{R})|n(\mathbf{R})\rangle, \tag{2}$$

where $|n(\mathbf{R})\rangle$ and $\epsilon_n(\mathbf{R})$ are the eigenstate and eigenvalue of the fast system that depends parametrically on the slow variable $\mathbf{R}$. The wave function of the composite system is

$$\Psi(\mathbf{r},\mathbf{R})=\sum_n \phi_n(\mathbf{R})\psi_n(\mathbf{r},\mathbf{R}), \tag{3}$$

with $\psi_n(\mathbf{r},\mathbf{R})=\langle\mathbf{r}|n(\mathbf{R})\rangle$ and $\phi_n(\mathbf{R})$ is the slow eigenfunction. When we integrate over the fast degrees of freedom, the effective Hamiltonian for the slow system contains a gauge potential $\mathbf{A}_n(\mathbf{R})$, whose flux gives the geometric phase. Thus, the effective Hamiltonian is given by [9,10]

$$H_{eff}=\frac{1}{2M}(\mathbf{P}-\hbar\mathbf{A}_n(\mathbf{R})^2)+\epsilon_n(\mathbf{R}), \tag{4}$$

where $\mathbf{A}_n(\mathbf{R})=i\langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle$ is the Berry potential. The presence of gauge potential leads to observable effects like a shift in quantum numbers and level splitting [15]. If we consider the time evolution of fast system, then during the cyclic change of the slow coordinate the wave function of the fast system acquires a geometric phase given by

$$\gamma_n(C)=i\oint_C\langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle\cdot d\mathbf{R}=\oint_C\mathbf{A}_n(\mathbf{R})\cdot d\mathbf{R}. \tag{5}$$

However, there could be situations where the slow coordinates need not undergo a cyclic variation. It has been shown by the author [8] that when the parameters are adiabatically changed along an arbitrary curve Γ, then the geometric phase is given by the line integral of a generalized gauge potential $\Omega_n(\mathbf{R})$,

$$\gamma_n(\Gamma)=i\int_\Gamma\langle\chi_n(\mathbf{R})|\nabla\chi_n(\mathbf{R})\rangle\cdot d\mathbf{R}=\int_\Gamma\Omega_n(\mathbf{R})\cdot d\mathbf{R}, \tag{6}$$

where $|\chi_n(\mathbf{R})\rangle$ is a ''reference-eigenstate'' introduced by the author, defined from the instantaneous eigenstate as $|\chi_n(\mathbf{R})\rangle=\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle/|\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle||n(\mathbf{R})\rangle$. The generalized gauge potential is related to the Berry potential as

$$\Omega_n(\mathbf{R})=\mathbf{A}_n(\mathbf{R})-\mathbf{P}_n(\mathbf{R}), \tag{7}$$

where $\mathbf{P}_n(\mathbf{R})$ is a new gauge potential, given by

$$\mathbf{P}_n(\mathbf{R})=\frac{i}{2|\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle|^2}[\langle n(\mathbf{R}(0))|\nabla n(\mathbf{R})\rangle \times\langle n(\mathbf{R})|-|n(\mathbf{R})\rangle\langle\nabla n(\mathbf{R})|)|n(\mathbf{R}(0))\rangle]. \tag{8}$$

The open-path adiabatic geometric phase is gauge invariant even if the nuclear coordinates do not come back to their original value. Because under a $U(1)$ gauge transformation of the fast eigenfunction $|n(\mathbf{R})\rangle$ is changed to $e^{i\alpha(\mathbf{R})}|n(\mathbf{R})\rangle$, whereas $\mathbf{A}_n(\mathbf{R})$ and $\mathbf{P}_n(\mathbf{R})$ transform as

$$\mathbf{A}_n(\mathbf{R})\to\mathbf{A}_n(\mathbf{R})-\nabla\alpha(\mathbf{R}),\quad \mathbf{P}_n(\mathbf{R})\to\mathbf{P}_n(\mathbf{R})-\nabla\alpha(\mathbf{R}), \tag{9}$$

and therefore the whole expression is gauge compensated.

Let us focus our attention on the generalized gauge potential $\Omega_n(\mathbf{R})$. In what follows, we express it in terms of the quantum fluctuation in some observable of the slow system. On defining a Hermitian operator $\mathbf{B}$ through the relation $|\nabla n(\mathbf{R})\rangle=i\mathbf{B}|n(\mathbf{R})\rangle$, we can express the potential $\mathbf{P}_n(\mathbf{R})$ as

$$\mathbf{P}_n(\mathbf{R})=-\frac{1}{2}\left[\frac{\langle n(\mathbf{R}(0))|\mathbf{B}|n(\mathbf{R})\rangle}{\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle}+\frac{\langle n(\mathbf{R})|\mathbf{B}|n(\mathbf{R}(0))\rangle}{\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle}\right]. \tag{10}$$

Using the fact [16] that the action of any Hermitian operator O on some state $|\Psi\rangle$ can be written as $O|\Psi\rangle=\langle O\rangle|\Psi\rangle+\Delta O|\Psi_\perp\rangle$, where $\langle O\rangle$ is the average and $\Delta O=\sqrt{(\langle O^2\rangle-\langle O\rangle^2)}$ is the uncertainty in the operator O, respectively. The state $|\Psi_\perp\rangle$ belongs to the orthogonal complement subspace of the Hilbert space, such that $\langle\Psi|\Psi_\perp\rangle=0$. For the adiabatic eigenstate and the operator $\mathbf{B}$, we have

$$\mathbf{B}|n(\mathbf{R})\rangle=-\mathbf{A}_n(\mathbf{R})|n(\mathbf{R})\rangle+\Delta\mathbf{B}|n_\perp(\mathbf{R})\rangle. \tag{11}$$

where $\langle n(\mathbf{R})|\mathbf{B}|n(\mathbf{R})\rangle=-\mathbf{A}_n$. With the help of the above equation, the potential $\mathbf{P}_n(\mathbf{R})$ can be expressed as

$$\mathbf{P}_n(\mathbf{R})=\mathbf{A}_n(\mathbf{R})-\mathrm{Re}\left(\frac{\langle n(\mathbf{R}(0))|n_\perp(\mathbf{R})\rangle}{\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle}\right)\Delta\mathbf{B}. \tag{12}$$

This shows that the Berry potential is just a part of the gauge potential $\mathbf{P}_n(\mathbf{R})$. Hence, the generalized potential can be expressed in terms of the fluctuation in the operator $\mathbf{B}$ as

$$\Omega_n(\mathbf{R})=\mathrm{Re}\left(\frac{\langle n(\mathbf{R}(0))|n_\perp(\mathbf{R})\rangle}{\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle}\right)\Delta\mathbf{B}. \tag{13}$$

This shows that the open-path geometric phase acquired by the fast eigenstate is related to the integral of the fluctuation in the operator $\mathbf{B}$.

On the other hand, the generalized gauge potential can also be expressed as a time-correlation function. To arrive at this, we relate it to the matrix elements of product of projection operators and force operator [the force operator is $-\nabla h(\mathbf{R})$]. Let us first introduce a complete set of eigenstates in the expression for $\mathbf{P}_n(\mathbf{R})$. Then we obtain

$$\mathbf{P}_n(\mathbf{R}) = -\mathrm{Im}\langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle - \mathrm{Im}\sum_{m\neq n}\frac{\langle n(\mathbf{R}(0))|m(\mathbf{R})\rangle}{\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle}\frac{\langle m(\mathbf{R})|\nabla h|n(\mathbf{R})\rangle}{(\epsilon_n(\mathbf{R})-\epsilon_m(\mathbf{R}))}, \tag{14}$$

where we have used the equality $\langle n|\nabla m\rangle = \langle n|\nabla h|m\rangle/(\epsilon_n - \epsilon_m)$ for $m\neq n$. Since $\mathbf{A}_n = -\mathrm{Im}\langle n(\mathbf{R})|\nabla n(\mathbf{R})\rangle$, we can write the generalized vector potential as

$$\Omega_n(\mathbf{R}) = \frac{1}{|\langle n(\mathbf{R}(0))|n(\mathbf{R})\rangle|^2} \times \mathrm{Im}\sum_{m\neq n}\frac{\langle n(\mathbf{R})|P_n(\mathbf{R}(0))P_m(\mathbf{R})\nabla h|n(\mathbf{R})\rangle}{(\epsilon_n(\mathbf{R})-\epsilon_m(\mathbf{R}))}, \tag{15}$$

where $P_n(\mathbf{R}(0))$ and $P_m(\mathbf{R})$ are instantaneous projection operators corresponding to the eigenstate $|n(\mathbf{R}(0))\rangle$ and $|m(\mathbf{R})\rangle$, respectively. By comparing the two expressions (13) and (15) for the gauge potential we obtain

$$\mathrm{Im}\sum_{m\neq n}\frac{\langle n(\mathbf{R})|P_n(\mathbf{R}(0))P_m(\mathbf{R})\nabla h|n(\mathbf{R})\rangle}{(\epsilon_n(\mathbf{R})-\epsilon_m(\mathbf{R}))} = \mathrm{Re}(\langle n(\mathbf{R}(0))|n_\perp(\mathbf{R})\rangle\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle)\Delta\mathbf{B}. \tag{16}$$

Further, we simplify the left hand side of the above expression using an integral representation of the energy denominator, viz. $1/(\epsilon_n - \epsilon_m) = 1/\hbar \lim_{s\to 0}\int_0^\infty dt\, e^{-st}\sin[(\epsilon_n - \epsilon_m)t/\hbar]$.

Therefore, we have

$$\frac{1}{\hbar}\lim_{s\to 0}\int_0^\infty dt\; e^{-st}\mathrm{Im}\sum_{m\neq n}\sin\left[(\epsilon_n-\epsilon_m)\frac{t}{\hbar}\right] \times\langle n(\mathbf{R})|P_n(\mathbf{R}(0))P_m(\mathbf{R})\nabla h|n(\mathbf{R})\rangle = \lambda(\mathbf{R})\Delta\mathbf{B}, \tag{17}$$

where $\lambda(\mathbf{R}) = \mathrm{Re}(\langle n(\mathbf{R}(0))|n_\perp(\mathbf{R})\rangle\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle)$ is a real scale factor. Then define a quantum correlation function between the instantaneous projection operator and the force operator as

$$Q(t) = \frac{1}{2}\langle n|(A_{-t}B + BA_{-t}) - (AB_t + B_tA)|n\rangle, \tag{18}$$

where $A = P_n(0)$, $B = \nabla h$ and A_{-t} is a time-evolved operator of A (with t replaced by $-t$) and $B_t = (\nabla h)_t$ is the time-evolved operator of ∇h at fixed $\mathbf{R}$. A similar quantum correlation function and its various moments have been studied in a different context [17], while discussing the quantum-classical discordance for chaotic systems. It can be shown that the quantum correlation function defined above is precisely what we have in the left hand side, i.e.,

$$Q(t) = -2\,\mathrm{Im}\sum_{m\neq n}\sin\left[(\epsilon_n-\epsilon_m)\frac{t}{\hbar}\right] \times\langle n(\mathbf{R})|P_n(\mathbf{R}(0))P_m(\mathbf{R})\nabla h|n(\mathbf{R})\rangle, \tag{19}$$

which is an almost periodic function, only if $\{\epsilon_n\}$ forms a complex sequence whereupon the convergence of such sums have to be treated nicely [18]. Thus, we arrive at our first result

$$-\frac{1}{2\hbar}\int_0^\infty dt Q(t) = \lambda\Delta\mathbf{B}, \tag{20}$$

where the convergence factor is left implicit.

To provide a physical meaning for the fluctuation in operator $\mathbf{B}$ we consider a family of Hamiltonians $h(\mathbf{R})$ which are unitarily related to a unparametrised Hamiltonian H [19], i.e.,

$$h(\mathbf{R}) = U(\mathbf{R})HU(\mathbf{R})^\dagger. \tag{21}$$

Then one can define the generators of the unitary operator $U(\mathbf{R})$ as $\mathbf{g}(\mathbf{R}) = i\hbar\nabla U(\mathbf{R})U(\mathbf{R})^\dagger$. These unitary operators $U(\mathbf{R})$ not necessarily constitute a group and provide a connection between the parameter dependent eigenbasis $|n(\mathbf{R})\rangle$ and parameter-independent basis $|n\rangle$ as defined through $|n(\mathbf{R})\rangle = U(\mathbf{R})|n\rangle$. Thus the generator of the parameter $\mathbf{g}$ is nothing but the operator $-\hbar\mathbf{B}$. Therefore, the quantity $\Delta\mathbf{B}$ is related to the fluctuation in the generator of the parameter dependent unitary operator. It is worth recalling that the usual Berry potential represents the average of the generator whereas the generalized gauge potential represents the fluctuation in the generator. Thus, Eq. (20) *relates the time integral of a quantum correlation function of the fast system to the quantum fluctuation of the generator of the unitary operator*, which is the central result of our paper. This may be called a *fluctuation-correlation theorem* analogous to the fluctuation-dissipation theorem in statistical mechanics.

Another meaning can be favored for the above expression. Note that the quantum correlation function $Q(t)$ defined above is actually a difference of two symmetrized time-correlation functions, i.e., $Q(t) = C_{AB}(-t) - C_{BA}(t)$, where $C_{AB}(t) = \frac{1}{2}\langle n|(A_{-t}B + BA_{-t})|n\rangle$ and $C_{BA}(t) = \frac{1}{2}\langle n|(AB_t + B_tA)|n\rangle$. Appealing to linear response theory of adiabatic many-body quantum system [20] we can define a generalized susceptibility in terms of the Laplace transform of the symmetrized time-correlation function, which is given by

$$\chi_{AB}(z) = \int_0^\infty e^{-zt}C_{AB}(t)dt. \tag{22}$$

With this idea Eq. (20) can be expressed, alternatively as

$$\lim_{z\to 0}[\chi_{AB}(z) - \chi_{BA}(z)] = -2\lambda\Delta\mathbf{P}. \tag{23}$$

This relates the quantum fluctuation to the susceptibility of the system in the limit $z\to 0$, which is statistical in nature.

Next, we express the open-path geometric phase in terms of the quantum metric tensor. The quantum metric tensor is an useful concept for studying the behavior of collective degrees of freedom of a many-body system. First we show that the uncertainty in $B_i, (i=1,\ldots,N)$ is nothing but the diagonal elements of the positive semidefinite quantum metric tensor g_{ij}, where g_{ij} is given by

$$g_{ij} = \mathrm{Re}\langle\partial_i n|\partial_j n\rangle - (i\langle n|\partial_i n\rangle)(i\langle n|\partial_j n\rangle). \tag{24}$$

The physical significance of the g_{ij} is that it defines the distance [21,22] between any two points along an arbitrary path in the parameter space corresponding to the evolution of the eigenstate in the Hilbert space. To see, this recall that the infinitesimal distance function between the adiabatic eigenstate $|n(\mathbf{R})\rangle$ and $|n(\mathbf{R}+d\mathbf{R})\rangle$ is given by

$$ds^2=[\langle\partial_i n|\partial_j n\rangle-(i\langle n|\partial_i n\rangle)(i\langle n|\partial_j n\rangle)]dR_i dR_j$$
$$=T_{ij}dR_i dR_j=g_{ij}dR_i dR_j, \qquad (25)$$

where $T_{ij}=g_{ij}+iv_{ij}$ is the Hermitian quantum geometric tensor, g_{ij} is the real symmetric tensor, and v_{ij} is real antisymmetric tensor. The quantum geometric tensor is manifestly gauge invariant. The antisymmetric tensor field is nothing but the phase two-form that gives the adiabatic Berry phase. The real symmetric tensor gives us the distance between the quantum states. But interestingly, the generalized vector potential (phase one-form) is related to the diagonal elements of the real symmetric tensor. The diagonal elements describe the uncertainties because $(\Delta B_i)^2=g_{ii}$ and off-diagonal elements describe the correlation between the operators B_i's. With the help of these metric structures we can recast the geometric phase as

$$\gamma_n(\Gamma)=\int\frac{\lambda}{|\langle n(\mathbf{R})|n(\mathbf{R}(0))\rangle|^2}\sqrt{g_{ii}(\mathbf{R})}\cdot dR_i. \qquad (26)$$

An immediate interpretation of the above result is that the open-path geometric phase for the ''fast'' system is the integral of the scaled symmetric tensor, during an arbitrary, adiabatic evolution of a ''slow'' quantal system.

Further, the diagonal elements of the real symmetric tensor can be expressed as a force-force correlation function, which is a very useful quantity in studying its classical limit when the fast motion is chaotic [17,25]. On writing g_{ii} as

$$g_{ii}=\mathrm{Re}\sum_{m\neq n}\frac{\langle n|\partial_i h|m\rangle\langle m|\partial_i h|n\rangle}{(\epsilon_n-\epsilon_m)^2}, \qquad (27)$$

and using the integral representation of the energy term, viz. $(\epsilon_n-\epsilon_m)^{-2}=-1/\hbar^2\int_0^\infty dt\, t\exp(i(\epsilon_n-\epsilon_m)t/\hbar)$ we have

$$g_{ii}=-\frac{1}{2\hbar^2}\int_0^\infty dt\, t[\langle n|(\partial_i h)_t(\partial_i h)|n\rangle+\langle n|(\partial_i h)(\partial_i h)_t|n\rangle, \qquad (28)$$

where $(\partial_i h)_t$ is the time-evolved operator. Since the geometric phase is related to the metric structure, it is also related to force-force correlation function, where the same components of the force operators are involved. However, we cannot say that the geometric phase is related to friction in pure quantum mechanics. Because friction in such situations is related to the symmetric part of the force-force correlation function (where different components of the force operators are involved). One can see in [24] that for chaotic quantum systems the phase two-form is related to the antisymmetric part of the force-force correlation function. Since for cyclic evolution of the adiabatic parameters our result [8] reduces to the result of Berry [1], it is natural to expect that the one-form is in some way related to force-force correlation function.

Before concluding this paper, we briefly discuss some issues related to the classical limits of this results. One can study the classical limit of the fluctuation-correlation relation, when the fast motion is chaotic. This is important, because the quantum fluctuations and correlations of large systems that have classical chaotic manifestations is of great interest. When the classical counterpart of the fast system is chaotic we assume that the mixing property holds. In this case the quantum expectation values of physical quantities correspond to the phase space average over the microcanonical distribution on an invariant energy surface [23–25], i.e.,

$$\langle n|O|n\rangle\rightarrow\langle O\rangle_E=\frac{\int d^N\mathbf{r}d^N\mathbf{p}O(\mathbf{r},\mathbf{p},\mathbf{R})\delta(E-h(\mathbf{R}))}{\int d^N\mathbf{r}d^N\mathbf{p}\delta(E-h(\mathbf{R}))}. \qquad (29)$$

The time-evolved operators in the quantum case correspond to the physical quantities at time evolved points on the trajectory generated by the fast Hamiltonian $h(\mathbf{R})$. Therefore, the classical analogue of fluctuation-correlation theorem can be expressed as

$$\int_0^\infty dt Q_c(t)=-2\hbar\lambda_c\langle(\mathbf{B}-\langle\mathbf{B}\rangle_E)^2\rangle_E, \qquad (30)$$

where $Q_c(t)=\int d^N\mathbf{r}d^N\mathbf{p}\delta(E-h(\mathbf{R}))(A_{-t}B-B_tA)/\int d^N\mathbf{r}d^N\mathbf{p}\delta(E-h(\mathbf{R}))$, is the classical valued correlation function and λ_c is the classical valued scale factor. Recently, it has been shown by Srednicki [26] that the time variation of the quantum fluctuation in the observable can be interpreted as an appropriate thermal fluctuation in that observable when the number of degrees of freedom N is large. Therefore, the fluctuation in the generator of the parameter change can be regarded as a thermal fluctuation in the classical statistical sense. The classical and quantum correlation functions can be useful in studying the various moments [17] and in analyzing the clash of limits $\hbar\rightarrow 0$ and $t\rightarrow\infty$, which may shed some light on the behavior of the geometric phase for classical chaotic systems. However, such studies are beyond the scope of this paper. One can remark that when the classical counterpart of the fast system is chaotic, then the mixing property would imply that for long times $(t\rightarrow\infty)$ the classical correlation function vanishes. This in turn implies that the fluctuation property of the slow system as seen by the ''fast'' system also vanishes. Therefore, to see the statistical fluctuation property of the ''slow'' system, study of long time behavior of the chaotic trajectory is not preferred.

If the classical counterpart of the fast system is integrable, then the quantum expectation values can be replaced by the torus average of the classical valued function over the surface of constant action $\mathbf{I}$, i.e.,

$$\langle n|O|n\rangle\rightarrow\langle O\rangle_I=\frac{1}{(2\pi)^N}\int d^N\theta\ \ O(\mathbf{r},\mathbf{p},\mathbf{R})$$
$$\times\delta(\mathbf{I}-\mathbf{I}(\mathbf{r},\mathbf{p},\mathbf{R})) \qquad (31)$$

and the fluctuation correlation theorem takes the form

$$\int_0^\infty dt Q_c(t) = -2\hbar\lambda_c\langle(\mathbf{B}-\langle\mathbf{B}\rangle_E)^2\rangle_I, \qquad (32)$$

where $Q_c(t) = 1/(2\pi)^N \int d^N\theta \;\; \delta(\mathbf{I}-\mathbf{I}(\mathbf{r},\mathbf{p},\mathbf{R}))(A_{-t}B - B_t A)$.

Although these discussions pertained to a collection of electrons (''fast'') and nuclei (''slow''), they are equally valid in the situations where one can separate the motion or energy scale to two regimes and BO approximation holds. We believe that the results found here will be applicable in other physical context, also. For example, recently, we [12] provided an explanation of the damping of collective excitations in finite-Fermi systems using the concept of adiabatic geometric phase and response functions when the system is fully chaotic. In such systems, the fluctuation and correlation theorem would be useful in quantifying various collective properties of the composite systems.

To summarize this paper, we studied the adiabatic geometric phase acquired by the ''fast'' system when the ''slow'' coordinates undergo a noncyclic variation, within the Born-Openheimer setting. This resulted in the formulation of a *fluctuation-correlation theorem*, which says that the time-integral of the correlation function of the ''fast'' system is proportional to the quantum fluctuation of the generator of the parameter change of the ''slow'' system as measured in the ''fast'' system. The fluctuation in the corresponding generator can be related to the generalized susceptibility of the ''fast'' system. Invoking the idea of geometric distance function, we have related the quantum one-form to the diagonal elements of the quantum metric tensor. The classical limit of the fluctuation-correlation theorem is discussed when the ''fast'' motion is chaotic and integrable. This work opens up the possibility of studying the spectral one-form $F(\epsilon)$ defined through $F(\epsilon) = \Sigma_n \delta(\epsilon - \epsilon_n)\Omega_n$ and in answering the classical limit of quantum one-form Ω_n for chaotic systems in the future.

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