[6]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 1027.

[7]The Laguerre polynomials L_p^k as employed here follow the conventions given; for example, by A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961); or E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961).

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Analytic Study of Pulse Chirping in Self-Induced Transparency*

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We analyze the phase modulation, or chirping, of coherent lossless light pulses propagating without distortion through resonant absorbers. In order to do this, we generalize the pioneering work of McCall and Hahn in two different directions. In the first place, of course, we abandon their assumption that the phase of the pulses has no temporal dependence. We prove that for slowly varying single pulses, chirping is not possible. However, we describe many multiple-pulse trains which are necessarily chirped, even under the slowly varying envelope restriction, and also describe the envelope modulations which produce large chirps. We show that certain zero-π chirped pulse trains are contained as special cases of our general results. Our second generalization of the McCall-Hahn work concerns the background material in which the two-level resonant atoms are suspended. We allow the host medium to possess significant nonresonant nonlinearities. We find that undistorted lossless single pulses are possible in such a medium and that they are necessarily chirped.

I. INTRODUCTION

The discovery of self-induced transparency (SIT) by McCall and Hahn[1] has focused attention again on the very old problem of light propagation in dielectrics. Even before 1920 the classical investigations of Sommerfeld and Brillouin,[2] among others, were sufficiently complete and in accord with experimental observation to discourage further serious study. However, the Lorentz linear model of dielectrics, the model used in those early studies, is adequate only if the light intensity is low, or if the light frequency is far from any of the atomic resonances of the dielectric medium.

It is only in the past decade that intense and practically monochromatic laser light has been available as a strong probe of optically resonant systems. The response of such systems, when strongly probed at resonance, is not well described by the Lorentz model of harmonically oscillating charges. Important nonlinearities arise in the light-dielectric interaction, and these are instrumental in producing a wide range of nonclassical effects, such as SIT,[1] photon echoes, optical nutation, and others.[3] The very recent achievement of continuously operating and continuously tunable dye lasers[4] promises to add further impetus to modern experimental studies of light propagation in resonant dielectrics.

One of the most interesting of the new phenomena observed in high-intensity coherent light propagation is frequency modulation of the electric field. By analogy with similar phenomena which are common at much lower frequencies, a frequency-modulated electromagnetic wave is said to be "chirped," and we will use this terminology frequently.

In this paper we analyze situations in which chirped optical waves may occur in SIT. In addition to a time-varying field phase we also allow nonresonant nonlinearities of the host medium in which the resonant atoms are imbedded.

We imply several restrictions on our work by the words chosen to describe it. By SIT we mean the propagation *without distortion* of an electric wave's envelope and phase through a medium containing resonant atoms. Because we will always work close to resonance, we assume, along with McCall and Hahn, that the resonant atoms may be treated as if they had only two energy levels separated by a transition frequency almost equal to the field-carrier frequency. In the most general case, these atoms are embedded in a host medium

which is not resonant but which nevertheless may have important nonlinear features.

In addition, the resonant atoms may have an *inhomogeneously broadened* absorption line. That is, the various resonant atoms may have slightly different values for their transition frequency because of slightly different local environments in the host. We assume, however, that *homogeneous broadening* is absent. In other words, we are interested in times short compared with, for example, collision times or the times of other effects which interrupt the individual atoms dipole-phase coherence, but in times which may be long compared with the macroscopic dipole-phase coherence of the entire collection of resonant atoms. In the language of magnetic resonance,[5] we may say $T_2^* \leq T_2' = \infty$.

We investigate first the limitations on chirping imposed by the special assumptions of McCall and Hahn in their development of the theory of SIT. There we confine ourselves to the restricted case that we call "bare" SIT. By this we mean that only the collection of near-resonant atoms is allowed to interact with the electromagnetic field. We find that chirping is not possible in this context.

However, in Sec. III we show that by lifting the explicit McCall-Hahn assumptions of time-independent ϕ, it is possible even in "bare" SIT to have chirped steady-state pulses. We find that such pulses are inevitably multipulse wave trains,[6] and never single pulses. We give the general dipole spectral-response function, as well as the pulse velocity and dispersion, and exhibit analytic expressions for atomic variables and pulse envelope and phase which are free of redundancies introduced in earlier work.[7] The natural interpolation between the results of Crisp[6(b)] and Eberly[6(c)] is presented.

We are also able to study, using our analytic expressions for electric field phase, those situations in which the phase changes very abruptly. As a special case we discuss the instantaneous phase jump $\Delta\phi = \pi$, and its connection with certain zero-π pulses.

We devote Sec. IV to an experiment-oriented interpretation of the very complicated general results of Sec. III.

In Sec. V the restriction to bare SIT is lifted. That is, we now allow the resonant atoms to be embedded in a background host medium which may also interact nonlinearly (although nonresonantly) with the electromagnetic field. We show in what sense chirping is then not only possible but inevitable, and extend related results already reported briefly by us.[8] A sample analytic solution is given of a single pulse, which propagates without shape change and which is chirped. Undistorted multipulse trains also occur in this context.

We conclude with a short summary in Sec. VI and an appendix which discusses the nonlinear susceptability of Sec. V as a function of frequency rather than time.

II. EQUATIONS OF MOTION

A. Schrödinger Equation for Active Atoms

In this section we state the basic equations of motion for the interaction of two-level atoms with resonant radiation. We will follow the Feynman-Vernon-Hellwarth[9] formalism in describing this interaction.

The interaction Hamiltonian of the active resonant atoms is taken to have the usual form $V = -\vec{E}\cdot\vec{P}$. Here $\vec{P}$ is the electric dipole moment of the atom and $\vec{E}$ is the electric field of the radiation. For $\Delta m = +1$ transitions, for example, between states $|a\rangle$ and $|b\rangle$, the matrix elements of V are

$$V_{ab} = -p(\hat{x} - i\hat{y})\cdot\vec{E}\,. \tag{2.1}$$

Following Feynman *et al.*[9] we may construct two real vectors $\vec{r}$ and $\vec{\omega}$, whose components along three fictitious Cartesian axes are defined by

$$r_1 = \rho_{ab} + \rho_{ba}\,, \quad r_2 = i(\rho_{ab} - \rho_{ba})\,, \quad r_3 = \rho_{aa} - \rho_{bb}\,, \tag{2.2}$$

$$\omega_1 = (V_{ab} + V_{ba})/\hbar, \quad \omega_2 = i(V_{ab} - V_{ba})/\hbar\,, \quad \omega_3 = \omega_0\,. \tag{2.3}$$

The ρ's are the elements of the density matrix in the Schrödinger picture, and ω_0 is the atomic transition frequency. When Schrödinger's equation is expressed in terms of these two vectors, it takes the following form:

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}\,. \tag{2.4}$$

It follows immediately from (2.4) and the normalization of the wave function that the magnitude of $\vec{r}$ stays constant and that its end point describes a curve on the surface of the unit sphere.[1,9]

B. Maxwell's Equation and Nonresonant Polarization

The complete description of our model of light-matter interaction will require in addition to the Schrödinger equation, Eq. (2.4), the Maxwell equations describing the field.

We first assume that, in the ideal case, the field is a plane circularly polarized wave propagating in the z direction, and may be described adequately without quantization. Thus we write it as the product of an envelope and a carrier as follows:

$$\vec{E}(z,t) = \mathcal{E}(z,t)[\hat{x}\cos\Phi(z,t) + \hat{y}\sin\Phi(z,t)]\,, \tag{2.5}$$

$$\Phi(z,t) = \omega t - (k + \Delta k)z + \phi(z,t)\,. \tag{2.6}$$

Here $\mathcal{E}(z,t)$ is the real envelope of the field and $\Phi(z,t)$ the total field phase. The real quantities $\mathcal{E}(z,t)$ and $\phi(z,t)$ are assumed to be slowly varying

functions of z and t, changing very little over many wavelengths $2\pi/k$ and periods $2\pi/\omega$. We have set $k=\eta_0\omega/c$, where η_0 is the linear part of the index of refraction of the host medium [see Eq. (2.9)], ω is the carrier frequency, and c is the velocity of light in vacuum. The quantity Δk reflects the possible existence of dispersive effects arising in the interaction between the light and the resonant atoms. It is usually much smaller than k. We will see that Δk can be defined in such a way that ω can be interpreted as the pulse-carrier frequency,[10] independent of amplitude or phase modulation.

The theory is obviously not restricted to circularly polarized waves. A linearly polarized wave can always be decomposed into two circularly polarized waves rotating in opposite senses. The rotating-wave approximation[5] then sanctions the elimination of the off-resonance component.

The electric field (2.5) must satisfy the Maxwell wave equation

$$\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\vec{E}+4\pi(\vec{P}_{nr}+\vec{P}_{res})], \tag{2.7}$$

where the total polarization that drives the electric field $\vec{E}$ has been written as a sum of two parts: $\vec{P}_{nr}$ which is the polarization due to any nonresonant dielectric present and $\vec{P}_{res}$ which is the polarization due to resonant, or "active," atoms embedded in the nonresonant dielectric. Although the description of resonant atoms embedded in a dielectric may be literally valid (in the case of ruby, for example), the active atoms might just as well constitute a gas, in which case the background dielectric is simply the vacuum.

In the theory of linear dielectrics, $\vec{P}_{nr}$ is related to $\vec{E}$ through the following relations:

$$\vec{E}+4\pi\vec{P}_{nr}=(1+4\pi\chi_{nr})\vec{E}=\epsilon\vec{E}=\eta^2\vec{E}\ . \tag{2.8}$$

Here χ_{nr} is the electric susceptibility of the material, ϵ its dielectric constant, and η its index of refraction. In linear optics η is assumed independent of the field strength $\vec{E}$. This is, of course, only an approximation, although an excellent one whenever the fields involved are not too strong. For high-intensity fields, the response of the dielectric is actually nonlinear. We translate this fact into our theory by allowing η to depend on $\vec{E}$.

Since we are interested in the qualitative effects on SIT which may arise due to a nonlinear η, rather than in accurately modeling pulses in a specific material, it should be sufficient to choose a simple form for the nonlinear part of η. For the purpose of this paper we assume η is given by

$$\eta=\eta_0(1+\beta\mathcal{E}^2). \tag{2.9}$$

An analogous dependence of the index of refraction on the field strength arises in studies of the Kerr effect. By analogy we will refer to β in Eq. (2.9) as the Kerr constant. In all materials β is a very small quantity of the order of 10^{-11} esu or less, so that the term $\beta\mathcal{E}^2$ is smaller than unity even in strong fields. For the field intensities used in experiments on bare SIT[1] it is justified to neglect this term altogether. Only in high-intensity ultrashort pulses,[11] or perhaps in self-trapped filaments, does the term $\beta\mathcal{E}^2$ become sizable. In any event we will assume that β is small enough that its higher powers may be neglected.

We now substitute (2.8) into (2.7) to arrive at the following form of the wave equation:

$$\frac{\partial^2\vec{E}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\eta^2\vec{E})=\frac{4\pi}{c^2}\frac{\partial^2\vec{P}_{res}}{\partial t^2}\ . \tag{2.10}$$

In linear theories η is usually defined as a function of frequency, while in our theory η depends on space and time through the solution for $\vec{E}$ of the wave equation itself. The consequences for η in the frequency domain are investigated in the Appendix.

C. Resonant Polarization and Coupled Nonlinear Dynamics

First let us turn our attention to the resonant part of the polarization appearing on the right-hand side of (2.10). We will express $\vec{P}_{res}$ in terms of the expectations of the dipole moments of the individual active atoms, and will explicitly allow for inhomogeneous broadening. Let $\mathfrak{N}$ be the number of resonant atoms per unit volume of the sample and let their resonant frequencies ω_0 exhibit a distribution about the field-carrier frequency ω given by $g(\gamma)=g(\omega_0-\omega)$, with $\int_{-\infty}^{\infty}g(\gamma)\,d\gamma=1$. The quantity $\gamma=\omega_0-\omega$ indicates how far an individual atomic frequency ω_0 is detuned from the field-carrier frequency ω.

The number per unit volume of these atoms in a frequency range $d\gamma$ is $\mathfrak{N}g(\gamma)\,d\gamma$, and the corresponding dipole moment is $\mathfrak{N}p(\hat{x}r_1+\hat{y}r_2)g(\gamma)\,d\gamma$. Therefore, the macroscopic polarization density of resonant atoms is given by

$$\vec{P}_{res}=\mathfrak{N}p\int_{-\infty}^{\infty}(\hat{x}r_1+\hat{y}r_2)\,g(\gamma)\,d\gamma\ . \tag{2.11}$$

We return now to Schrödinger's equation (2.4) and transform from the laboratory system $(\hat{x},\hat{y},\hat{z})$ to a rotating coordinate system whose unit vectors $\hat{e}_1$ and $\hat{e}_2$ rotate about $\hat{e}_3\equiv\hat{z}$ with angular speed $\dot{\Phi}$. In this rotating frame the components of $\vec{\omega}$ are quasistationary and always lie in the $(\hat{e}_1,\hat{e}_2)$ plane. Equation (2.4) itself becomes

$$\left(\frac{d\vec{r}}{dt}\right)_{rot}=(\vec{\omega}-\hat{k}\dot{\Phi})\times\vec{r}\ . \tag{2.12}$$

If we write u, v, and w for the components of $\vec{r}$ along the axes $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, respectively, then Eq. (2.12) can be written in component form as follows:

$$\dot{u} = -(\gamma - \dot{\phi})v \,, \tag{2.13}$$

$$\dot{v} = (\gamma - \dot{\phi})u + \kappa \mathcal{E} w \,, \tag{2.14}$$

$$\dot{w} = -\kappa \mathcal{E} v \,, \tag{2.15}$$

where $\kappa = 2p/\hbar$. Since $\vec{r}$ had magnitude unity, its components in the rotating frame must also satisfy

$$u^2 + v^2 + w^2 = 1 \,, \tag{2.16}$$

which is seen to be consistent with Eqs. (2.13)–(2.15).

Finally, it is easy to show that the resonant polarization $\vec{P}_{res}$ has the same form in the rotating as in the laboratory system, namely,

$$\vec{P}_{res} = \mathfrak{N} p \int_{-\infty}^{\infty} (\hat{e}_1 u + \hat{e}_2 v) g(\gamma)\, d\gamma \,. \tag{2.17}$$

Since $\vec{E} = \hat{e}_1 \mathcal{E}$ we see that u is the in-phase or dispersion component of the polarization and that v is its out-of-phase or absorption component.

We are especially interested in studying the propagation of undistorted shape-preserving pulses. By shape-preserving pulses we mean solutions of (2.10) and (2.13)–(2.15) in which the quantities $\mathcal{E}$, ϕ, u, v, and w depend on their arguments z and t only through the combination $\zeta = t - z/V$. V is the velocity with which these shape-preserving pulses travel through the medium. Although such pulses are not the most general loss-free pulses (see Lamb, Ref. 1) they are, in a certain sense, the most basic, since other loss-free pulses gradually evolve toward a shape-preserving state. Physically speaking, they are asymptotic pulses in the sense that we expect an experimental pulse to travel many absorption lengths before its shape stops changing and settles down to its steady form. Because all parts of such pulses move at the same velocity V, we will frequently refer to them as "steady-state" pulses.

The existence of such undistorted solutions of the quantum system, (2.10) and (2.13)–(2.15), representing single pulses, as well as pulse trains of a variety of shapes, is of obvious interest from both physical and mathematical points of view. It is well known, [2] according to the *classical* theory based on the Lorentz model of harmonically oscillating charges, that only pure sinusoidal waves can propagate without change in shape.

We now write the wave equation, Eq. (2.10), in the rotating coordinate system, perform the indicated differentiations and apply the slowly varying approximation (by neglecting second derivatives of $\mathcal{E}$, ϕ, and η, as well as the products of their first derivatives), and express the space and time derivatives in terms of ζ derivatives. Finally, we equate the coefficients of the orthogonal rotating vectors $\hat{e}_1$ and $\hat{e}_2$ separately to arrive at the following pair of scalar equations, sometimes called the reduced Maxwell equations:

$$\dot{\mathcal{E}} = -(\kappa/m^2) \int_{-\infty}^{\infty} v(\zeta, \gamma) g(\gamma)\, d\gamma \,, \tag{2.18}$$

$$(\dot{\phi} - \omega\beta\mathcal{E}^2/c_0\delta + \Delta k/\delta)\mathcal{E} = +(\kappa/m^2) \int_{-\infty}^{\infty} u(\zeta, \gamma) g(\gamma)\, d\gamma \,, \tag{2.19}$$

where

$$1/m^2 = \pi \mathfrak{N} \hbar \omega / c \eta_0 \delta$$

and

$$\delta = 1/V - 1/c_0 \,.$$

Thus, Maxwell's wave equation for the electric field driven by both resonant and nonresonant dipole sources, together with the Schrödinger equation governing the resonant dipoles, can be cast into the relatively simple forms exhibited in (2.18), (2.19), and (2.13)–(2.15). Of course we must keep in mind the restrictions to steady-state pulses, and slowly varying envelopes and phases.

Despite the deceptively simple form of (2.13)–(2.15) and (2.18) and (2.19), they are obviously thoroughly nonlinear and tightly coupled. One of the real achievements of McCall and Hahn was to present analytic solutions even in the simplest special case when $\dot{\phi} = 0$ and $\beta = 0$. Building on their lead, we will exhibit a much wider class of analytic solutions, and interpret the new solutions physically.

III. CHIRPING IN BARE SIT

A. Solution of Equations of Motion

We will recall that by "bare" SIT we mean SIT of the asymptotic steady-state type described by the original McCall-Hahn equations, unmodified by additional nonlinearities, higher-order effects, or dissipative losses. Bare SIT includes the 2π-sech pulse of McCall and Hahn, and the $\infty\pi$-pulse and 0π-pulse trains described subsequently. Still more general steady-state fields are possible within the framework of bare SIT if one merely allows for the possibility of a time-varying phase. [12] Equations (2.13)–(2.15), (2.18), and (2.19) are equivalent to the original McCall-Hahn equations when $\dot{\phi} = 0$, and when the Kerr constant β is ignored.

The difficulty in solving this system, even when $\beta = 0$, lies in the fact that the pulse $\vec{E}$ is supported by an inhomogeneously broadened atomic line $g(\gamma)$. In the search for solutions, one may first assume that the line is very sharp, in effect unbroadened, so that it can be represented by a δ function centered at some particular value of γ, say $\gamma_1 = \omega_{01} - \omega$. Then $g(\gamma) = \delta(\gamma - \gamma_1)$ and the solving of the system is intrinsically simplified. The solution is now easily obtained and one observes that the result for $v(\zeta, \gamma_1)$ appears in a factorized form: $F(\gamma_1)\, v(\zeta, 0)$. We have retained this factorization as a basic assumption even for the case of an arbitrary, continuous atomic line-shape function $g(\gamma)$. To the best of our knowledge, only under this

special assumption or its equivalent[7] have analytic solutions been found for arbitrary $g(\gamma)$. Let us also remark that this assumption does not always work, as we will see both in this section and in Sec. V.

Therefore, we start by assuming that

$$v(\zeta, \gamma) = F(\gamma)\, v(\zeta, 0) \tag{3.1}$$

with an implied normalization $F(0) = 1$. We will call $F(\gamma)$ the *dipole spectral-response function.* Using Eqs. (3.1), (2.15), and (2.18), we arrive at the following first integral:

$$\mathcal{E}^2 = [2/\mu^2 F(\gamma)]\,[w(\zeta, \gamma) - w_0(\gamma)]\,, \tag{3.2}$$

where w_0 is an integration constant smaller than or equal to 1, since $|w| \leq 1$ and $w - w_0 \geq 0$, and where we have set $1/\mu^2 = (1/m^2)\int_{-\infty}^{\infty} F(\gamma)\, g(\gamma)\, d\gamma$. This first integral expresses the conservation of energy which is exchanged between the field and the atomic system. It is merely a manifestation of the Poynting theorem. Despite appearances, it does not depend on the detuning frequency γ, of course [cf. Eqs. (3.19)–(3.23)].

Note that one can determine already the maximum modulation of intensity possible in the SIT wave. $(\Delta\mathcal{E}^2)_{\max}$ is $4/\mu^2 F$ since the maximum excursion in w is from -1 to $+1$ [cf. Eq. (2.16)].

By differentiating Eq. (2.19) with $\beta = 0$ and using Eq. (2.13) as well as Eq. (3.1), we obtain the following important relation between the envelope of the field and the modulation of its phase:

$$2\dot{\phi}\dot{\mathcal{E}} + \ddot{\phi}\mathcal{E} = (\bar{\gamma} - \Delta k/\delta)\dot{\mathcal{E}}\,. \tag{3.3}$$

The existence of some such relation is necessary, of course, since the phase ϕ and envelope $\mathcal{E}$ were introduced in place of the single variable $\vec{E}$. Here $\bar{\gamma} = \bar{\omega}_0 - \omega$ is an average of the detuning frequency with respect to a weight function $F(\gamma)\, g(\gamma)$, i.e.,

$$\bar{\gamma} = \int_{-\infty}^{\infty} \gamma F(\gamma)\, g(\gamma)\, d\gamma \Big/ \int_{-\infty}^{\infty} F(\gamma)\, g(\gamma)\, d\gamma\,.$$

From (3.3) we see immediately that for any unchirped pulse we must have $\Delta k = \bar{\gamma}\delta$. To see that corresponding solutions indeed exist we observe that $\mathcal{E}^2$ is an integrating factor for Eq. (3.3), and that it can be integrated with the result

$$\dot{\phi} = \tfrac{1}{2}(\bar{\gamma} - \Delta k/\delta) + C_1/\mathcal{E}^2\,, \tag{3.4}$$

C_1 being a new integration constant which we evaluate in (3.20) and interpret in Sec. IV.

Recall the discussion following Eq. (2.6). Our interpretation of ω is as the *carrier* frequency. By this definition, those parts of $d\Phi/dt$ depending on the *envelope* $\mathcal{E}$ in any way are not associated with ω, but with ϕ. Similarly, the parts of $d\Phi/dt$ independent of $\mathcal{E}$ must add up to ω. Therefore, we have to have

$$\Delta k = \bar{\gamma}\delta \tag{3.5}$$

and

$$\dot{\phi} = C_1/\mathcal{E}^2\,. \tag{3.6}$$

Thus, the chirp in bare SIT, when it is present, is inversely proportional to the intensity. This result appears already in Eberly,[13a] Matulic,[10] Barone,[13b] and Dialetis.[7] We will sometimes refer to the constant C_1 as the "chirping constant" because only when it does not vanish do we get nontrivial phase modulations, $\dot{\phi} \neq \text{const}$, in bare SIT.

Substitution of Eqs. (2.18) and (3.6) into (2.13), followed by an integration, yields another first integral:

$$u = \frac{\mu^2 F\gamma}{\kappa}\,\mathcal{E} + \frac{C_1\mu^2 F}{\kappa}\,\frac{1}{\mathcal{E}} + C_3\,, \tag{3.7}$$

where C_3 is another constant of integration, one that has apparently been overlooked altogether in other work on the subject.[7,13]

We now show that one may proceed directly to expressions of interest. We first obtain an equation for $\mathcal{E}$ itself. In this way we avoid the apparent multiplicity of cases to be considered separately which have unnecessarily complicated previous work (cf. Ref. 7, for example). By differentiating (2.18) once, and using (2.14) for $\dot{v}$, as well as (3.2), (3.6), and (3.7) for w, $\dot{\phi}$, and u, we find a nonlinear differential equation for $\mathcal{E}$:

$$\ddot{\mathcal{E}} = -\frac{\kappa^2}{2}\,\mathcal{E}^3 - \left(\frac{\kappa^2 w_0}{\mu^2 F} + \gamma^2\right)\mathcal{E} + C_1^2\,\mathcal{E}^{-3} + \frac{C_1 C_3 \kappa}{\mu^2 F}\,\mathcal{E}^{-2} - \frac{C_3\kappa\gamma}{\mu^2 F}\,. \tag{3.8}$$

The relation (3.8) is the most general for distortionless propagation in bare SIT.[14] An integration of Eq. (3.8) followed by a multiplication by $\mathcal{E}^2$ yields

$$\mathcal{E}^2\dot{\mathcal{E}}^2 = \tfrac{1}{4}\kappa^2(-\mathcal{E}^6 + M\mathcal{E}^4 + R\mathcal{E}^3 + N\mathcal{E}^2 + T\mathcal{E} + Q)\,, \tag{3.9}$$

where

$$M = -(4/\kappa^2)(\kappa^2 w_0/\mu^2 F + \gamma^2)\,,$$

$$R = -8C_3\gamma/\kappa\mu^2 F\,,$$

$$N = 4C_2/\kappa^2\,,$$

$$T = -8C_1 C_3/\kappa\mu^2 F\,,$$

$$Q = -4C_1^2/\kappa^2\,,$$

and where C_2 is still another integration constant which remains to be interpreted.

Note that (3.9) can be trivially rearranged into the form of an "energy" integral in classical mechanics: $\frac{1}{2}\dot{\mathcal{E}}^2 + V(\mathcal{E}) = \text{const}$. Thus we already could consider our job over, because the basic equation has been reduced to quadratures. However, we are more interested in the physical nature of the solutions than in their mere existence.

Equation (3.9) leads in general to an integral containing a square root of an "irreducible" poly-

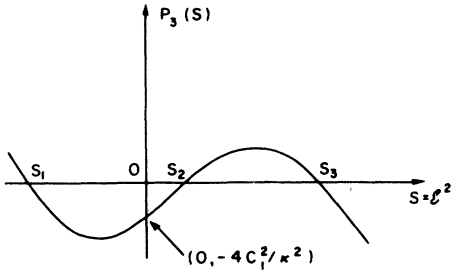

FIG. 1. General position of zeroes of the polynomial $P_3(S)$.

nomial of degree six in $\mathcal{E}$, and consequently cannot be expressed in terms of known functions. [Dialetis's claims[7] of "exactness" and "completeness" are in error because he has found an expression equivalent not to (3.9), but only to its most interesting special case (3.10)] However, there are two rather general cases in which numerical solution is unnecessary. One of them is obtained when the "chirping constant" C_1 vanishes, which is the case mentioned by Jaynes.[14] The integration of (3.9) then leads to elliptic functions for $\mathcal{E}$ and (3.4) gives $\dot{\phi}=0$. From our present point of view the opposite case, in which $C_3=0$ but $C_1\neq 0$, is more interesting simply because it allows us to study chirping. What is more, it can be shown that there is no *factorable* solution when $C_3\neq 0$.

Under this condition, $R=T=0$, and the polynomial in (3.9) contains only even powers of $\mathcal{E}$ and is "reducible" to a cubic in $S=\mathcal{E}^2$. Thus, Eq. (3.9) can be written as

$$\int_{S_0}^{S}[P_3(S)]^{-1/2}dS=\kappa\zeta\ , \tag{3.10}$$

where $S_0=S(0)$ and $P_3(S)=-S^3+MS^2+NS+Q=-(S-S_1)(S-S_2)(S-S_3)$. Equation (3.10) now leads in general to elliptic functions and furnishes a whole range of analytic solutions to our problem.

We assume the following ordering of the zeroes of $P_3(S)$: $S_1<S_2,\ S_3$. The nature of the solutions depends on the relative position of these zeroes which, in turn, depends on the two constants of integration C_1 and C_2. Both of these constants, and especially C_1, play important roles in the theory. Because of this we will investigate separately the case in which $C_1=C_2=0$.

B. Uniqueness of Single-Pulse Solutions

If $C_1=C_2=0$, then Eq. (3.6) gives $\dot{\phi}=0$, and only trivial solutions for the phase ϕ are possible. But with $\dot{\phi}=0$, Eqs. (2.13)–(2.15) and (2.18) and (2.19) are the McCall-Hahn equations and they naturally give the McCall-Hahn single-pulse solution. Thus in the absence of the Kerr-effect interaction[8] or some other effect that leads to a modification of the slowly varying envelope and phase equations (cf. Marth and Eberly, Ref. 12), *a single bare* SIT *pulse cannot exhibit chirping.*[10] Moreover, the McCall-Hahn hyperbolic secant pulse is a unique solution of the dynamical equations in the sense that it is the only *single-pulse* steady-state solution of these equations.

C. Steady-State Pulse Trains

Let us return to Eq. (3.10). The integral on the left-hand side of this expression is real only for those values of S for which $P_3(S)>0$ and then it leads in general to elliptic functions. It is known that the nature of these functions depends on the zeroes of the cubic $P_3(S)$. Since the coefficient of S^3 in $P_3(S)$ is -1 and $S_1S_2S_3=Q=-4C_1^2/\kappa^2\leq 0$, either all three zeroes are nonpositive or only one of them is nonpositive. The first possibility is ruled out on physical grounds since $S=\mathcal{E}^2>0$. Similarly, when two of the zeroes are complex, there are no positive values of S for which $P_3(S)>0$. Hence, we must have $S_1<0<S_2,\ S_3$. We illustrate in Fig. 1 one such arrangement.

The square of the field will therefore oscillate between $\mathcal{E}^2=S_2$ and $\mathcal{E}^2=S_3$, and the field itself between $\mathcal{E}=(S_2)^{1/2}$ and $\mathcal{E}=(S_3)^{1/2}$. (We make the arbitrary choice of taking the positive sign for these square roots. The opposite choice will simply give the symmetric boundary of the envelope.)

When $C_1=0$ and there is no chirping, then $S=0$ is a root of $P_3(S)$ and the curve in Fig. 1 must go through the origin. The physically meaningful situations when this can happen are indicated in Fig. 2. The integral in the left-hand side of (3.10) is a standard elliptic integral that we can write in the following way:

$$S(\zeta)=S_3[1-l^2\,\mathrm{sn}^2(\zeta/\tau;k)]\ , \tag{3.11}$$

where the new parameters τ, k, and l will be shown in Sec. IV to have a direct physical meaning for the pulse train itself. They can therefore be taken to be independent of any particular atom's detuning frequency. One consequence is the restrictions given below on $C_1,\ldots,F(\gamma)$. In terms of the polynomial roots, τ, k, and l are

$$1/\tau=\tfrac{1}{2}\kappa\,(S_3-S_1)^{1/2}\ , \tag{3.12}$$

$$k^2=(S_3-S_2)/(S_3-S_1)\leq 1\ , \tag{3.13}$$

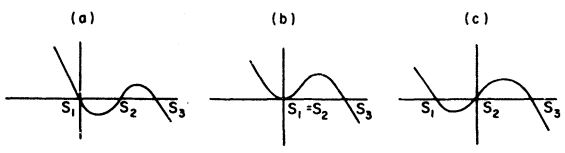

FIG. 2. Position of zeroes of the polynomial $P_3(S)$ leading to unchirped pulses.

$$l^2 = (S_3 - S_2)/S_3 \leq 1 \,. \tag{3.14}$$

Clearly, the character of our solutions will depend as much on the parameter l, which we may call the chirping parameter, as on the more familiar elliptic modulus k and the "pulse length" τ. We know that k satisfies the relation $|k^2| \leq 1$. On the other hand, it is easy to show that the chirping parameter l satisfies the following relation:

$$|k^2| \leq |l^2| \leq 1. \tag{3.15}$$

One may notice at this point that when $l = k$ we have $S_1 = 0$, which leads to the first-type solution of Crisp,[6] and when $l = 1$, then $S_2 = 0$ and we get Eberly's solution.[6]

Using (3.12)–(3.14), we obtain $S_3 = 4k^2/l^2\kappa^2\tau^2$, with the help of which (3.11) can be rewritten as

$$\mathcal{E}(\xi) = \mathcal{E}_0(k/l)\,[1 - l^2\,\mathrm{sn}^2(\xi/\tau; k)]^{1/2} \,, \tag{3.16}$$

where $\mathcal{E}_0 = 2/\kappa\tau$. Equation (3.16) contains as a special case the envelopes of all undistorted pulses previously found, including both of Dialetis's instances of chirped pulse trains,[7] one of which corresponds to positive and the other to negative values of k^2 and l^2. All the functions given by Eq. (3.16) are positive and periodic with period $2K$ [$K \equiv K(k)$ is the complete elliptic integral of the first kind]. Because of possible phase variations we still have the possibility of finding "zero-π" pulses even though $\mathcal{E}$ is always positive according to our convention. We will have more to say about this question at the end of this section.

Corresponding to our solution in (3.16) for the real field envelope, one obtains the following expressions for the components of the atomic polarization and for the atom's energy:

$$u(\xi, \gamma) = F(\gamma)\left(\frac{2k\mu^2\gamma}{l\kappa^2\tau}\,(1 - l^2\,\mathrm{sn}^2)^{1/2} + \frac{C_1 l\tau\mu^2}{2k}\,(1 - l^2\,\mathrm{sn}^2)^{-1/2}\right), \tag{3.17}$$

$$v(\xi, \gamma) = F(\gamma)\,\frac{2lk\mu^2}{\kappa^2\tau^2}\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}\,(1 - l^2\,\mathrm{sn}^2)^{-1/2} \,, \tag{3.18}$$

$$w(\xi, \gamma) = w_0(\gamma) + \frac{2k^2\mu^2}{l^2\kappa^2\tau^2}\,(1 - l^2\,\mathrm{sn}^2) \,, \tag{3.19}$$

where the "argument" of the elliptic functions is understood to be $(\xi/\tau; k)$. Finally, the coefficients $w_0(\gamma)$, $F(\gamma)$, and μ, as well as C_1^2, may be given explicitly as functions of k, l, and τ:

$$C_1^2 = \frac{16k^4}{l^6\kappa^4\tau^6}\,(1 - l^2)(l^2 - k^2) \,, \tag{3.20}$$

$$w_0(\gamma) = \frac{l^2(1 - \gamma^2\tau^2) - k^2(3 - l^2)}{\{[l^2(1 - \gamma^2\tau^2) - k^2(1 - l^2)]^2 + 4k^2[l\gamma\tau + (1 - l^2)^{1/2}(l^2 - k^2)^{1/2}]^2\}^{1/2}} \,, \tag{3.21}$$

$$\mu = \frac{l\kappa\tau}{[l^4(1 - k^2)^2 + 2l^2k^2(1 + k^2) - 3k^4]^{1/4}} \,, \tag{3.22}$$

$$F(\gamma) = \left(\frac{l^4(1 - k^2)^2 + 2l^2k^2(1 + k^2) - 3k^4}{[l^2(1 - \gamma^2\tau^2) - k^2(1 - l^2)]^2 + 4k^2[l\gamma\tau + (1 - l^2)^{1/2}(l^2 - k^2)^{1/2}]^2}\right)^{1/2} . \tag{3.23}$$

These relations embody all of the restrictions needed to ensure that the solutions for $\mathcal{E}$ and ϕ are independent of the detuning frequency γ, as is required on physical grounds. Of course, these formidable expressions still require interpretation in physical terms. The interpretation, which is equivalent to a prescription for adjusting k and l experimentally, will be given in Sec. IV.

Substituting (3.16) and (3.19) into (3.6), we obtain

$$(\pm)\dot{\phi} = \frac{1}{\tau}\,\frac{[(1 - l^2)(l^2 - k^2)]^{1/2}}{l(1 - l^2\,\mathrm{sn}^2)} \,, \tag{3.24}$$

the sign of ϕ being as yet undetermined. Integrating (3.24) we obtain the phase function itself:

$$(\pm)\phi = \frac{[(1 - l^2)(l^2 - k^2)]^{1/2}}{l}\,\Pi\left(\frac{\xi}{\tau}\,; l^2; k^2\right), \tag{3.25}$$

where $\Pi(u, l^2, k^2)$ is the normal or incomplete elliptic integral of the third kind.

The velocity of these pulses is given by

$$\frac{1}{V} = \frac{1}{c_0} + \frac{\alpha_B'\tau^2}{2\pi g(0)}\int_{-\infty}^{\infty} F(\gamma)g(\gamma)\,d\gamma \,, \tag{3.26}$$

where

$$\alpha_B' = \frac{l^2\alpha_B}{[l^4(1 - k^2)^2 + 2k^2l^2(1 + k^2) - 3k^4]^{1/2}} \,,$$

α_B being the reciprocal of Beer's length for single pulses.[1] The pulse trains are, therefore, also delayed in propagation. V may be much smaller than c_0.

The dispersion relation Eq. (3.5) can now be written as

$$\Delta k = \frac{\alpha_B'\tau^2}{2\pi g(0)}\int_{-\infty}^{\infty} \gamma g(\gamma)F(\gamma)\,d\gamma \,, \tag{3.27}$$

and in general Δk is different from zero, even for symmetric $g(\gamma)$.

The area[1] of these pulses, defined as $\theta = \kappa \int_{-K\tau}^{K\tau} \mathcal{E}(\xi) d\xi$, i.e., the area under one oscillation of the envelope, can be expressed as follows:

$$\theta = \frac{4k}{l(1-k^2)^{1/2}} \left\{ \frac{l^2}{k^2} \Pi\left[\frac{\pi}{2}; \frac{k^2}{k^2-1}; \left(\frac{l^2-k^2}{1-k^2}\right)^{1/2}\right] - \frac{l^2-k^2}{k^2} K\left[\left(\frac{l^2-k^2}{1-k^2}\right)^{1/2}\right] \right\}, \quad (3.28)$$

Π being the complete elliptic integral of the third kind and K the complete elliptic integral of the first kind.

The width of these pulses, defined as full width at half-maximum of one oscillation, may be written as

$$T_{1/2} = 2\tau \, \mathrm{sn}^{-1}([2 + l^2 - 2(1-l^2)^{1/2}]^{1/2}/2l; k) . \quad (3.29)$$

Some comments on these solutions are in order. In the first place, we have been able to find them only by assuming that $v(\zeta, \gamma)$ can be factored into a product of a function of ζ alone and a function of γ alone. As far as we know, there have been found no analytic solutions of inhomogeneously broadened optical pulse equations which do not satisfy this factorization condition.

Next, from (3.20) we see that $C_1 = 0$ only when $l = 1$ or $l = k$. Then $\dot{\phi} = 0$ and these are the only two unchirped pulse trains. For these values of l the general solutions have been reported previously[6,7] and we give only the expressions for $F(\gamma)$:

$$F = k^2/[(k^2 - \gamma^2\tau^2)^2 + 4\gamma^2\tau^2]^{1/2}, \quad l = k \quad (3.30)$$

$$F = 1/[(1 - \gamma^2\tau^2)^2 + 4k^2\gamma^2\tau^2]^{1/2}, \quad l = 1 . \quad (3.31)$$

If from either of these expressions we eliminate the parameter k in favor of the constant w_0 of Eq. (3.2), then the expression for the spectral response function reduces to the inclusive form given already by one of us[6c]:

$$F(\gamma) = (\kappa/\mu\gamma\xi)^2 [w_0 + (w_0^2 + \xi^2)^{1/2}], \quad (3.32)$$

where $\xi^2 = 1/(\gamma\tau)^4 - 1$. When $|w_0| < 1$ then (3.32) gives the F in (3.31), and when $w_0 < -1$ one gets the F of (3.30). The McCall-Hahn solutions are special cases of either (3.30) or (3.31) and are obtained by putting $k = 1$.

From the general expressions of the spectral response function $F(\gamma)$ [Eq. (3.23)], we see that it is far from having a Lorentzian shape, as in the 2π-sech pulses of McCall and Hahn. In general, it is not symmetric, and a detailed study shows that it can have strong maxima for which $F > F(0)$ (see Fig. 3). At the same points at which F has maxima, the function $\Delta w(\gamma)$, which represents the variation of the energy of an atom whose detuning is γ, also has a maximum. This is consistent with the existence of a chirp and shows that the notion of an atom "at resonance" is no longer unambiguous. The atoms with $\gamma = 0$ are not always those which are most strongly interacting with the field. It is possible to show that $\Delta w(\gamma)$ has a strong maximum near $\gamma = -|\dot{\phi}|_{\max}$. This fact suggests that the as yet undetermined sign of the chirping constant C_1 in Eq. (3.20) be given a negative value. This implies that the instantaneous frequency of the field is $d\Phi/dt = \omega - |\dot{\phi}|$. Thus, the atoms with maximum excursion of their energies are those which are nearest to this instantaneous field frequency.

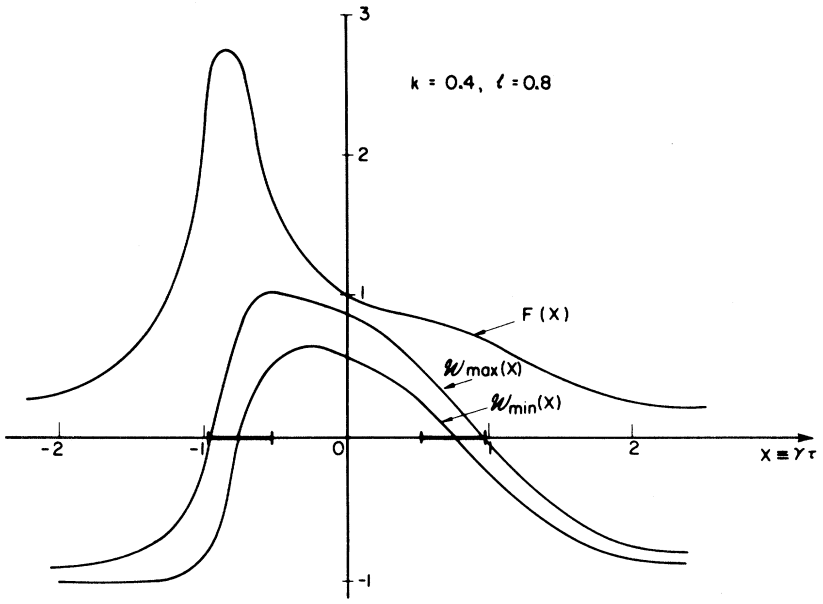

FIG. 3. F, $w_{\max}$, and $w_{\min}$ for $k = 0.4$ and $l = 0.8$ as a function of $x \equiv \gamma\tau$. Heavy lines on the x axis indicate the excursion of $\dot{\phi}\tau$.

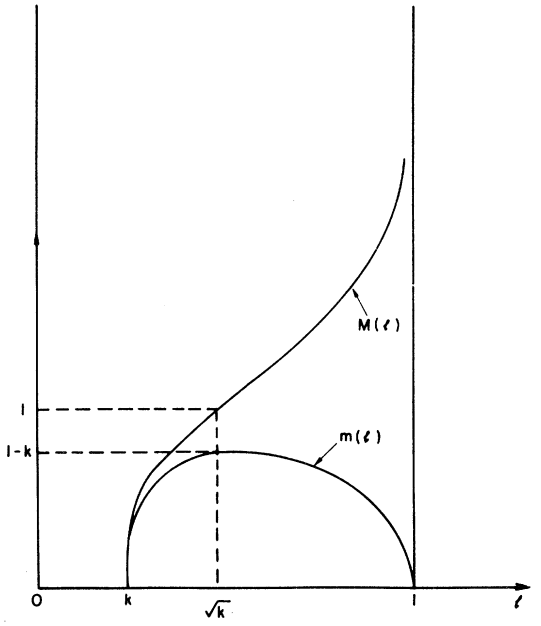

FIG. 4. Plot of $M(l)$ and $m(l)$ as a function of l, for $k=\frac{1}{4}$.

D. Phase Modulation and Zero-π Pulses

In order to discuss further our results, especially those concerning the phase modulation $\dot{\phi}$, we define the following quantities:

$$M(l)=(1/l)\,[(l^2-k^2)/(1-l^2)]^{1/2},$$

$$m(l)=(1/l)\,[(1-l^2)(l^2-k^2)]^{1/2},$$

which are, respectively, the maximum and minimum of $|\tau\dot{\phi}|$ as functions of the chirping parameter l. They are plotted in Fig. 4 for a fixed value of k.

The function $M(l)$ is monotonically increasing from 0 to ∞ in the interval $l=k$ to $l=1$. This implies that $\dot{\phi}(\zeta)$ is undefined for $l=1$. Indeed for this particular value of l, $\dot{\phi}(\zeta)=0$ for all ζ except for $\zeta=K\tau$, where it has a singularity. This is precisely the situation for which the envelope $\mathcal{E}(\zeta)$ vanishes. The fact that the phase must have a singularity when the envelope vanishes follows, of course, directly from Eq. (3.2).

In the immediate neighborhood of $l=1$, $\dot{\phi}$ is not a slowly varying function, so, strictly speaking, our theory would not apply. Nevertheless, it is interesting to notice that this solution coincides with those obtained by *completely neglecting the temporal modulation of the phase of the field.*[6]

The $l=1$ case is the only case in which the field envelope is not smooth. It is the case in which $\mathcal{E}\to 0$ with a sharp cusp as $\zeta\to(2n+1)K\tau$. In order to elucidate this behavior further we calculate the change in ϕ, given by (3.25), during one period of oscillation from 0 to $2K$. This change is given by $\Delta\phi=(2/l)[(1-l^2)(l^2-k^2)]^{1/2}\Pi(K;l^2;k^2)$, which for $l=1$ reduces to $\Delta\phi=\pi$. In the case when the field envelope has zeroes ($l=1$), there is no chirping strictly speaking [$C_1=0$, Eq. (3.20)], and the phase stays constant during the whole period. The phase, however, changes abruptly by π at the points at which the envelope has its cusps, a possibility that has been suggested previously by computer studies of optical pulse propagation,[15] and that can be interpreted by saying that $\mathcal{E}$ changes sign. Such a simple interpretation fails, of course, in the majority of cases, in which a phase change equal to π occurs gradually, not abruptly.

Thus we find ourselves able to investigate analytically one of the interesting phenomena of computer studies, namely "zero-π" pulses, pulses in which the envelope becomes negative periodically, thereby allowing zero net area under the envelope. In our approach we can avoid the problems of interpretation associated with negative envelopes because we find none. However, the real part $\mathcal{E}\cos\phi$ of the *complex* envelope does become negative periodically, in just such a way that $\mathcal{E}\cos\phi$ is a smooth physically realistic function. Neither $\mathcal{E}$ nor ϕ is always smooth, but no physical significance requires smoothness of either of them separately.

To see how this comes about, let us consider the x component of the field equation (2.5),

$$E_x(T)=\mathcal{E}(T)\cos[\omega t+\phi(T)]\,. \tag{3.33}$$

(For simplicity we have set $z=0$ and $T=t/\tau$.) Equation (3.33) reduces to

$$E_x(T)=\mathcal{E}(T)\cos\phi(T)\cos\omega t \quad \text{for } l=1\,,$$

since then $\phi(T)$ is a step function that increases in steps of π, and we choose the origin of $\phi(T)$ at zero, i.e., $\phi(0)=0$. Now, with reference to Fig. 5, we see that $E_x(T)$ and its derivatives are indeed continuous for all T, because the discontinuities in $\mathcal{E}$ and ϕ counteract each other. Thus the imposition of the physical requirement of smoothness on the physical field strength $\mathcal{E}\cos\phi$ eliminates the need for the introduction of "negative envelope" solutions.

Finally, we show in Fig. 6 the graphs of the envelope $\mathcal{E}$, the "real field" $\mathcal{E}\cos\phi$, and the phase function ϕ for a near-critical case for which $l=0.991$. The envelope is obviously positive, while the field goes negative at the appropriate values of ϕ (at $\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$). It is interesting to notice how in this case the "slowly" varying envelope is modulated by an even more slowly varying function $\cos\phi$. Of course, inside the "total envelope" $\mathcal{E}\cos\phi$ we must still imagine the fast oscillations of the factor $\cos\omega t$.

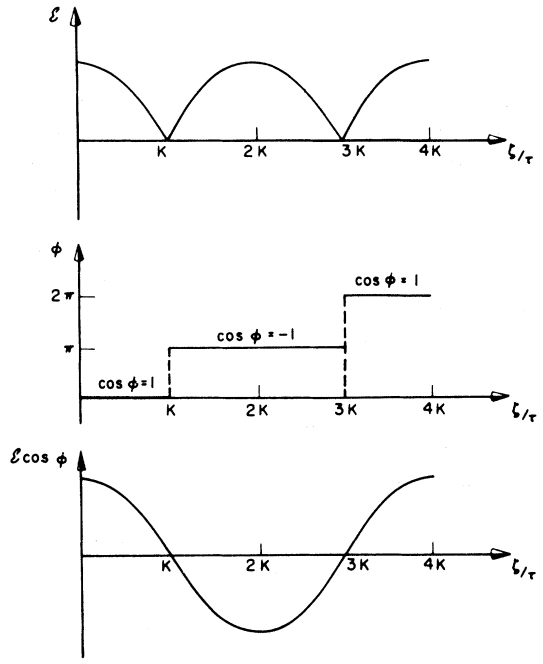

FIG. 5. Envelope, phase, and field for the critical case $l=1$.

IV. PHYSICAL MEANING OF PARAMETERS k, l, AND τ

In this section we give a physical interpretation of the amplitude modulation parameter k and the phase modulation or chirping parameter l. We also indicate a possible way of producing experimentally the amplitude and phase modulated pulse trains in bare SIT discussed in Sec. III.

Figure 7 shows a period, from $\zeta = -K\tau$ to $\zeta = K\tau$, of the pulse intensity $\mathcal{E}^2(\zeta)$, where $\mathcal{E}(\zeta)$ is given by (3.16). As shown in this figure we define

$$A^2 \equiv \mathcal{E}_M^2 - \mathcal{E}_m^2 \tag{4.1}$$

to be the amplitude of the modulation of the intensity function $\mathcal{E}^2(\zeta)$. Then, using Eqs. (3.12)–(3.14), we obtain

$$k = A/\mathcal{E}_0\,, \quad l = A/\mathcal{E}_M\,, \quad \mathcal{E}_0 = 2/\kappa\tau\,. \tag{4.2}$$

In terms of these new physical parameters A, $\mathcal{E}_M$, and $\mathcal{E}_0$ (or τ), the envelope of our oscillatory pulses and their chirp [Eqs. (3.16) and (3.24)] can be written as follows:

$$\mathcal{E}(\zeta) = \mathcal{E}_M[1 - (A/\mathcal{E}_M)^2\,\mathrm{sn}^2(\zeta/\tau;\, A/\mathcal{E}_0)]^{1/2}\,, \tag{4.3}$$

$$\dot{\phi}(\zeta) = \frac{\kappa}{2}\,\frac{[(\mathcal{E}_M^2 - A^2)(\mathcal{E}_0^2 - \mathcal{E}_M^2)]^{1/2}}{\mathcal{E}_M[1 - (A/\mathcal{E}_M)^2\,\mathrm{sn}^2(\zeta/\tau;\, A/\mathcal{E}_0)]}\,. \tag{4.4}$$

It is seen, therefore, that the nature of these oscillatory solutions depends on three physical parameters which, in principle, can be experimentally controlled: the amplitude of the intensity modulation, A^2, the peak value $\mathcal{E}_M$ of the pulse, and the pulse width $\tau = 2/\kappa\mathcal{E}_0$. These parameters are arbitrary except for the limitations implied by Eq. (3.15), namely,

$$A \le \mathcal{E}_M \le \mathcal{E}_0\,. \tag{4.5}$$

Thus, if we want to excite only one of these waves, we must impose on a resonant medium the dynamical conditions characteristic of this special wave. Note that it is sufficient to control *only the pulse envelope*, since all the atomic variables as well as the phase are determined by k and l, and thus by the envelope parameters A, $\mathcal{E}_M$, and $\mathcal{E}_0$ (or τ). In practice this might be done by using as a source a tunable cw dye laser whose output could be modulated, for instance with the help of a Pockels cell, the required frequency of this modulation being well inside the realm of experimental capabilities (of the order of 10^9 Hz if sodium vapor is used for the resonant medium). It also seems possible, once such steady-state pulses are achieved, to detect directly their phase variation by interferometric experiments.

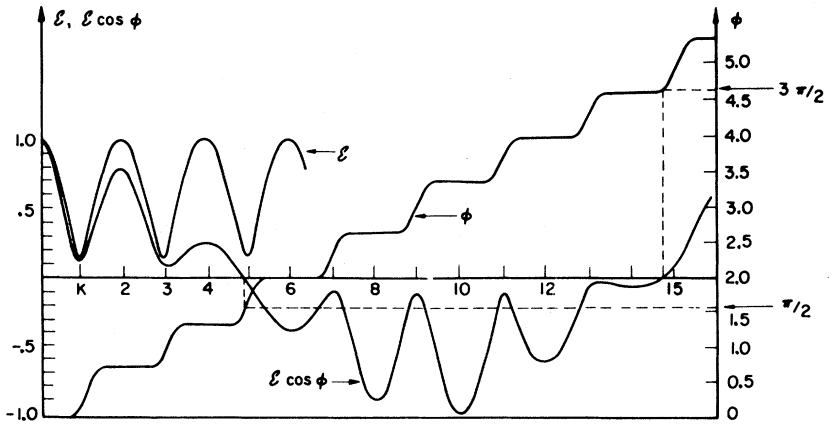

FIG. 6. Envelope, phase, and "field" for a near-critical case: $k=0.99$, $l=0.991$.

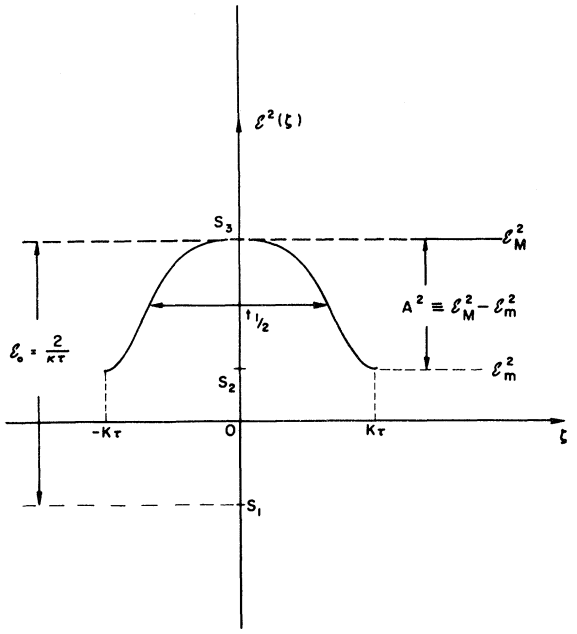

FIG. 7. Pulse intensity $\mathcal{E}^2$ as a function of $t - z/V$. S_1, S_2, and S_3 are the zeroes of $P_3(S)$ defined in Sec. III.

Notice that if the envelope modulation can be made almost complete (i.e., if $A \approx \mathcal{E}_M$ can be achieved), then the maximum chirp can become very large. In fact, this is the *only* way large chirps can be found, in apparent contrast to (3.6) which suggests that large C_1, independent of small $\mathcal{E}$, also means large $\dot{\phi}$. The explanation is that C_1 enters the equation for $\mathcal{E}$ and is not independent of it [see (3.8)].

The exact physical meaning of the parameter τ can be given by the transcendental expression

$$t_{1/2}/2\tau = \mathrm{sn}^{-1}(\tfrac{1}{2}\sqrt{2};\ \tfrac{1}{2}A\kappa\tau).$$

where $t_{1/2}$ is the full width at half-*amplitude* of one oscillation of the intensity pattern (Fig. 7).

V. STEADY-STATE PULSE CHIRPING AND NONLINEAR INDEX OF REFRACTION

In this section we will discuss the effect on SIT pulses of the background medium in which the two-level resonant atoms are embedded. We will assume that we are far from all resonances of this medium and that we can neglect its linear dispersion. Therefore, the dispersion of the host medium is assumed to arise from the *field dependence* of its index of refraction. Thus, we are confronted with a problem that includes two kinds of nonlinearities of quite different origin; one due to the interaction of the electromagnetic field with two-level resonant atoms and the other due to the nonlinear response of the passive medium.[8]

For a qualitative understanding of chirping in such a situation it should be sufficient to examine specifically the simplest case, typical of isotropic media, in which the index of refraction η is quadratically dependent on the field intensity, and write $\eta = \eta_0(1 + \beta\mathcal{E}^2)$. Assuming that the term $\beta\mathcal{E}^2$ is small enough so that its higher powers can be neglected, the equations of motion for the interaction of the light pulse with the two-level resonant atoms embedded in the nonlinear dispersive medium are only slightly different from the equations governing bare SIT. Indeed, the only difference is in the dispersive equation (2.19) on the left-hand side of which now appears the term $-\omega\beta\mathcal{E}^3/c_0\delta$.

A. Chirped Single Pulses in Nonlinear Absorbers

If we assume again, as we did in Sec. III, that the absorptive component of the resonant polarization $v(\zeta, \gamma)$ factors into a product of a function of ζ and a function of $\gamma = \omega_0 - \omega$ alone, *we obtain formal "solutions" which contradict the original equations*. Since the only arbitrary assumption made in deriving these formal "solutions" is the factorization of $v(\zeta, \gamma)$, we conclude that this factorization is not possible in the presence of a nonlinearly dispersive medium, as expected.[12] One way of avoiding this difficulty is to reduce the problem to one of an extremely sharp atomic line (i.e., one with $T_2^* \gg \tau$), which we discuss in the remainder of this section.

We now assume that the separation between the energy levels of all atoms is the same, $\omega_0^0 = (E_a - E_b)/\hbar$. Then $g(\gamma)$ is described by a δ function, $g(\gamma) = \delta(\gamma - \gamma_0) = \delta(\omega_0 - \omega_0^0)$. Substitution of this δ function into the Maxwell equations leads to the following simplified field equations:

$$\dot{\mathcal{E}} = -(\kappa/m^2)v\ , \tag{5.1}$$

$$(\dot{\phi} - \omega\beta\mathcal{E}^2/c_0\delta + \Delta k/\delta)\mathcal{E} = (\kappa/m^2)u\ . \tag{5.2}$$

At the same time we must change γ to γ_0 in the Bloch equations which are unaffected by β.

As we have already sketched briefly,[8] steady-state pulses can be shown to exist under the circumstances envisioned here. Following the same method used in solving the bare SIT equations in Sec. III we arrive at a relation which looks very much the same as Eq. (3.9). We will write it as follows:

$$\mathcal{E}^2\dot{\mathcal{E}}^2 = \tfrac{1}{4}\kappa^2(1 - 2\gamma_0\omega\beta/\kappa^2 c_0\delta) \times(-\mathcal{E}^6 + L\mathcal{E}^5 + M\mathcal{E}^4 + R\mathcal{E}^3 + N\mathcal{E}^2 + T\mathcal{E} + Q)\ . \tag{5.3}$$

To first order in β the coefficients of the new "irreducible" sixth-degree polynomial appearing in this expression are given by the following formulas:

$$L = 2C_3\omega\beta/m^2\kappa c_0\delta\ ,$$

$$M = 4C_1\omega\beta/\kappa^2 c_0\delta - (4/\kappa^2)(\kappa^2 w_0/m^2 + \gamma_0^2)A ,$$

$$R = -8C_3\gamma_0 A/m^2\kappa , \quad N = 4C_2 A/\kappa^2 ,$$

$$T = -8C_1C_3 A/m^2\kappa , \quad Q = -4C_1^2 A/\kappa^2 , \tag{5.4}$$

where $A = 1 + 2\gamma_0\omega\beta/\kappa^2 c_0\delta$.

Equations analogous to Eqs. (3.5)–(3.7) now read

$$\Delta k = \gamma_0\delta , \tag{5.5}$$

$$\dot{\phi} = 3\omega\beta\mathcal{E}^2/4c_0\delta + C_1/\mathcal{E}^2 , \tag{5.6}$$

$$u = (m^2/\kappa)(\gamma_0\mathcal{E} - \omega\beta\mathcal{E}^3/4c_0\delta + C_1/\mathcal{E}) + C_3 . \tag{5.7}$$

Equation (5.6) shows that in the presence of a host nonlinearity we will have chirping even if the chirping constant C_1 is zero. The phase modulation is then directly, rather than inversely, proportional to the intensity of the pulse.

The nonvanishing of C_1 and C_2 is again related to oscillatory envelopes. For the present we are interested only in single pulses and set these constants as well as C_3 equal to zero. We will also assume the asymptotic initial conditions corresponding to an attenuator, i.e., that there is no field in the distant past and that all the atoms are in the ground state: $\mathcal{E}(-\infty) = 0$, $w(-\infty) = w_0 = -1$. Under these conditions Eq. (5.3) reduces simply to

$$\dot{\mathcal{E}}^2 = \tfrac{1}{4}\kappa^2(1 - 2\gamma_0\omega\beta/\kappa^2 c_0\delta)(M - \mathcal{E}^2)\mathcal{E}^2 . \tag{5.8}$$

Equation (5.8) can be immediately integrated with the result

$$\mathcal{E} = (2/\kappa\tau)(1 + \gamma_0\omega\beta/\kappa^2 c_0\delta)\,\mathrm{sech}(\zeta/\tau) , \tag{5.9}$$

where

$$1/\tau = (\kappa/m)(1 - m^2\gamma_0^2/\kappa^2)^{1/2} . \tag{5.10}$$

The solution for the phase is obtained by substituting Eq. (5.9) into Eq. (5.6) (with $C_1 = 0$) and is given by

$$\phi = (3\omega\tau/c_0\delta)(\beta/\kappa^2\tau^2)\tanh(\zeta/\tau) . \tag{5.11}$$

The expressions for the components of the Bloch vector can easily be derived and we write them in the following way:

$$u = \frac{2\gamma_0\tau}{1+\gamma_0^2\tau^2}\left(1 + \frac{\gamma_0\omega\beta}{\kappa^2 c_0\delta}\right)\mathrm{sech}\left(\frac{\zeta}{\tau}\right) - \frac{2\omega\beta}{c_0\delta\kappa^2\tau(1+\gamma_0^2\tau^2)}\,\mathrm{sech}^3\left(\frac{\zeta}{\tau}\right) , \tag{5.12}$$

$$v = \frac{2}{1+\gamma_0^2\tau^2}\left(1 + \frac{\gamma_0\omega\beta}{\kappa^2 c_0\delta}\right)\mathrm{sech}\left(\frac{\zeta}{\tau}\right)\tanh\left(\frac{\zeta}{\tau}\right) , \tag{5.13}$$

$$w = -1 + \frac{2}{1+\gamma_0^2\tau^2}\left(1 + \frac{2\gamma_0\omega\beta}{\kappa^2 c_0\delta}\right)\mathrm{sech}^2\left(\frac{\zeta}{\tau}\right) . \tag{5.14}$$

Equations (5.9)–(5.14) constitute the solution of our problem to first order in β. It is apparent that it reduces to the solution of McCall and Hahn specialized to the case $g(\gamma) = \delta(\gamma - \gamma_0)$ when $\beta = 0$.

In Fig. 8, for a very sharp pulse (such that $T_2^* > \tau$ would be satisfied for most materials), we show the solutions for the atomic inversion w and the pulse envelope $\mathcal{E}$ for both $\beta = 0$ and $\beta \neq 0$. In the same figure we also show the relative frequency shift $\dot{\phi}/\omega$. Note that the chirp is large enough to produce a very substantial phase shift. In fact, $\dot{\phi}\tau \sim 1$.

These results concerning steady-state pulse propagation in nonlinear absorbers can now be summarized in the following way: (i) Undistorted or steady-state pulses can propagate in resonant absorbers even if the nonresonant host medium is nonlinearly dispersive. (ii) All such pulses are inevitably chirped if the most important nonlinearity is of the Kerr-effect type.

The only previous instances known to us in which analytic expressions for frequency modulation (chirping) in the solution of the coupled Schrödinger and Maxwell equations was predicted are an early study by one of us (J. H. E.) of atoms in external fields,[13a] the studies of Crisp and Jaynes[16] and Stroud and Jaynes[16] which discuss the frequency modulation expected to accompany spontaneous emission, the investigations of Armstrong and Courtens[17] who have found the existence of chirping in amplifying media, and that of Dialetis.[7]

Some comments on our solutions are now in order. For "asymptotic resonance," i.e., for the

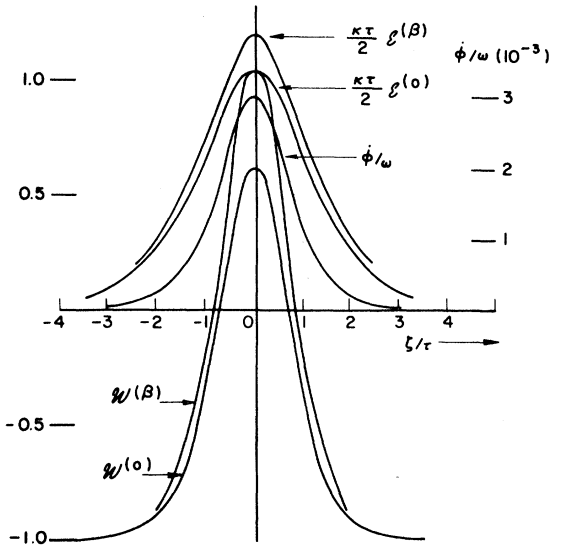

FIG. 8. Sketches of analytic solutions for pulse envelopes and atomic inversion for $\beta = 0$ and $\beta \neq 0$, and for time-dependent relative frequency shift $\dot{\phi}/\omega$. For the purpose of illustration, extreme values have been assigned: $\tau = 10^{-12}$ sec, $2\gamma_0\omega\beta/\kappa^2 c_0\delta = \frac{1}{4}$, $\gamma_0\tau = \frac{1}{2}$, and $2/\kappa\tau = 6\times 10^6$ V/cm.

case in which the single atomic transition ω_0 coincides at $t \to -\infty$ with the field-carrier frequency $\dot{\Phi} \to \omega$, the only difference between the present solution and that of the corresponding bare SIT is the presence of the phase modulation (which is unaffected by the detuning frequency γ), and the nonvanishing of the in-phase component of the resonant polarization. It is interesting to notice that the absorption component of this polarization factors: $v(\zeta, \gamma_0) = F(\gamma_0)v(\zeta, 0)$.

Inspection of the energy of the atom, Eq. (5.14), makes it clear that the effect of the pulse is to take each resonant two-level atom from its ground state to an excited superposition state and back again to the ground state. Direct integration of $\kappa\mathcal{E}$ shows that the pulse is a 2π pulse only on resonance, when $\gamma_0 = 0$.

By inspection of the expression (5.10), we see that τ is not affected by β. The same is true for the pulse velocity V, which is given by

$$1/V = 1/c_0 + (\pi \mathfrak{N}\hbar\omega\kappa^2\tau^2/\eta_0 c)(1+\gamma^2\tau^2)^{-1}.$$

The pulse delay is, therefore, due exclusively to the interaction with resonant atoms, to first order in β.

It is useful to compare more closely the variation of $\dot{\phi}$ with that of $\dot{w}$. From the equations of motion it follows immediately that $\mathcal{E}^2 = (2/m^2) \times (w - w_0)$. Eliminating $\mathcal{E}$ between this equation and Eq. (5.6) with $C_1 = 0$, and differentiating the result, one obtains

$$\ddot{\phi} + (3\pi\mathfrak{N}\hbar\omega^2\beta/2c^2\kappa^2)\dot{w} \quad . \tag{5.15}$$

Therefore, if $\beta > 0$ (as is usually the case), the slope of $\dot{\phi}$ is proportional to and of the same sign as that of w. We may say that chirping follows the inversion of the atoms and is maximum when this inversion is maximum (Fig. 8).

Crisp and Jaynes[16] and Stroud and Jaynes[16] have obtained similar relations to that of Eq. (5.15), although the origin of the frequency modulation in their theories is different from its origin in our theory. In the Crisp-Jaynes and Stroud-Jaynes cases the chirping originates in the reaction of the emitted field on the atoms, while in our case, as in that of Armstrong and Courtens,[17] the frequency modulation appears as a consequence of the dispersive characteristics of the host dielectric. The relationship between ϕ and w corresponding to Eq. (5.15) derived by Armstrong and Courtens, written in our notation, is

$$\ddot{\phi} = \mathfrak{N}\kappa\omega_0^2 A\dot{w}/(\sigma^2 + \omega_0^2 A^2), \tag{5.16}$$

σ being the nonresonant loss parameter.

There is, however, an important difference between the Armstrong-Courtens result and ours. While the Kerr constant β is usually positive, the dispersion parameter A turns out to be in general negative. Thus the direction of chirping in the Armstrong-Courtens case is opposite to ours. The source of this difference can be traced to the fact that the assumed origins of the host dispersion in both of these theories are very dissimilar, as we show in the Appendix. In practice, however, both kinds of nonresonant dispersion may be expected to contribute to the frequency modulation.[18]

Since the chirping follows the inversion, it is clear that the frequency sweep in SIT pulses cannot be monotonic, as it is for the pulses in the Stroud-Jaynes and Crisp-Jaynes investigations, or for the π pulses propagating in amplifying lossy host media of Armstrong and Courtens.

B. Oscillating Pulses in Nonlinear Absorbers

It is of course possible to obtain analytic solutions of a more general type even in the presence of the Kerr constant β. Assuming that the integration constants C_1 and C_2 are different from zero (but $C_3 = 0$), we obtain a solution for the field envelope that can be written as

$$\mathcal{E} = (k\mathcal{E}_0/l)[1 - l^2\,\mathrm{sn}^2(\zeta/\tau; k)]^{1/2}, \tag{5.17}$$

where

$$1/\tau = \tfrac{1}{2}\kappa(1 - \gamma_0\omega\beta/\kappa^2 c_0\delta)(S_3 - S_1)^{1/2}. \tag{5.18}$$

Here $S_1 < S_2$, S_3 are the zeroes of the cubic $P_3(S) = -S^3 + MS + NS + Q$ $(S = \mathcal{E}^2)$. The values of the parameters k and l will, of course, reflect the fact that the coefficients M, N, and Q now depend on the Kerr constant β.

The essential difference between these oscillatory pulses and those of the bare SIT found in Sec. III is that the oscillatory pulses traveling in nonlinear host media are all chirped without exception.

The phase function itself can be written as follows:

$$\phi = (3\omega\tau/c_0\delta)(\beta/\kappa^2\tau^2)E(\zeta/\tau; k) + (1/l)[(1-l^2)(l^2-k^2)]^{1/2}\,\Pi(\zeta/\tau; l^2; k^2) \quad , \tag{5.19}$$

where $E(u, k)$ and $\Pi(u; l^2; k^2)$ are the normal elliptic integrals of the second and third kind, respectively. When $k = 1$, $E(u, 1) = \mathrm{sn}(u, 1) = \tanh u$ and Eq. (5.19) goes over to Eq. (5.11).

It is also interesting to observe from Eq. (5.18) that the width of these pulses, and consequently their velocities, is affected by β, a situation that is in contrast to the single pulse in the presence of of nonzero β.

VI. SUMMARY AND CONCLUSIONS

In Secs. I–V we have given a detailed account of the phase modulation expected in slowly varying shape-preserving pulses propagating in resonant nonlinear absorbers. We have found as one of our most general results that the propagation of such

pulses is universally accompanied by phase modulation of their field vector. The only one of these pulses that has been experimentally observed so far, the sech solitary pulse, is not chirped (at least not in the slowly varying envelope approximation[12]).

The only unchirped multiple pulses are those found previously by Arecchi-Degiorgio-Someda, by Crisp, and by one of us (J. H. E.). These solutions as well as that of McCall and Hahn, are particular cases of more general, generally chirped, shape-preserving pulses in bare SIT derived here and also given by Dialetis. We have interpreted physically the extra parameters appearing in these more general pulses, and have suggested how one might produce pulses with a desired degree of chirping by appropriate envelope modulation. We have obtained a general form for the spectral response function, which departs drastically from Lorentzian shape, and have also established the equivalence of special forms obtained by Crisp and by Eberly. Finally, we have shown explicitly that, in the particular case of those pulses whose envelopes vanish periodically, the phase stays constant during a period, but changes suddenly by π at all points at which the envelope vanishes. This behavior is seen to be the natural limit of cases in which the envelope periodically approaches, but does not equal, zero.

If nonlinear dispersion is present, the chirping is then not only possible but necessary. We have investigated shape-preserving pulses when the host nonlinearity has a Kerr-effect origin and found that the pulse envelope has a sech dependence while the chirp is proportional to $\mathrm{sech}^2(\xi/\tau)$. The frequency sweep may be, under favorable circumstances, quite large in spite of the smallness of the Kerr constant.

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APPENDIX: FREQUENCY DEPENDENCE OF SUSCEPTIBILITY IN TERMS OF KERR CONSTANT

According to Eq. (2.8) the relation between the electric field E and the polarization P in the time domain is

$$P_{\mathrm{nr}}(t)=\frac{\eta_0^2-1}{4\pi}E(t)+\frac{\beta}{2\pi}\mathcal{E}^2(t)E(t)\ . \tag{A1}$$

We will derive the corresponding relation in the frequency domain when the field $E(t)$ is that of the single pulse found in Sec. V. We will write it in the following form:

$$E(t)=\mathcal{E}_0(1+\beta\lambda)\,\mathrm{sech}(t/\tau)e^{i\Gamma(t)}, \tag{A2}$$

where

$$\Gamma=\omega t+\beta\sigma\tanh(t/\tau),\quad \sigma=3\omega/\tau\kappa^2c_0,$$

$$\lambda=\tfrac{1}{3}\gamma_0\tau\sigma,\qquad \mathcal{E}_0=2/\kappa\tau\ .$$

We first compute the Fourier transform of the electric field in (A2):

$$E(\omega')=\mathcal{E}_0(1+\beta\lambda)\int_{-\infty}^{\infty}\mathrm{sech}(t/\tau)e^{i[\Gamma(t)-\omega' t]}dt\ .$$

The integral in this expression is evaluated through terms of first order in β. Making the change of variables $T=t/\tau$ and letting $q=\tau(\omega-\omega')$, we have

$$I=\tau\int_{-\infty}^{\infty}\mathrm{sech}(T)e^{i(qT+\beta\sigma\tanh T)}dT\ .$$

Writing the exponential in trigonometric form and keeping terms of up to first power in β, we get

$$E(\omega')=\pi\mathcal{E}_0\tau[1+\beta\lambda(1+3q/\gamma_0\tau)]\,\mathrm{sech}\tfrac{1}{2}(\pi q)\ . \tag{A3}$$

In a similar manner we obtain the Fourier transform of the nonresonant polarization (A1). The result is

$$P_{\mathrm{nr}}(\omega')=\tfrac{1}{4}\mathcal{E}_0\tau\{(\eta_0^2-1)[1+\beta\lambda(1+3q/\gamma_0\tau)]+\beta\mathcal{E}_0^2(1+q^2)\}\mathrm{sech}\tfrac{1}{2}(\pi q)\ . \tag{A4}$$

Using (A2) and (A4), we derive an expression for the susceptibility $\chi(\omega')=P_{\mathrm{nr}}(\omega')/E(\omega')$ which, to the first order in β and near the field frequency ω, is given by

$$4\pi\chi(\omega')=\eta_0^2-1+\beta\mathcal{E}_0^2[1+\tau^2(\omega-\omega')^2]\ . \tag{A5}$$

It is interesting to compare the form of the susceptibility given by (A5) with the one assumed by Armstrong and Courtens in their studies of pulse propagation in dispersive amplifying media.[17] They expand the host susceptibility near resonance in inverse powers of ω' and write

$$4\pi\chi_{\mathrm{AC}}(\omega')=a_0-a_1(\omega'/\omega')+a_2(\omega/\omega')^2, \tag{A6}$$

where a_0, a_1, and a_2 are phenomenological constants, characteristic of the medium. They treat the host dispersion by *linear*-response theory in terms of this frequency-dependent susceptibility.

In order to compare (A5) with (A6), we must find an expansion of our susceptibility (A5) in inverse powers of ω'. The required expression is

$$4\pi\chi(\omega')=\eta_0^2-1+\beta\mathcal{E}_0^2[(1+\omega^2\tau^2)-2\omega^2\tau^2(\omega/\omega')+\omega^2\tau^2(\omega/\omega')^2]. \tag{A7}$$

It is clear from the inspection of (A6) and (A7) that the host dispersion in the Armstrong-Courtens

theory is of quite different nature from the one that is incorporated in our model of the host dielectric. Our susceptibility in (A7) clearly has a nonlinear term.

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