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Application of Green's Functions to Radiative Cooperative Effects in Multiatom Systems*

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Spontaneous-radiation processes associated with a number of multiatom systems are studied. Green's-function techniques are used in conjunction with a model of N two-level atoms interacting with a quantized radiation field to investigate the assumption of the independence of the spontaneous-radiation properties of a given atom from the states of the other atoms in the system. It is shown that the natural linewidth of an excited atom in the presence of a deexcited atom is different from that of an isolated excited atom. For interatomic separations smaller than a critical separation, the two-atom system is best considered as a collective unit with regard to its spontaneous-radiation properties. The influence of these "radiative cooperative effects" is studied also for the systems of two initially excited atoms. Also the influence of radiative cooperative effects in the scattering of a photon by a system of two deexcited atoms is studied. It is shown that the frequency distribution of the scattered radiation exhibits a double-peak structure for interatomic separations smaller than $\frac{3}{4}\lambda$. The techniques of Green's functions are then applied to the study of radiative cooperative effects in several many-atom systems. It is shown that within the context of the model certain information is obtained exactly via the Green's-function techniques as applied to the many-atom problems of a single excited atom in the presence of $N \gg 1$ deexcited atoms and the scattering of a photon by a system of $N \gg 1$ deexcited atoms. The necessity of treating many-atom systems from a "collective" point of view is easily seen from our results.

I. INTRODUCTION

The emission of radiation by a single excited atom is one of the classic problems of quantum electrodynamics. An approximate quantum-mechanical solution was given quite early by Weisskopf and Wigner (WW).[1] The approach of WW has by now become a standard approximation method for computing the lifetime of an excited atomic state, not only for an isolated excited atom but also for systems of many atoms. That is, the usual treatment of the spontaneous radiation emitted by an extended system of atoms is based on the assumption that the individual atoms in the system emit radiation at a rate characteristic of the spontaneous emission rate of a single isolated atom. Consequently, implicit in these treatments is the assumption that the radiation emitted by the individual atoms is independent of the state of the other atoms in the system.

Spontaneous emission of radiation from many-atom systems has been considered by Dicke[2] who recognized the analogy between a system of two-level atoms and a system of spins, and used it to describe a many-atom system interacting through a common radiation field. He showed that under certain conditions the atoms may cooperate in a manner so as to emit radiation at a rate much larger than what would be expected assuming independent emission. Recently, in an impressive work, Rehler and Eberly[3] have reviewed and extended virtually all of the previous results of Dicke and others[4] with regard to the treatment of superradiance. On a much smaller scale, Stephen[5] and Hutchinson and Hameka[6] investigated the problem of a pair of two-level atoms interacting with each other via their common radiation field when one atom is initially excited and the other is deexcited. By extending the methods of the perturbation theory of Heitler[7] to this two-atom system they were able to draw the conclusion that the excited atom radiates at a rate different from that of an isolated atom. In a similar spirit Ernst and Stehle[8] and Ernst[9] "extended" the WW theory of the natural linewidth to a system of N identical nonoverlapping two-level atoms and discussed the time evolution of the system from various initial states for the atoms and radiation field.

In this paper we present a unified approach to many of the above problems utilizing Heisenberg picture Green's functions and the equation-of-motion method of solution. In contrast to most previous treatments, this technique does not resort to perturbation methods. The results of our approach are in general accord with the results derived by other methods, thus substantiating the use of previous perturbation schemes. A recent paper by Chang and Stehle[10(a)] has treated the problem of the resonant interaction of two neutral atoms using a Bethe–Salpeter equation. The solution of this problem using the equation-of-motion method is given in this paper and was carried out independently before this author was aware of this result.

We make use of a model Hamiltonian which incorporates all of the assumptions used by previous authors[2,3,5,6,8,9] to describe the interaction of a quantized many-mode radiation field with a system of N two-level atoms located at fixed points in space. Having established the model in Sec. II, we next give a definition of the kinds of Green's functions that are relevant to spontaneous-radiation processes and discuss briefly the physical information that may be obtained once a Green's function, or set of Green's functions, is known. We then proceed in Sec. III with the solution of problems in which there are two atoms centered around fixed positions in space. Since we are interested in the properties of spontaneous radiation that would be emitted by the two-atom system, we limit our considerations to those Green's function which serve to answer questions concerning the role of "radiative cooperative effects" in the modulation of the decay of one or both atoms. First we consider the case when one atom is initially excited while the other is deexcited. It is found that the initially excited atom does not decay at the rate characteristic of an isolated atom, in agreement with the results of others,[5,6,8–10(a)] but rather the decay rate is a sensitive function of the interatomic separation.

The next problem considered is that for which the initial state of the two-atom system is such that both atoms are excited. In this case, we find that we must resort to approximation methods, since the equations for the Green's functions are not exactly soluble. A graphical technique[11] is employed to find an expression for the two-atom Green's function which includes to lowest order the radiative cooperative effects. We close Sec. III by presenting a solution to the problem of the scattering of a photon by a system of two deexcited atoms.[12] Here we are interested in the frequency spectrum of the outgoing radiation as a function of the interatomic separation. We solve this problem and interpret the results for several values of the interatomic separation.

Section IV is devoted exclusively to systems with a large number of atoms. We generalize the one excited one deexcited problem to the case in which initially we have a single excited atom located at some point inside a system of N deexcited atoms and ask what the lifetime of the excited atom becomes under these conditions. Second, we generalize the scattering problem to the case when a photon is scattered by a system of N deexcited atoms. Finally, in Sec. V, we discuss the results of our approach with regard to its novelty and usefulness.

II. MODEL AND GREEN'S FUNCTIONS

A. Model

Consider a collection of two-level atoms centered around fixed positions $\vec{R}_\alpha$, $\alpha = 1, \ldots, N$ enclosed in a volume of finite extent but otherwise of arbitrary configuration. We assume that the atoms are separated from one another by distances sufficiently large such that there is no appreciable overlap of the spatial wave functions between any given pair of atoms. Thus we shall treat the atoms as distinguishable objects and neglect any requirements of symmetry. We shall represent the creation and destruction of an atom in a state $|n, \alpha\rangle$ centered about the point $\vec{R}_\alpha$ by the boson creation and destruction operators $b^\dagger_{n,\alpha}(t)$ and $b_{n,\alpha}(t)$ in the Heisenberg picture. These operators are assumed to satisfy the following equal time commutation relations:

$$[b_{n,\alpha}(t),\ b^\dagger_{n',\beta}(t)] = \delta_{nn'}\,\delta_{\alpha\beta}\,, \qquad (1)$$
$$[b_{n,\alpha}(t),\ b_{n',\beta}(t)] = 0, \quad [b^\dagger_{n,\alpha}(t),\ b^\dagger_{n',\beta}(t)] = 0$$

for all $\alpha, \beta = 1, \ldots, N$ and $n, n' = 1, 2$.

We take as our model Hamiltonian

$$H = \sum_{\alpha=1}^{N} \epsilon(b^\dagger_{e,\alpha} b_{e,\alpha} - b^\dagger_{g,\alpha} b_{g,\alpha}) + \sum_{\vec{k}\sigma} \hbar\omega_{\vec{k}} a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} + \sum_{\alpha=1}^{N} \sum_{\vec{k}\sigma} [g_{k\sigma} U_{\vec{k}}(R_\alpha) b^\dagger_{e,\alpha} b_{g,\alpha} a_{\vec{k}\sigma} + g^*_{k\sigma} U^*_{\vec{k}}(R_\alpha) b^\dagger_{g,e} b_{e,\alpha} a^\dagger_{\vec{k}\sigma}]\,. \qquad (2)$$

Here the subscripts e and g stand for the excited and ground states, respectively. The energy of the excited state is chosen to be $+\epsilon$, while the ground-state energy is chosen to be $-\epsilon$. The second sum represents the free electromagnetic field Hamiltonian. The operators $a^\dagger_{\vec{k}\sigma}$ and $a_{\vec{k}\sigma}$ represent, respectively, the creation and destruction of a photon with wave vector $\vec{k}$ and polarization σ. The $a_{\vec{k}\sigma}$ and $a^\dagger_{\vec{k}\sigma}$ satisfy the following boson commutation relations:

$$[a_{\vec{k}\sigma}, a^\dagger_{\vec{k}'\sigma'}] = \delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'},$$
$$[a_{\vec{k}\sigma}, a_{\vec{k}'\sigma'}] = 0, \qquad [a^\dagger_{\vec{k}\sigma}, a^\dagger_{\vec{k}'\sigma'}] = 0 \,. \tag{3}$$

The coupling between atoms and field is taken to include only "resonant" interactions between the atoms and the field. Terms proportional to $b^\dagger_e b_g \times a^\dagger_k$ and $b^\dagger_g b_e a_k$ are neglected as are terms that would be proportional to A^2. Thus our model is a version of what is historically called the "rotating wave approximation."[13] This model has been used as a basis for a number of semiphenomenological laser theories,[14-16] usually, however, in a version which does not explicitly include the dependence of the interaction upon the positions of the two-level atoms. This dependence is reflected in our model by the explicit appearance of the atomic centers R_α in the mode functions $U_{\vec{k}}(R_\alpha)$. In the present work we choose the mode functions $U_{\vec{k}}(R_\alpha) = e^{i\vec{k}\cdot\vec{R}_\alpha}$ appropriate to running waves. However, other choices may be made depending upon the geometry of the problem.

B. Green's Functions

We shall throughout this paper always work in the Heisenberg picture. Let $A_{\{\vec{k}\}}(t)$ be a product of arbitrary combinations of atom and photon destruction operators. Similarly, let $B_{\{\vec{k}'\}}(t)$ be an arbitrary combination of atomic and photon creation operators. We define a Green's function to be

$$G_{A_{\{\vec{k}\}}B_{\{\vec{k}'\}}}(t, t') = \langle 0 | T(A_{\{\vec{k}\}}(t) B_{\{\vec{k}'\}}(t')) | 0 \rangle, \tag{4}$$

where the state $|0\rangle$ is the vacuum of photons and atoms. That is

$$a_{\vec{k}\sigma}(t)|0\rangle = 0 \,, \qquad b_{n,\alpha}(t)|0 = 0 \,.$$

The symbol T refers to the usual time-ordering symbol:

$$T(X(t)Y(t')) = \theta(t-t')X(t)Y(t') \pm \theta(t'-t)Y(t')X(t) \,,$$

where the plus sign is chosen when the operators constituting X and Y satisfy boson commutation relations. The minus sign is chosen when the operators satisfy fermion commutation relations. Here we choose the plus sign since both the atomic and photon creation and destruction operators satisfy the boson commutation relations (1) and (3).

Green's functions have the following physical interpretation. From the definition (4), $G_{AB}(t, t')$ represents the probability amplitude that if a system of atoms is prepared in a state $|B\rangle_{t'} \equiv B(t')|0\rangle$ at time t', it will be found in a state $|A\rangle_t$ at time t. That is

$$G_{AB}(t, t') = \theta(t-t')\, {}_t\langle A | B\rangle_{t'} \,.$$

We shall employ the equation-of-motion method for the determination of the Green's functions in all the problems we consider. This method is by now a standard method for determining Green's functions.[17] The relation of our approach to the usual perturbation expansions is also well known.[18] We shall have occasion to consider this connection in relation to the exact solutions we obtain.

III. TWO-ATOM SYSTEMS

In the usual treatment of the emission of radiation from excited many-atom systems it is assumed that the individual atoms emit spontaneous radiation at a rate which is independent of the states of the other atoms in the system. This assumption is usually justified by arguing that as a result of the large distance between atoms and subsequent weak interactions, the probability of a given atom emitting a photon should be independent of the states of excitation of the other atoms. We shall demonstrate quantitatively in this section by studying the Heisenberg picture Green's functions pertinent to the problem that such an "independent emission hypothesis" is in principle incorrect. While such effects have been considered before,[19] it is noteworthy that our conclusions are based on a nonperturbative approach. We are able to demonstrate what radiative processes are taken into account in the description of these effects.

A. One Excited One Deexcited Atom System

Let us consider a two-atom system prepared initially so that one atom is excited and centered around a point $\vec{R}_1$ and a second atom is deexcited and is placed at some fixed point $\vec{R}_2$. We desire to find how this system propagates in time. In particular we are interested to see what effect the presence of the second deexcited atom has on the natural linewidth of the excited atom, relative to the natural linewidth of an isolated excited atom. This leads us to consider the following two-atom Green's function:

$$G_{ge}(t, t') = \langle 0 | T(b_{g,2}(t) b_{e,1}(t) b^\dagger_{g,2}(t')\, b^\dagger_{e,1}(t')) | 0 \rangle \,. \tag{5}$$

In order to find $G_{ge}(t, t')$, we employ the equation-of-motion method. That is, we compute $i\partial_t G_{ge}(t,t')$ using the Heisenberg equations of motion for the atom and photon creation and destruction operators.[20] This leads us to introduce a new Green's function. We find its equation of motion. The procedure is continued until the set of equations for the Green's functions closes. In the present model closure occurs, while in general it does not. After some manipulations we find

$$i\partial_t G_{ge}(t, t') = i\delta(t-t') + \sum_{\vec{k}\sigma} g_{k\sigma} e^{-i\vec{k}\cdot\vec{R}_1} A_{\vec{k}\sigma}(t, t') \,, \tag{6}$$

$$(i\partial_t + 2\epsilon - \omega_k)A_{\vec{k}\sigma}(t, t') = g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_2} G_{eg}(t, t') + g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_1} G_{ge}(t, t'), \quad (7)$$

$$i\partial_t G_{eg}(t, t') = \sum_{\vec{k}_1\sigma_1} g_{k_1\sigma_1} e^{-i\vec{k}\cdot\vec{R}_2} A_{\vec{k}_1\sigma_1}(t, t') , \quad (8)$$

where

$$A_{\vec{k}\sigma}(t, t') = \langle 0 | T(b_{g,1}(t) a_{\vec{k}\sigma}(t) b_{g,2}(t) b^\dagger_{g,2}(t') b^\dagger_{e,1}(t')) | 0\rangle , \quad (9)$$

$$G_{eg}(t, t') = \langle 0 | T(b_{g,1}(t) b_{e,2}(t) b^\dagger_{g,2}(t') b^\dagger_{e,1}(t')) | 0\rangle. \quad (10)$$

The two new propagators that have appeared may be interpreted as follows. The Green's function $A_{\vec{k}\sigma}(t, t')$ describes the process whereby the initially excited atom emits a photon, while $G_{eg}(t, t')$ is the amplitude that the initially excited atom at $\vec{R}_1$ becomes deexcited and the initially deexcited atom at $\vec{R}_2$ becomes excited. These propagators describe the only other physical processes that may occur within the context of our model.

Thus, in order to find $G_{ge}(t, t')$, we proceed to solve the equations of motion as follows. First we take the Fourier transform of all the propagators. The equations of motion become

$$zG_{ge}(z) = 1 + \sum_{\vec{k}\sigma} g_{k\sigma} e^{-i\vec{k}\cdot\vec{R}_1} A_{\vec{k}\sigma}(z) , \quad (11)$$

$$(z + 2\epsilon - \omega_k)A_{\vec{k}\sigma}(z) = g^*_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{R}_2} G_{eg}(z) + g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_1} G_{ge}(z) , \quad (12)$$

$$zG_{eg}(z) = \sum_{\vec{k}\sigma} g_{k\sigma} e^{-i\vec{k}\cdot\vec{R}_2} A_{\vec{k}\sigma}(z). \quad (13)$$

After some manipulation, we find that these equations may be solved exactly for $G_{ge}(z)$ (and all the other Green's functions). The result is

$$G_{ge}(z) = \left(z - \Sigma(z) - \frac{\Lambda^{12}(z)\Lambda^{21}(z)}{z - \Sigma(z) + i\mu} + i\delta \right)^{-1}. \quad (14)$$

Here

$$\Sigma(z) = \Delta(z) - \tfrac{1}{2} i\,\Gamma(z) , \quad (15)$$

with

$$\Gamma(z) = \tfrac{4}{3}(z^3/c^3)|d|^2 , \quad (16)$$

and

$$\Delta(z) = P\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\omega)}{z + 2\epsilon - \omega} . \quad (17)$$

We also have

$$\Lambda_{\alpha\beta}(z) = \int_0^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\omega)\chi(\omega R_{\alpha\beta}/c)}{z + 2\epsilon - \omega + i\eta} , \quad (18)$$

with

$$R_{\alpha\beta} = |\vec{R}_\alpha - \vec{R}_\beta| .$$

The function $\chi(x)$ is defined as

$$\chi(x) = 3[\sin(x) - x\cos(x)]/x^3 . \quad (19)$$

It should be noted that in solving the Fourier-transformed equation of motion for $G_{ge}(z)$, we have transformed all sums over wave vector to the continuum limit. Also it may easily be seen that $\chi(x)$ as defined in (19) is an even function of x. Consequently, $\Lambda^{12}(z) = \Lambda^{21}(z)$. We assume at this point that since we are interested in the behavior of $\Lambda^{\alpha\beta}(z)$ for values of z near physical values, i.e., for z around the unperturbed energy $z = 0$, when z is in the neighborhood of $z = 0$ the integral (18) has a pole near $\omega = 2\epsilon - i\eta$. If we further assume that $\Gamma(\omega)$ is also peaked about $\omega = 2\epsilon$ and falls off rapidly away from the point, then we may extend the lower limit of integration to $-\infty$ with only a small error.[21] Thus we may write (18) as

$$\Lambda^{\alpha\beta}(z) = \Delta^{\alpha\beta}(z) - \tfrac{1}{2} i\Gamma^{\alpha\beta}(z) , \quad (20)$$

where

$$\Delta^{\alpha\beta}(z) = P\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\omega)\chi(\omega R_{\alpha\beta}/c)}{z + 2\epsilon - \omega} , \quad (21)$$

$$\Gamma^{\alpha\beta}(z) = \Gamma(z)\chi(zR_{\alpha\beta}/c) . \quad (22)$$

Using (15) and (20) in (14), we have

$$G_{ge}(z) = \frac{z - \Delta + \frac{1}{2}i\,\Gamma + i\lambda}{[z - \Delta - \Delta^{12} + \frac{1}{2}i(\Gamma + \Gamma^{12}) + i\lambda][z - \Delta + \Delta^{12} + \frac{1}{2}i(\Gamma - \Gamma^{12}) + i\lambda]} . \quad (23)$$

At this point we make the usual "slowly varying" assumption[22] with respect to the behavior of the functions $\Delta(z)$, $\Delta^{12}(z)$, $\Gamma(z)$, and $\Gamma^{12}(z)$ near the unperturbed energy $z = 0$. We denote these functions evaluated at $z = 0$ by Δ, $\tilde{\Delta}$, Γ and $\tilde{\Gamma}$, respectively. Performing the inverse Fourier transform we find

$$G_{ge}(t, t') = \tfrac{1}{2}\theta(t - t')\{\exp[-i(\Delta + \tilde{\Delta})(t - t')] \times \exp[-\tfrac{1}{2}(\Gamma + \tilde{\Gamma})(t - t')] + \exp[-i(\Delta - \tilde{\Delta})(t - t')] \times \exp[-\tfrac{1}{2}(\Gamma - \tilde{\Gamma})(t - t')]\} . \quad (24)$$

It is apparent that (24) is considerably different from what would be obtained if there were just a single isolated atom, or if we assume that the presence of the second deexcited atom has no influence upon the decay properties of the excited atom. It is clear that the effect of the second

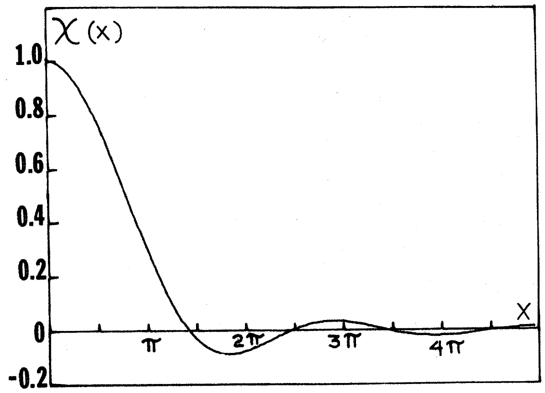

FIG. 1. Plot of the function $\chi(x)$.

atom is to introduce a further energy shift and a new linewidth. From a consideration of the properties of the function $\chi(x)$, a plot of which is shown in Fig. 1, we may conclude in the limit that $R_{12} \to 0$ that $\tilde{\Gamma} \to \Gamma$, while in the limit that $R_{12} \to \infty$ that $\tilde{\Gamma} \to 0$. Thus in the small separation limit, we have from (24)

$$\lim_{R_{12}\to 0} G_{ge}(t, t') = \tfrac{1}{2}\theta(t-t')\{\exp[-i(2\Delta)(t-t')] \times \exp[-\Gamma(t-t')]+1\} \qquad (25)$$

while in the large separation limit

$$\lim_{R_{12}\to\infty} G_{ge}(t, t') = \theta(t-t')e^{-i\Delta(t-t')}e^{-(\Gamma/2)(t-t')} . \qquad (26)$$

Thus we conclude that strictly only in the infinite-separation limit does the effect of the second deexcited atom completely disappear. Note that (25) indicates that in the small-separation limit the linewidth of the radiation that would be emitted is doubled. In addition, by computing the probability $|G_{ge}(t,t')|^2$, we find there is a probability of $\frac{1}{2}$ that the system does not radiate at all after a long time. These effects have been considered by many authors. The Green's-function approach of the present work shows that radiative processes have been taken into account in other treatments. Within the context of our model, the results of the calculations of the Fourier-transformed Green's functions are exact. We may, however, easily make contact with other approaches. In particular, it is instructive to demonstrate the connection between the Heisenberg Green's-function approach and the usual Feynman–Dyson expansion.[23] The results for the case of the one-excited, one deexcited atom system is illustrated in the graphs of Fig. 2.

In Fig. 2 the propagators are represented by straight vertical lines with a label e or g to denote the state. The propagation of a photon in an intermediate state is represented by a wiggly line. That the only kinds of diagrams which may occur are the "bubble" and "ladder" diagrams is a consequence of the model.[24] In the absence of the second deexcited atom, the bubble diagrams would contribute alone to the interactions of the excited atom with the field. It is thus clear that the extra linewidth and level shifts $\tilde{\Gamma}$ and $\tilde{\Delta}$ are to be understood as arising from the "ladder" diagrams exclusively. When the two atoms are close together the effects due to photons shuffling back and forth between the atoms become comparable to the pure self-energy effects.

In addition, we note that for the one-excited-one-deexcited atoms system the equation which determines the two-atom Green's function is also easily expressed in the form indicated in Fig. 3. The Green's function is thus seen to be determined by solving a Bethe–Salpeter-type equation.[10,18] In this language it is clear that the "interaction operator" for the two-atom Green's function $G_{ge}(t, t')$ is simply that part of the irreducible kernel which includes only the infinite subset of "ladder" graphs.

B. Two Excited Atoms

Let us now consider an initial state for which both atoms are excited and placed at points $\vec{R}_1$ and $\vec{R}_2$, respectively. As with the one excited one deexcited atoms system, we pose the same questions with respect to the spontaneous-emission properties. We begin by defining the two excited atoms Green's function as follows:

$$G_{ee}(t, t') = \langle 0|T(b_{e,1}(t)b_{e,2}(t)b^{\dagger}_{e,1}(t')b^{\dagger}_{e,2}(t'))|0\rangle . \qquad (27)$$

Employing the equation-of-motion method, we find the following set of equations must be satisfied:

$$(i\partial_t - 2\epsilon)\, G_{ee}(t, t') = i\delta(t-t') + \sum_{\vec{k}\sigma} g_{k\sigma} e^{-i\vec{k}\cdot\vec{R}_1} B_{\vec{k}\sigma}(t, t')$$

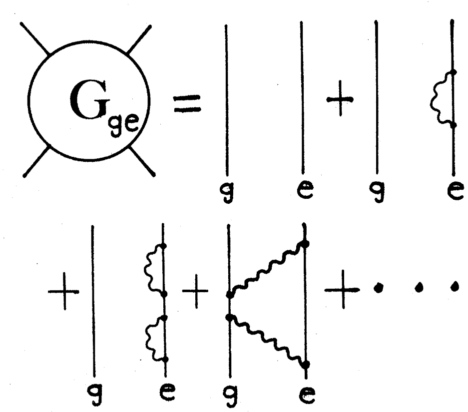

FIG. 2. Graphical representation of G_{ge}. The vertical lines represent the one-atom propagators with initial *and* final states indicated at the bottom. The wiggly line represents the propagation of a photon.

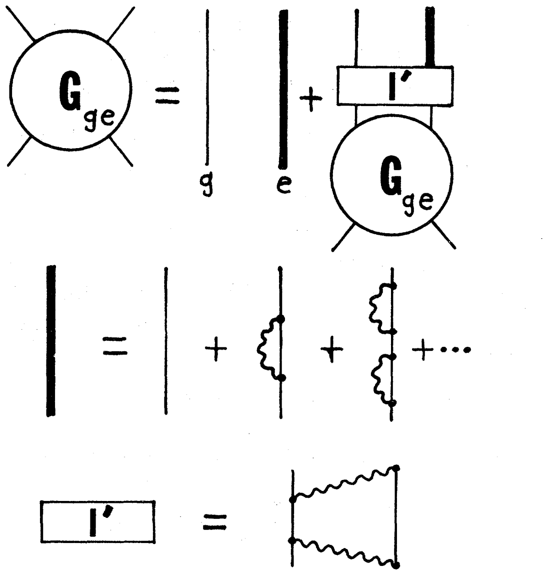

FIG. 3. Bethe–Salpeter-type equation satisfied by G_{ge}. The heavy vertical line represents the sum of all graphs including the "bubble" diagrams only. I' is that part of the irreducible kernel which contributes to the time evolution of the one excited one deexcited atoms system.

$$+\sum_{\vec k\sigma} g_{k\sigma}\, e^{-i\vec k\cdot\vec R_2} H_{\vec k\sigma}(t,t')\ , \quad (28)$$

$$(i\partial_t-\omega_k)\,B_{\vec k\sigma}(t,t')=g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_1}\,G_{ee}(t,t') + \sum_{\vec k'\sigma'} g_{k'\sigma'}\,e^{-i\vec k'\cdot\vec R_2} I_{\vec k'\sigma',\vec k\sigma}(t,t')\ , \quad (29)$$

$$(i\partial_t-\omega_k)\,H_{\vec k\sigma}(t,t')=g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_2}\,G_{ee}(t,t') + \sum_{\vec k'\sigma'} g_{k'\sigma'}\,e^{-i\vec k'\cdot\vec R_1} I_{\vec k'\sigma',\vec k\sigma}(t,t')\ , \quad (30)$$

$$(i\partial_t+2\epsilon-\omega_k-\omega_{k'})\,I_{\vec k'\sigma',\vec k\sigma}(t,t')=g^*_{k'\sigma'}\,e^{i\vec k\cdot\vec R_1}H_{\vec k\sigma}(t,t') \times g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_1}H_{\vec k'\sigma'}(t,t')+g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_2}B_{\vec k'\sigma'}(t,t') + g^*_{k'\sigma'}\,e^{i\vec k'\cdot\vec R_2}B_{\vec k\sigma}(t,t')\ , \quad (31)$$

where

$$B_{\vec k\sigma}(t,t')=\langle 0|\,T(b_{g,1}(t)\,a_{\vec k\sigma}(t)\,b_{e,2}(t)\,b^\dagger_{e,1}(t')\,b^\dagger_{e,2}(t'))\,|0\rangle\ , \quad (32)$$

$$H_{\vec k\sigma}(t,t')=\langle 0|\,T(b_{e,1}(t)\,b_{g,2}(t)\,a_{\vec k\sigma}(t)\,b^\dagger_{e,1}(t')\,b^\dagger_{e,2}(t'))\,|0\rangle\ , \quad (33)$$

$$I_{\vec k'\sigma',\vec k\sigma}(t,t')=\langle 0|\,T(b_{g,1}(t)\,a_{\vec k'\sigma'}(t)\,b_{g,2}(t)\,a_{\vec k\sigma}(t) \times b^\dagger_{e,1}(t')\,b^\dagger_{e,2}(t'))\,|0\rangle\ . \quad (34)$$

The Green's functions $B_{\vec k\sigma}(t,t')$, $H_{\vec k\sigma}(t,t')$, and $I_{\vec k'\sigma',\vec k\sigma}(t,t')$, respectively, describe processes whereby the atom at $\vec R_1$ emits a photon, the atom at $\vec R_2$ emits a photon, and both atoms decay, emitting photons. These Green's functions are the only ones that may contribute to the time evolution of the two excited atoms system. Recall that in the previous section there appeared Green's functions describing amplitudes for final states consisting of a photon and both atoms deexcited, along with a Green's function describing the excitation of the initially deexcited atom and deexcitation of the initially excited atom, with no photons present. In the present case, the situation is more complicated in the sense that in the final state of one atom excited and one deexcited a photon accompanies the one excited one deexcited atoms system. Consequently, it is possible in the present system for a photon emitted by one of the excited atoms to contribute to the amplitude that the other atom decays by "stimulating" the other atom to decay or by being absorbed by the other atom.

We proceed as before by taking the Fourier transform of the equations of motion (28)–(31). Then solving for $I_{\vec k'\sigma',\vec k\sigma}$ in terms of $H_{\vec k\sigma}(z)$ and $B_{\vec k\sigma}(z)$, we find that the Fourier-transformed Green's functions $B_{\vec k\sigma}(z)$ and $H_{\vec k\sigma}(z)$ must satisfy

$$\left(z-\omega_k-\sum_{\vec k'\sigma'}\frac{|g_{k'\sigma'}|^2}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}\right)B_{\vec k\sigma}(z)=g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_1}\,G_{ee}(z)+\sum_{\vec k'\sigma'}\frac{g_{k'\sigma'}e^{-i\vec k'\cdot\vec R_2}g^*_{k\sigma}e^{i\vec k\cdot\vec R_2}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}B_{\vec k'\sigma'}(z)$$
$$+\sum_{\vec k'\sigma'}\frac{|g_{k'\sigma'}|^2e^{-i\vec k'\cdot(\vec R_2-\vec R_1)}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}H_{\vec k\sigma}(z)+\sum_{\vec k'\sigma'}\frac{g_{k'\sigma'}e^{-i\vec k'\cdot\vec R_2}g^*_{k\sigma}e^{i\vec k\cdot\vec R_1}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}H_{\vec k'\sigma'}(z)\ , \quad (35)$$

$$\left(z-\omega_k-\sum_{\vec k'\sigma'}\frac{|g_{k'\sigma'}|^2}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}\right)H_{\vec k\sigma}(z)=g^*_{k\sigma}\,e^{i\vec k\cdot\vec R_2}\,G_{ee}(z)+\sum_{\vec k'\sigma'}\frac{g_{k'\sigma'}e^{-i\vec k'\cdot\vec R_1}g^*_{k\sigma}e^{i\vec k\cdot\vec R_1}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}H_{\vec k'\sigma'}(z)$$
$$+\sum_{\vec k'\sigma'}\frac{g_{k'\sigma'}e^{-i\vec k'\cdot\vec R_1}g^*_{k\sigma}e^{i\vec k\cdot\vec R_2}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}B_{\vec k'\sigma'}(z)+\sum_{\vec k'\sigma'}\frac{|g_{k'\sigma'}|^2e^{-i\vec k'\cdot(\vec R_1-\vec R_2)}}{z+2\epsilon-\omega_k-\omega_{k'}+i\eta}B_{\vec k\sigma}(z)\ . \quad (36)$$

After proceeding to the continuum limit in $\vec k$, it is clear by inspection of Eqs. (35) and (36) that in order to find $B_{\vec k\sigma}(\sigma)$ and $H_{\vec k\sigma}(z)$ we must solve two coupled singular integral equations, assuming $G_{ee}(z)$ to be a "known" function. After solving these equations, we then must substitute the solu-

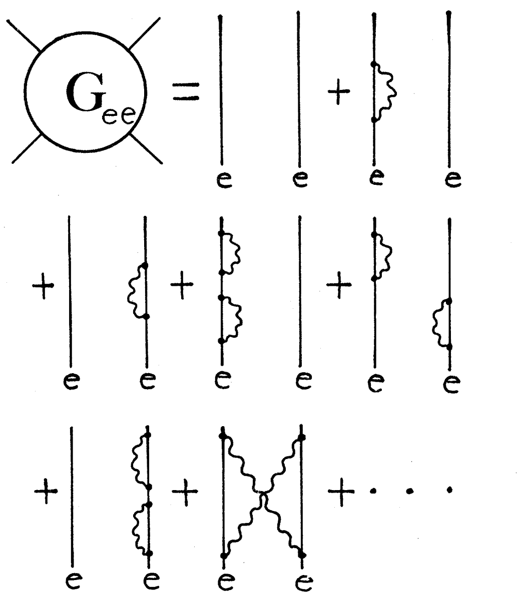

FIG. 4. Graphical representation of G_{ee}. Note that in a Bethe–Salpeter equation language the irreducible part of the interaction is represented by the "crossed" graph.

tions into the Fourier transform of Eq. (28) and then solve possibly another singular integral equation for $G_{ee}(z)$. We have been unable to find the general solution for $G_{ee}(z)$. Therefore, we resort to an approximation method of solution. We are interested in obtaining an approximate solution which includes to lowest order the "radiative cooperative" effects. In Fig. 4 we have given a schematic graphical representation of the radiative processes which contribute to the time evolution of $G_{ee}(t, t')$. We obtain an approximate solution to Eqs. (35) and (36) as follows. We notice that if only the terms $g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_1} G_{ee}(z)$ and $g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_2} \times G_{ee}(z)$ on the right-hand side of (35) and (36) are kept and we subsequently solve for $B_{\vec{k}\sigma}(z)$ and $H_{\vec{k}\sigma}(z)$, we obtain for $G_{ee}(z)$ the following result:

$$G_{ee}(z) = \left(z - 2\epsilon - 2 \sum_{k\sigma} \frac{|g_{k\sigma}|^2}{z - \omega_k - \Sigma(z - \omega_k) + i\mu} + i\lambda \right)^{-1}, \tag{37}$$

where

$$\Sigma(z - \omega_k) = \sum_{k'\sigma'} \frac{|g_{k'\sigma'}|^2}{z + 2\epsilon - \omega_k - \omega_{k'} + i\eta} . \tag{38}$$

If we now make the approximation that $\Sigma(z - \omega_k)$ may be neglected, then the result (37) becomes

$$G_{ee}(z) = [z - 2\epsilon - 2\Sigma(z) + i\lambda]^{-1} . \tag{39}$$

This approximate expression for $G_{ee}(z)$ would yield a time dependence for $G_{ee}(t, t')$ indicative of two independent excited atoms with natural line with Γ. We include to lowest order the "crossed" interaction as follows. $B^*_{\vec{k}\sigma}(z)$ and $H^*_{\vec{k}\sigma}(z)$ are computed by substituting the expressions

$$B^*_{\vec{k}\sigma}: \; g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_1} G_{ee}(z)[z - \omega_k - \Sigma(z - \omega_k) + i\eta]^{-1},$$

$$H^*_{\vec{k}\sigma}: \; g^*_{k\sigma} e^{i\vec{k}\cdot\vec{R}_2} G_{ee}(z)[z - \omega_k - \Sigma(z - \omega_k) + i\eta]^{-1}$$

on the right-hand side of (35) and (36), with the exception of the term in (35) involving $H^*_{\vec{k}'\sigma'}$ summed $\vec{k}'\sigma'$ and the term in (36) involving $B^*_{\vec{k}'\sigma'}(z)$ summed over $\vec{k}'\sigma'$, which are excluded. We obtain for $G_{ee}(z)$

$$G_{ee}(z) = [z - 2\epsilon - 2\tilde{\Sigma}(z) - 2I_1(z) - 2I_2(z) + i\beta]^{-1}, \tag{40}$$

where, after going to the continuum limit in $\vec{k}$:

$$\tilde{\Sigma}(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\omega)}{z - \omega - \Sigma(z - \omega) + i\eta} . \tag{41}$$

The functions $I_1(z)$ and $I_2(z)$ are defined as

$$I_1(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \times \frac{\Gamma(\omega)\,\chi(\omega R/c)\,\Gamma(\omega')\,\chi(\omega' R/c)}{[z - \omega - \Sigma(z - \omega) + i\mu](z + 2\epsilon - \omega - \omega' + i\eta)} \times \frac{1}{z - \omega' - \Sigma(z - \omega') + i\mu} , \tag{42}$$

$$I_2(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \times \frac{\Gamma(\omega)\,\chi(\omega R/c)\Gamma(\omega')\chi(\omega' R/c)}{[z - \omega - \Sigma(z - \omega) + i\mu]^2 [z + 2\epsilon - \omega - \omega' + i\eta]} . \tag{43}$$

We are interested in the properties of our approximate expression for $G_{ee}(z)$ in two limits. First in the limit of very large interatomic separation we may effectively put $I_1 = 0$ and $I_2 = 0$. Consequently,

$$\lim_{R\to\infty} G_{ee}(z) \simeq [z - 2\epsilon - 2\tilde{\Sigma}(z) + i\beta]^{-1} . \tag{44}$$

Second, in the limit of very small interatomic separation

$$\lim_{R\to 0} G_{ee}(z) \simeq [z - 2\epsilon - 2\tilde{\Sigma}(z) - 2\bar{I}_1(z) - 2\bar{I}_2(z) + i\beta]^{-1}, \tag{45}$$

where $\bar{I}_1$ and $\bar{I}_2$ are the $R = 0$ limits of I_1 and I_2. Considering the large separation limit, we see that the Green's function effectively reduces to a form indicative of two independently evolving excited atoms. In the very small separation limit, however, there are extra terms present which represent the contribution of the radiative cooperative effects. It is seen that the additional contributions are of "order" $\Gamma(\Gamma/\omega)$, whereas Σ is of "order" Γ. While these terms are small we can-not in principle ignore them.

C. Scattering of Radiation by Two Deexcited Atoms

The system of two deexcited atoms is easily shown to have a trivial time evolution within the context of the present model.[25] However, if we consider the scattering of a photon by a system of two deexcited atoms, then we expect that interesting cooperative effects may be induced by the incoming photon.[10(a),12]

Thus consider a system of two deexcited atoms centered, respectively, around the points $\vec{R}_1$ and $\vec{R}_2$. Let a photon of wave vector $\vec{k}'$ and polarization σ' be scattered by the atoms and consider the amplitude that finally there is a photon with wave vector $\vec{k}$ and polarization σ and again two deexcited atoms. This amplitude is given by the following Green's function

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')=\langle 0|\,T(b_{g,1}(t)\,b_{g,2}(t)\,a_{\vec{k}\sigma}(t)\,b^\dagger_{g,1}(t')\times b^\dagger_{g,2}(t')\,a^\dagger_{\vec{k}'\sigma'}(t'))|\,0\rangle\,. \quad (46)$$

Proceeding as before, the equations of motion are found to be

$$(i\partial_t+2\epsilon-\omega_k)\,P_{\vec{k}\sigma\vec{k}'\sigma'}(t,t') = i\delta_{\vec{k}\vec{k}'}\,\delta_{\sigma\sigma'}\,\delta(t-t')+g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_2}Q_{\vec{k}'\sigma'}(t,t') + g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_2}R_{\vec{k}'\sigma'}(t,t')\,, \quad (47)$$

$$i\partial_t\,Q_{\vec{k}'\sigma'}(t,t')=\sum_{\vec{k}_1\sigma_1} g_{k_1\sigma_1}e^{-i\vec{k}_1\cdot\vec{R}_1}P_{\vec{k}_1\sigma_1,\vec{k}'\sigma'}(t,t')\,, \quad (48)$$

$$i\partial_t\,R_{\vec{k}'\sigma'}(t,t')=\sum_{\vec{k}_2\sigma_2} g_{k_2\sigma_2}e^{-i\vec{k}_2\cdot\vec{R}_2}P_{\vec{k}_2\sigma_2,\vec{k}'\sigma'}(t,t')\,, \quad (49)$$

where

$$Q_{\vec{k}'\sigma'}(t,t') =\langle 0|\,T(b_{e,1}(t)\,b_{g,2}(t)\,b^\dagger_{g,1}(t')\,b^\dagger_{g,2}(t')\,a^\dagger_{k'\sigma'}(t'))|\,0\rangle\,, \quad (50)$$

$$R_{\vec{k}'\sigma'}(t,t') =\langle 0|\,T(b_{g,1}(t)\,b_{e,2}(t)\,b^\dagger_{g,1}(t')b^\dagger_{g,2}(t')\,a^\dagger_{\vec{k}'\sigma'}(t'))|\,0\rangle\,. \quad (51)$$

After taking the Fourier transform of the equations of motion (47)–(49), we find that $P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')$ may be found by solving an integral equation with a separable kernel. After some manipulation, we find

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(z) =\frac{\delta_{\vec{k}\vec{k}'}\,\delta_{\sigma\sigma'}}{z+2\epsilon-\omega_k+i\eta}+\Bigg(\frac{g^*_{k\sigma}g_{k'\sigma'}\,e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}_1}}{(z+2\epsilon-\omega_k+i\eta)(z+2\epsilon-\omega_{k'}+i\eta)}+\frac{\beta^{1,2}(z)\,g^*_{k\sigma}\,e^{i\vec{k}\cdot\vec{R}_1}g_{k'\sigma'}\,e^{-i\vec{k}'\cdot\vec{R}_2}}{(z+2\epsilon-\omega_k+i\eta)[\,\alpha(z)+i\lambda]\,(z+2\epsilon-\omega_{k'}+i\eta)(z+i\mu)}$$
$$+\frac{g^*_{k\sigma}g_{k'\sigma'}\,e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}_2}}{(z+2\epsilon-\omega_k+i\eta)[\alpha(z)+i\lambda]\,(z+2\epsilon-\omega_k+i\eta)}+\frac{\beta^{2,1}(z)\,g^*_{k\sigma}g_{k'\sigma'}\,e^{-i\vec{k}'\cdot\vec{R}_1}e^{i\vec{k}\cdot\vec{R}_2}}{[\alpha(z)+i\lambda](z+2\epsilon-\omega_k+i\eta)(z+2\epsilon-\omega_{k'}+i\eta)}$$
$$+\frac{\beta^{1,2}(z)\,g^*_{k\sigma}g_{k'\sigma'}\,e^{-i(\vec{k}'-\vec{k})\cdot\vec{R}_2}}{[\,\alpha(z)+i\lambda](z+2\epsilon-\omega_k+i\eta)(z+2\epsilon-\omega_{k'}+i\eta)}\Bigg)\left(\alpha(z)-\frac{\beta^{1,2}(z)\,\beta^{2,1}(z)}{\alpha(z)+i\mu}+i\gamma\right)^{-1}, \quad (52)$$

where

$$\alpha(z)=1-\sum_{\vec{k}_1\sigma_1}\frac{|g_{k_1\sigma_1}|^2}{(z+2\epsilon-\omega_{k_1}+i\eta)(z+i\mu)} \quad (53)$$

and

$$\beta^{i,j}(z)=\sum_{\vec{k}\sigma}\frac{|g_{k\sigma}|^2\,e^{-i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)}}{(z+2\epsilon-\omega_k+i\eta)(z+i\mu)}\,,\qquad i,j=1,2. \quad (54)$$

The inverse Fourier transform of $P_{\vec{k}\sigma,\vec{k}'\sigma'}(z)$ gives

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')=\theta(t-t')\Bigg(\delta_{\vec{k}\vec{k}'}\,\delta_{\sigma\sigma'}\,e^{-i\Omega_k(t-t')}+\frac{1}{2}\,\frac{e^{-i\Delta^+(t-t')}e^{-(\Gamma/2)(t-t')}\,\Phi_{kk'}(\Omega_k-\Delta^+-\tfrac{1}{2}i\,\Gamma^+)(\Omega_{k'}-\Delta^+-\tfrac{1}{2}i\,\Gamma^+)}{\delta^+_k\delta^+_{k'}}$$
$$+\frac{1}{2}\,\frac{e^{-i\Delta^-(t-t')}e^{-(\Gamma/2)(t-t')}\,\Psi_{kk'}(\Omega_k-\Delta^--\tfrac{1}{2}i\Gamma^-)(\Omega_{k'}-\Delta^-\tfrac{1}{2}i\,\Gamma^-)}{\delta^-_k\,\delta^-_{k'}}+\frac{e^{-i\Omega_k(t-t')}\,\Lambda_{kk'}(\omega_k)(\Omega_k-\Delta^+-\tfrac{1}{2}i\,\Gamma^+)(\Omega_k-\Delta^--\tfrac{1}{2}i\,\Gamma^-)}{\delta^+_k\,\delta^-_k\,(\omega_k-\omega_{k'})}$$
$$-\frac{e^{-i\Omega_{k'}(t-t')}\,\Lambda_{kk'}(\omega_k)(\Omega_{k'}-\Delta^+-\tfrac{1}{2}i\Gamma^+)(\Omega_{k'}-\Delta^--\tfrac{1}{2}i\Gamma^-)}{\delta^+_{k'}\,\delta^-_{k'}\,(\omega_k-\omega_{k'})}\Bigg)\,, \quad (55)$$

where

$$\Phi_{\vec{k}\vec{k}'} = g^*_{k\sigma} g_{k'\sigma'}\{ \exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_1] + \exp(-i\vec{k}'\cdot\vec{R}_1 + i\vec{k}\cdot\vec{R}_2) + \exp(-i\vec{k}\cdot\vec{R}_2 + i\vec{k}'\cdot\vec{R}_1) + \exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_2]\}, \tag{56}$$

and

$$\Psi_{\vec{k}\vec{k}'} = g^*_{k\sigma} g_{k'\sigma'}\{ -\exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_1] + \exp(-i\vec{k}\cdot\vec{R}_1 + i\vec{k}\cdot\vec{R}_2) + \exp(-i\vec{k}\cdot\vec{R}_2 + i\vec{k}'\cdot\vec{R}_1) - \exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_2]\}, \tag{57}$$

$$\Lambda_{\vec{k}\vec{k}'}(\omega_k) = g^*_{k\sigma} g_{k\sigma'}\,(\{\exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_1] + \exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_2]\}(\Omega_k - \Delta + \tfrac{1}{2}i\Gamma) + [\exp(+i\vec{k}\cdot\vec{R}_1 - i\vec{k}'\cdot\vec{R}_2) + \exp(i\vec{k}\cdot\vec{R}_2 - i\vec{k}'\cdot\vec{R}_1)](\Delta^{12} - \tfrac{1}{2}i\Gamma)), \tag{58}$$

$$\delta^{\pm}_k = (\Omega - \Delta^{-}_{+}\Delta^{12})^2 + \tfrac{1}{4}[(\Gamma \pm \Gamma^{12})^2], \qquad \Delta^{\pm} = \Delta \pm \Delta^{12}, \qquad \Gamma^{\pm} = \Gamma \pm \Gamma^{12}. \tag{59}$$

We now consider times long compared with $(\Gamma^+)^{-1}$, $(\Gamma^-)^{-1}$ and $\vec{k}$ and $\vec{k}'$s such that $\vec{k} \neq \vec{k}'$. Under these conditions $P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')$ becomes

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t') = \theta(t-t')\left(\frac{e^{-i\Omega_k(t-t')}F_{\vec{k}\vec{k}'}(\omega_k)}{\omega_k - \omega_{k'}} - \frac{e^{-i\Omega_{k'}(t-t')}F_{\vec{k}\vec{k}'}(\omega_{k'})}{\omega_k - \omega_{k'}}\right), \tag{60}$$

where

$$F_{\vec{k}\vec{k}'}(\omega_k) = \frac{\Lambda_{\vec{k}\vec{k}'}(\omega_k)(\Omega_k - \Delta^+ - \frac{1}{2}i\Gamma^+)(\Omega_k - \Delta^- - \frac{1}{2}i\Gamma^-)}{\delta^+_k\,\delta^-_k}. \tag{61}$$

Now expanding $F_{\vec{k}\vec{k}'}(\omega_k)$ in a Taylor series about $\omega_k = \omega_{k'}$:

$$F_{\vec{k}\vec{k}'}(\omega_k) = F_{\vec{k}\vec{k}'}(\omega_{k'}) + (\omega_k - \omega_{k'})R_{\vec{k}\vec{k}'}, \tag{62}$$

where $R_{\vec{k}\vec{k}'}$ denotes the rest of the terms in the expansion. Using (62) in (60), we then collect terms and compute $|P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')|^2$. After some manipulation we find, upon averaging over the outgoing photon frequency (considered as continuous) while keeping $\vec{k}'$ fixed

$$\langle |P_{\vec{k}\vec{k}'}(t,t')|^2\rangle_{\text{av}k} = \frac{t\pi|\Lambda_{\vec{k}\vec{k}'}(\omega_{k'})|^2\rho(\omega_{k'})}{\delta^+_{k'}\,\delta^-_{k'}} + \text{other terms}, \tag{63}$$

where the "other terms" exhibit a time dependence proportional to t^0 and thus do not dominate for long times. Thus for long times only the first term in (63) is important. We may draw the following general conclusions regarding the long-time properties of $\langle |P_{\vec{k}\vec{k}'}|^2\rangle_{\text{av}k}$. There is an anomalous two-peaked structure for $\langle |P_{\vec{k}\vec{k}'}|^2\rangle_{\text{av}k}$ when plotted as a function of $\omega_{k'} - 2\epsilon$. For small interatomic separations there are two peaks centered symmetrically about the points $2\epsilon \pm \Delta$, one peak with half-width of approximately $\Gamma^+ \simeq 0$. As the interatomic separation increases, the peaks tend to merge, in the limit of infinitely large separation, into a single Lorentzian. These general features are in accord with our previous discussion of the effects of radiative cooperation in the one excited one deexcited atoms problem. Thus it is clear from the present example that the assumption that the deexcited atoms scatter the photon independently is not valid. This indicates the ordinary treatment of resonance fluorescence may not be of general validity in multiatom systems.

IV. MANY-ATOM SYSTEMS

In this section we extend the treatment from two atoms to many atoms. In particular, we consider generalizations of two of the problems treated in Sec. III. It is of interest, perhaps, to point out that these problems are solved without recourse to perturbation theory. It will be seen, moreover, that a complete solution of these problems can be of interest in themselves, in addition to their relevance to the more difficult N-atom problems which we will not consider in the present paper.

A. One Excited N Deexcited

Consider a single excited atom centered around some point $\vec{R}_0$ in the presence of N deexcited atoms located at points $\vec{R}_1, \ldots, \vec{R}_N$. In the light of the results of Sec. III with regard to the lifetime of the excited atom in the presence of another deexcited atom, we now study the effect of the system of N deexcited atoms upon the lifetime of the excited atom.[9] Here we do not specify N. For N large the present problem approximates in a crude way a system of atoms that has been very inefficiently excited by some means at a time t'. More appropriately we might consider the present system as constituting a region of a larger system and consider the actual system as made up of these subsystems.

We define the one-excited-N-deexcited-atoms Green's function to be

$$G_{e\{g\}}(t,\ t') = \langle 0|T\Big(b_{e,0}(t)\prod_{\alpha=1}^{N} b_{g,\alpha}(t)b^{\dagger}_{e,0}(t')$$

$$\times \prod_{\beta=1}^{N} b^{\dagger}_{g,\beta}(t')\Big)\Big|0\rangle . \quad (64)$$

The equation-of-motion method yields the following *closed* set of equations of motion:

$$[i\partial_t-(N-1)\epsilon]\,G_{e\{g\}}(t,\ t')=i\delta(t-t') +\sum_{\vec{k}\sigma} g_{k\sigma}e^{-i\vec{k}\cdot\vec{R}_0}K_{g\vec{k}_\sigma\{g\}}(t,\ t') , \quad (65)$$

$$[i\partial_t+(N+1)\epsilon-\omega_k]K_{gk\sigma\{g\}}(t,\ t')=g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_0} \times G_{e\{g\}}(t,\ t')+\sum_{\alpha=1}^{N} g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_\alpha}\tilde{G}^{\alpha}_{g\{g|e|g\}}(t,\ t') , \quad (66)$$

$$[i\partial_t+(N+1)\epsilon]\,\tilde{G}^{\alpha}_{g\{g|e|g\}}(t,\ t')=\sum_{\vec{k}\sigma} g_{k\sigma}e^{i\vec{k}\cdot\vec{R}_\alpha} \times M^{\alpha}_{gk\sigma\{g\}}(t,\ t') , \quad (67)$$

$$[i\partial_t+(N-1)\epsilon-\omega_k]M^{\alpha}_{gk_\sigma\{g\}}(t,\ t') =g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_0}G_{e\{g\}}(t,\ t')+\sum_{\beta=1}^{N} g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_\beta} \times \tilde{G}_{g\{g|e|g\}}(t,\ t'), \quad (68)$$

where

$$K_{g\vec{k}\sigma\{g\}}(t,\ t')=\langle 0|\,T\Big(b_{g,0}(t)a_{\vec{k}\sigma}(t)\prod_{\alpha=1}^{N} b_{g,\alpha}(t) \times b^{\dagger}_{e,0}(t')\prod_{\beta=1}^{N} b^{\dagger}_{g,\beta}(t')\Big)|0\rangle , \quad (69)$$

$$\tilde{G}^{\alpha}_{g\{g|e|g\}}(t,\ t')=\langle 0|\,T\Big(b_{g,0}(t)\prod_{\lambda=1}^{\alpha-1} b_{g,\lambda}(t)\,b_{e,\alpha}(t) \times \prod_{\delta=\alpha+1}^{N} b_{g,\delta}(t)\ b^{\dagger}_{e,0}(t')\prod_{\beta=1}^{N} b^{\dagger}_{g,\beta}(t')\Big)|0\rangle , \quad (70)$$

and

$$M^{\alpha}_{g\vec{k}\sigma\{g\}}(t,\ t')=\langle 0|\,T\Big(b_{g,0}(t)\prod_{\gamma=1}^{\alpha-1} b_{g,\gamma}(t)\,b_{g,\alpha}(t)\,a_{\vec{k}\sigma}(t) \times \prod_{\delta=\alpha+1}^{N} b_{g,\delta}(t)\ b^{\dagger}_{e,0}(t')\prod_{\beta=1}^{N} b^{\dagger}_{g,\beta}(t')\Big)|0\rangle . \quad (71)$$

Proceeding as in previous problems, we find the following Fourier-transformed Green's function:

$$G_{e\{g\}}(z)=\Big(z+(N-1)\epsilon-\Lambda^{00}(z)-\sum_{\alpha,\beta=1}^{N}\Lambda^{0\alpha}(z) \times(\Xi(z)^{-1})^{\alpha\beta}\Lambda^{\beta 0}(z)+i\delta\Big)^{-1} , \quad (72)$$

$$\Xi^{\alpha\beta}(z)\equiv[z+(N-1)\epsilon]\,\delta^{\alpha\beta}-\Lambda^{\alpha\beta}(z) , \quad (73)$$

where the $\Lambda^{ij}(z)$ with $i,\ j=0,\ 1,\ 2,\ \ldots,\ N$ were defined in (18). In order to proceed further, we must know the explicit form of the inverse of the matrix $\Xi^{\alpha\beta}$. Only then may we find the singularities of $G_{e\{g\}}(z)$ and perform the inverse Fourier transformation to find $G_{e\{g\}}(t,\ t')$. We have been unable to find a form for $\Xi^{\alpha\beta}$ for arbitrary N which is useful for obtaining an analytic solution. We may, however, consider the following limiting case. We recall that in the limit $R_{\alpha\beta}\to\infty$ that $\chi\to 0$. Consequently, if we have a system of one excited atom in N deexcited atoms such that $R_{\alpha 0} \gg \frac{3}{4}\lambda$ then effectively $\chi(\omega R_{\alpha 0}/c)\simeq 0$.[26] With these assumptions, the inverse Fourier transform is found to yield

$$G_{e\{g\}}(t,\ t')=\theta(t-t')\exp\{-i[-(N-1)\epsilon+\Delta](t-t')\} \times\exp[-\tfrac{1}{2}\Gamma(t-t')] .$$

This result indicates that in the limit where all the deexcited atoms are far separated from the initially excited atom the system evolves as though there were a single isolated excited atom and N freely propagating deexcited atoms. Thus, in analogy with the one excited one deexcited case we see that an excited atom in a many-deexcited-atom system only strictly radiates at a rate characteristic of an isolated excited atom in the limit where the deexcited atoms are infinitely separated from the excited atom.

B. Scattering of Radiation by N Deexcited Atoms

Consider now the scattering of a photon by a system of N two-level atoms all in the ground state. Thus define the Green's function

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,\ t')=\langle 0|\,T\Big(\prod_{\alpha=1}^{N} b_{g,\alpha}(t)\,a_{\vec{k}\sigma}(t) \times\prod_{\beta=1}^{N} b_{g,\beta}(t')\,a^{\dagger}_{\vec{k}'\sigma'}(t')\Big)|0\rangle . \quad (74)$$

The equations of motion are

$$(i\partial_t+N\epsilon-\omega_k)P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,\ t') =\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}+\sum_{\lambda=1}^{N} g^*_{k\sigma}e^{i\vec{k}\cdot\vec{R}_\lambda}Q^{\lambda}_{\vec{k}'\sigma'}(t,\ t') , \quad (75)$$

$$[i\partial_t+(N-2)\epsilon]\,Q^{\lambda}_{\vec{k}'\sigma'}(t,\ t') =\sum_{\vec{k}_1\sigma_1} g_{k_1\sigma_1}e^{-i\vec{k}_1\cdot\vec{R}_\lambda}P_{\vec{k}_1\sigma_1,\vec{k}'\sigma'}(t,\ t') . \quad (76)$$

Note that the Green's function $Q^{\lambda}_{k'\sigma'}(t,\ t')$ describes the process whereby the incoming photon has excited the λth atom. It is defined as follows:

$$Q^{\lambda}_{\vec{k}'\sigma'}(t,\ t')=\langle 0|\,T\Big(\prod_{\alpha=1}^{\lambda-1} b_{g,\alpha}(t)\,b_{e,\lambda}(t)\prod_{\mu=\lambda+1}^{N} b_{g,\mu}(t) \times\prod_{\beta=1}^{N} b^{\dagger}_{g,\beta}(t')\,a^{\dagger}_{k'\sigma'}(t')\Big)|0\rangle .$$

Proceeding as before, the Fourier-transformed equations of motion for $P_{\vec{k}\sigma,\vec{k}'\sigma'}$ and $Q^{\lambda}_{\vec{k}'\sigma'}$ are found to be separable integral equations in the continuum limit. We find

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(z)=\frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}}{z+N\epsilon-\omega_k+i\eta}+\frac{g^*_{k\sigma}}{z+N\epsilon-\omega_k+i\eta}\times\sum_{\alpha,\beta}e^{i\vec{k}\cdot\vec{R}_\alpha}(L^{-1}(z))^{\alpha\beta}e^{-i\vec{k}'\cdot\vec{R}_\beta}\frac{g_{k'\sigma'}}{z+N\epsilon-\omega_{k'}+i\eta}, \tag{77}$$

where

$$L^{\alpha\beta}(z)=[z+(N-2)\epsilon]\,\delta^{\alpha\lambda}-\Psi^{\alpha\lambda}(z), \tag{78}$$

with

$$\Psi^{\alpha\lambda}(z)=\sum_{\vec{k}\sigma}\frac{|g_{k\sigma}|^2e^{-i\vec{k}\cdot(\vec{R}_\alpha-\vec{R}_\lambda)}}{z+N\epsilon-\omega_k+i\eta}.$$

This expression for $P_{\vec{k}\sigma,\vec{k}'\sigma'}$ is exact. However, in order to proceed further, some approximation would have to be made, as the exact expression contains the inverse of the matrix $L(z)$. The contribution of $L(z)$ to the singularities of $P_{\vec{k}\sigma,\vec{k}'\sigma'}(z)$ in the z plane must be known before the time evolution may be computed. This becomes an unmanageable task for more than three of four atoms. Consequently, we shall consider here certain limits.

First let us consider the "point system" limit. In this case, the matrix Ψ has the property that the off-diagonal elements are all equal to the diagonal elements. Thus $\Psi^{\alpha\beta}(z)\to\Psi^{\alpha\alpha}(z)$ for all α, $\beta=1,\ldots,N$. Using this property of $\Psi^{\alpha\beta}(z)$, it may be easily shown that

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(z)=\frac{\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}}{z+N\epsilon-\omega_k+i\eta}+\frac{Ng^*_{k\sigma}g_{k'\sigma'}}{(z+N\epsilon-\omega_k+i\eta)[z+(N-2)\epsilon-N\Psi(z)+i\eta](z+N\epsilon-\omega_{k'}+i\eta)}.$$

Note that $\Psi(z)$ is just the self-energy. The inverse Fourier transform yields

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,\ t')=\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}\exp[-i(-N\epsilon+\omega_k)(t-t')]+\frac{Ng^*_{k\sigma}g_{k'\sigma'}\exp[-i(-N\epsilon+\omega_k)(t-t')]}{(\omega_k-2\epsilon-N\Delta+\frac{1}{2}iN\Gamma)(\omega_k-\omega_{k'})}$$
$$+\frac{Ng^*_{k\sigma}g_{k'\sigma'}\exp[-i(-N\epsilon+2\epsilon+N\Delta)(t-t')]\exp[-\frac{1}{2}N\Gamma(t-t')]}{(2\epsilon-\omega_k+N\Delta-\frac{1}{2}iN\Gamma)(2\epsilon-\omega_{k'}+N\Delta-\frac{1}{2}iN\Gamma)}+\frac{Ng^*_{k\sigma}g_{k'\sigma'}\exp[-i(-N\epsilon+\omega_{k'})(t-t')]}{(\omega_{k'}-2\epsilon-N\Delta+\frac{1}{2}iN\Gamma)(\omega_{k'}-\omega_k)}. \tag{79}$$

From this we note that the probability $|P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,t')|^2$ have terms proportional to N^2, indicating an enhancement in the probability due to the extreme radiative cooperative effects prevalent in the "point-system" limit. Also, by inspection of the denominators in (79) we see that the level shift and linewidth are proportional to N in the point-system limit.[27]

Next consider the opposite limit. That is, suppose the system is infinitely dilute so that *only* the diagonal $\Psi^{\alpha\beta}$'s survive: $\Psi^{\alpha\beta}\to\Psi\delta^{\alpha\beta}$. Recalling the definition of the matrix $L^{\alpha\beta}(z)$ in (78), we find

$$L^{\alpha\beta}(z)=[z+(N-2)\epsilon-\Psi(z)]\,\delta^{\alpha\beta}.$$

Thus

$$[L^{-1}(z)]^{\alpha\beta}=[z+(N-2)\epsilon-\Psi(y)+i\eta]^{-1}(I_N)^{\alpha\beta},$$

where I_N is the $N\times N$ identity matrix. The inverse Fourier transform then yields

$$P_{\vec{k}\sigma,\vec{k}'\sigma'}(t,\ t')=\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'}\exp[-i(-N\epsilon+\omega_k)(t-t')]+g^*_{k\sigma}g_{k'\sigma'}\sum_{\beta=1}^{N}\exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_\beta]\{\exp[-i(-N\epsilon+\omega_k)(t-t')]$$
$$\times(\omega_k-2\epsilon-\Delta+\tfrac{1}{2}i\Gamma)^{-1}(\omega_k-\omega_{k'})^{-1}+\exp[-i(2\epsilon-\omega_k+\Delta)(t-t')]\exp[\tfrac{1}{2}\Gamma(t-t')](2\epsilon-\omega_k+\Delta+\tfrac{1}{2}i\Gamma)^{-1}$$
$$\times(2\epsilon-\omega_{k'}+\Delta-\tfrac{1}{2}i\Gamma)+\exp[-i(-N\epsilon+\omega_{k'})(t-t')](\omega_k-\omega_{k'})^{-1}(\omega_{k'}-2\epsilon-\Delta+\tfrac{1}{2}i\Gamma)^{-1}\}. \tag{80}$$

We note that (80) differs from the amplitude for the scattering of the photon by a single deexcited atom only in that the energy of the unperturbed system is of course $-N\epsilon$ instead of $-\epsilon$ and instead of the term $\exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}]$ we have

$$\sum_{\beta=1}^{N}\exp[-i(\vec{k}'-\vec{k})\cdot\vec{R}_\beta].$$

V. DISCUSSION

In this paper we have studied the effect of the state of the other atoms in a multiatom system upon the spontaneous-emission properties of a given atom. We have utilized Heisenberg-picture Green's functions and the equation-of-motion method of solution. The interaction picture has not been used in the present work. Thus, no recourse has been made to the standard perturbation schemes.[11,18] In fact, we have shown that within the present approach certain multiatom systems may be treated exactly. Herein lies one advantage of the Green's-function approach employed here. We point out that we, as well as previous workers, have been able to exactly compute the time evolution of certain systems because of the properties of the model employed. Incidentally, it is clear that the application of the Green's-function approach to the one-atom problem within the context of the model used here yields results identical to those of WW. Hence the use of the WW approximation for a single two-level atom is equivalent to summing only the laddered series of "bubble" graphs.[25] Our results for the two atom systems are generally in accord with the Bethe–Salpeter treatment of Chang and Stehle.[10a] We arrive at results similar to these authors and others for the one-excited, one deexcited atoms system.[5,6,8,9,12] However, our results for the two-excited atoms system constitute a correction to the time evolution relative to previous treatments.[8,10a] The correction takes into account in lowest order the radiative correlations that are present for the two excited atoms system. Within the context of the model radiative cooperative effects for this system are contained in the "crossed graph" of Fig. 4. Also, another advantage of the present approach is that the straightforward application of the equation-of-motion method allows one to see at each stage of the calculation just what physical processes contribute to the given system. For more complicated systems it may be possible to make approximations more systematically by using these methods. It thus appears feasible to apply these techniques to systems of more than two atoms based on our results for the many-atom systems considered in Sec. IV. We conclude that the results of the present investigation indicate that the Heisenberg Green's-function approach can be useful in treating radiative correlations in many-atom systems. We shall apply these techniques to more complicated systems in a future publication.

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