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Critique of Electromagnetic Turbulent-Plasma Scattering Theories*†

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The scattering of electromagnetic waves by a large-scale turbulent plasma has recently become of interest again in connection with the formation and study of the nonequilibrium plasma (often turbulent) of controlled fusion machines. Other than the usual Born approximation, it has also become fashionable to use the equivalent of the Dyson and Bethe-Salpeter equations, as well as a transport formulation for the scattering wave. The use of these methods has recently been justified by Frisch. These results are considered in a critical fashion. It has been possible to conclude that the results obtained from a solution of the stochastic transport theory are valid in both the high- and low-frequency limits, while the results of the smoothing approximation for the Dyson and Bethe-Salpeter equations are limited to the low-frequency limit. In addition, it has also been concluded that the secular effects that exist in the ordinary Born series may be fictional.

INTRODUCTION

Historically, the interaction of electromagnetic waves with a large-scale turbulent plasma was first of interest in connection with wave propagation studies in the disturbed ionosphere.[1] In this connection, I proposed that some of the many problems associated with the multiple scattering of waves by a turbulent medium might perhaps be resolved by a stochastic formulation based on Maxwell's equations which could be used to derive a *stochastic* transport model for the radiation.[2–4]

More recently the wave problem has been of interest in connection with studies of instabilities and turbulence in the nonequilibrium plasma of the controlled thermonuclear fusion machines. A paper by Tatarskii and Gertsenshtein[5] was specifically directed towards a solution of this problem. They used the bilocal, or first-order smoothing approximation, an extension of the work of Bourrett.[8]

The wave-scattering problem has also been of interest in connection with the diagnostics of the forced turbulence of a weakly ionized plasma.[7] We should like to call the reader's attention to the continued work of Granatstein,[8–10] Feinstein and Granatstein,[11] Salpeter and Treiman,[12] Barabanenkov and Finkel'berg,[13] Vedenov,[14] Watson,[15] and some recent comments by the present author.[16,17]

In addition the problem of wave scattering by a random medium has attracted the attention of the applied mathematicians. Papers by Keller,[18,19] and the text by Frisch,[20] are of particular interest.

While the bilocal or first-order smoothing approximation was first introduced by Tatarskii and Gertsenshtein[5] and specifically restricted by them to a wave-number or frequency region where

$$ka_0 \ll 1 ,$$

where a_0 is the fluctuation scale size or mean eddy size of the turbulence, their work has in many cases been neglected and in other cases improperly applied, with equally bad results. For example an entire issue of an engineering journal devoted to the question of partial coherence fails to mention this particular paper or properly apply the smoothing theory.[21] In a more recent paper,[22] the smoothing theory was applied without any mention of the frequency limitations.

In addition to these difficulties in the literature, the Keller theory[18,19] has also been applied to the high-frequency limit [for example, see Keller, Ref. 19, Eq. (31)]. This might well be questioned as Frisch[20] has shown that Keller's method is equivalent to a first smoothing approximation of the Dyson equation for the mean field, and consequently must be limited to the low-frequency limit in this applications, as observed by Tatarskii.[5]

In this paper, I have directed my attention to a review of all of these difficulties as well as a study of the differences between the first-order smoothing approximations for the Dyson and Bethe-Salpeter equations and the solutions of the stochastic transport theory in the first stochastic approximation. This comparison is carried out for both the low- and high-frequency limits of the theories. As a result of this study it seems reasonable to conclude that the transport theory produces a reasonable result to a first stochastic approximation, in the high-frequency limit, while the first-order smoothing theory does not.

In addition to these results, I have made a number of suggestions which should prove useful for those, who like myself, might wish to probe further into these questions.

FORMULATION AND REVIEW

We shall consider solutions of the wave equation for the electric field intensity $\vec{E}$, in a turbulent plasma,

$$\nabla^2 \vec{E} + \bar{k}^2 \vec{E} = 4\pi r_e \, n_e(\vec{r}) \vec{E} \;, \qquad (1)$$

characterized by the electron density N_e, whose mean value is denoted by $\overline{N}_e$, such that the electron fluctuation density resulting from the turbulence is given by

$$n_e(\vec{r}) = N_e - \overline{N}_e \;. \qquad (2)$$

We shall restrict the following to cases where the plasma is well underdense in the frequency range of interest, as this is sufficient to serve our purpose. Other cases follow directly. Hence we have

$$\bar{k}^2 = \omega^2 \mu_0 \bar{\epsilon} \,, \quad \bar{\epsilon} = \epsilon_0 (1 - \bar{\omega}_p^2/\omega^2) \;, \qquad \bar{\omega}_p^2 = e^2 \overline{N}_e / m_e \epsilon_0 \,, \qquad (3)$$

and r_e is the classical electron radius. The integral form for Eq. (1) is

$$E(\vec{r}) = G_0(\vec{r}) - 4\pi r_e \int_V G_s(\vec{r}, \vec{r}_1') n_e(\vec{r}_1') E(\vec{r}_1') \, d\vec{r}_1' \;, \qquad (4)$$

where G_0 and G_s are the appropriate Green's functions for the free-space geometry and for the scattering region, and where the geometry is illustrated by Fig. 1. The integral equation may be written as a perturbation series of the Born form,

$$E(\vec{r}) = G_0(\vec{r}) - 4\pi r_e \int_V G_s(\vec{r}, \vec{r}_n') \sum_{n=1}^{\infty} E_n(\vec{r}_n') \, d\vec{r}_n' \;, \qquad (5)$$

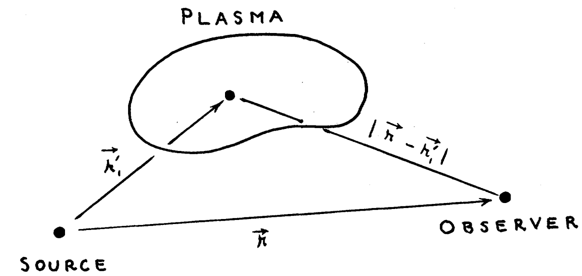

FIG. 1. Geometry for scattering by the plasma turbulence.

with

$$E_1(\vec{r}\,') = n(\vec{r}_1') E_0(\vec{r}_1')$$

and

$$E_{n+1}(\vec{r}_{n+1}') = -4\pi r_e n_e(\vec{r}_{n+1}') \int_V G_s(\vec{r}_{n+1}', \vec{r}_n') E_n(\vec{r}_n') \, d\vec{r}_n' \,.$$

Since the electron density is a stochastic variable, this equation may only be solved in a statistical sense. The solution may be represented graphically by the undressed diagram. For example, for the mean value or expectation of the field shown in Fig. 2, the $\langle \rangle$ indicate that an ensemble average must be taken of each term of the diagram. In drawing this diagram, we have considered the case of a Gaussian random *process* for the amplitude fluctuations of the electron density of the plasma. In short-lived experiments that do not reach a steady state, the ensemble consists of the totality of all experiments starting from the same initial conditions, same fields H_0, etc.

As noted by Frisch,[20] a bare or undressed diagram will contribute to the scattered field in a number of different ways, depending on the number of possible two point correlation functions, for the case of a Gaussian process. As this is a well-known property of such a process, I shall not dwell on it here. It is sufficient to note that bare diagrams of moment $2p$ may be written as the sum of

$$(2p)!/2^p p!$$

terms, each of which is the product of two point correlation functions. All moments of odd order vanish for the Gaussian process. This, in fact, is a good experimental test for such a process. As an example of this property, let us consider the case for $p = 2$, as shown in Fig. 3. The diagrams on the right-hand side of the figure are referred to as *dressed*, in that the two point space-correlation functions for the plasma fluctuations have been indicated by dotted lines.

The set of all *dressed* diagrams for a given field variable, such as the expectation of $\vec{E}$, will consist of both *connected* and *unconnected* diagrams.

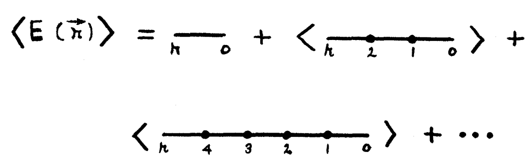

FIG. 2. Mean value or statistical expectation of the electromagnetic field.

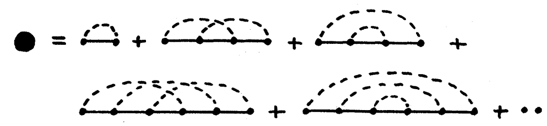

FIG. 4. Effective wave-number operator for the mean field intensity.

These are defined as follows:

A connected diagram is a diagram without terminals, that is not factorizable. In other words, one that *cannot* be cut into two or more diagrams without cutting any *dotted* lines.

The definition of an unconnected diagram is the simple converse and follows directly. While the above definition is essentially that of Frisch,[20] with our emphasis on the key words, it should be noted that his example contains an error (Ref. 20, p. 110).

With this definition for the connected diagram, it is possible to define the effective wave-number operator and construct the Dyson equation for the expectation of the electric field intensity.

Effective wave-number operator. This is defined as the sum of all possible connected diagrams (see Fig. 4). If the expectation, or mean field intensity, is denoted by Fig. 5(a), then the Dyson equation for the mean field is given by Fig. 5(b). This equation is a "formal" solution, as it has never been solved for E because of the many difficulties associated with the problem of finding a closed form expression for the effective wave-number operator. The literature to date has been restricted to finding the bilocal or first-order smoothing approximation for the expectation. This is obtained from Fig. 5(b) by using the first approximation for the effective wave-number operator, i.e., the first term in Fig. 4 [see Fig. 5(c)]. We shall return to Fig. 5(c) later in this paper.

The diagram method may also be used to write an expression for the space correlation of the field intensity E. The Born series for this is given by

$$\langle E(\vec{r}_1)E^*(\vec{r}_2)\rangle = \left\langle\left(G_0 - 4\pi r_e \int_V G_S(\vec{r}_1, \vec{r}_n')\sum_{n=1}^{\infty} E_n(\vec{r}_n')\, d\vec{r}_n'\right)\right. \times \left.\left(G_0^* - 4\pi r_e \int_V G_s^*(\vec{r}_2, \vec{r}_m^{\,1})\sum_{m=1}^{\infty} E_m^*(\vec{r}_m')\, d\vec{r}_m'\right)\right\rangle \,, \tag{6}$$

where E^* denotes the complex conjugate of E. When the amplitude fluctuations of the plasma are Gaussian, the first few terms of the dressed diagram are given by Fig. 6. The first two terms are just the first Born approximation, so often used in the calculation of scattering by a turbulent plasma. The equivalent of a Bethe-Salpeter equation for the space-correlation function of the electric field intensity may be constructed by using a method similar to that used for the Dyson equation.

Intensity operator. This is defined as the sum of all possible connected double diagrams contributing to $\langle E_1 E_2^* \rangle$ (see Fig. 7). A Bethe-Salpeter equation for the space-correlation function of the electric field intensity may be written as shown in Fig. 8. It should be noted that this result differs from that obtained by Frisch[20] in that I have not used the Dyson equation in the construction of the Bethe-Salpeter equation. As a consequence, the intensity operator differs from that given by Frisch.[20] I prefer this form for the Bethe-Salpeter equation as I am of the opinion that the other

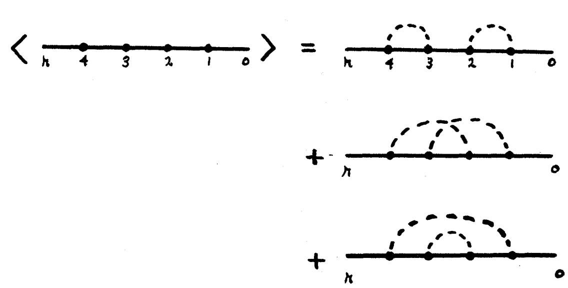

FIG. 3. Example of a dressed diagram using the third term on the right-hand side of Fig. 2.

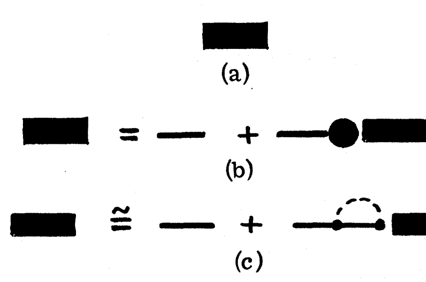

FIG. 5. (a) Diagram for the mean field intensity of the electric field. (b) Dyson equation (after Frisch) for the mean field intensity of the electromagnetic field. Actually the name is not to be applied in a strict sense as this equation is not directly related to Dyson's earlier work in the quantum theory (Refs. 23 and 24). (c) Bilocal or first-order smoothing approximation for the statistical expectation of the electromagnetic field.

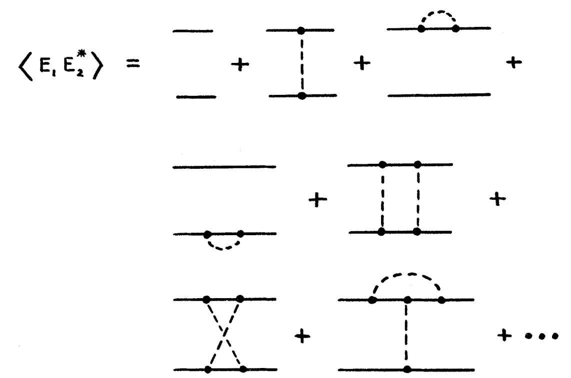

FIG. 6. First few terms of the dressed diagram for the space-correlation function of the electromagnetic field intensity.

is an example of "bootstrapping." It does, however, appear to be permissible in a first-order smoothing approximation.

As was the case for the Dyson equation, this is only a "formal" solution for $\langle E_1 E_2^*\rangle$, because of the difficulty of finding a closed form solution for the intensity operator. The literature to date has been restricted to the solution of the bilocal or first smoothing approximation for $\langle E_1 E_2^*\rangle$, by using the first term in Fig. 7. This might also be referred to as the ladder approximation, as it is in other applications of Feynman diagrams (see Fig. 9).

In order to compare the results of the bilocal or first-order smoothing approximations to the results of the transport theory, it is necessary to consider a number of examples.

EXPECTATION OF FIELD

The bilocal or first-order smoothing approximation for the mean field intensity may be obtained from Fig. 5(c). The first few terms are shown in Fig. 10. It is self-evident that this approximation includes all simple humps but neglects all of the crossed humps in Figs. (2) and (3). Nevertheless, the approximation does include some of the multiple-scattering effects. This, of course, is the reason for its usefulness. Frisch's method has been used to obtain the following differential equation for the expectation of the field intensity, for a point source in the turbulent plasma:

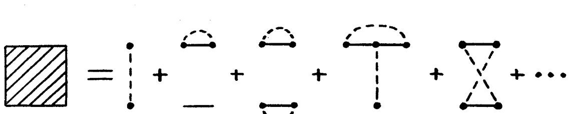

FIG. 7. Field intensity operator.

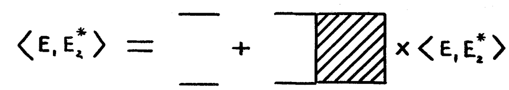

FIG. 8. Bethe-Salpeter equation (title after Frisch) for the space-correlation function of the electric field intensity. As in the case of the Dyson equation, this name is not to be applied in a strict sense as this equation is not directly related to the earlier work in the quantum theory (Ref. 25).

$$(\nabla^2+\bar{k}_0^2)\langle E(\vec{r})\rangle + 16\pi^2 r_e^2 \int_V \frac{e^{i\bar{k}_0|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \times \langle n_e(\vec{r}) n_e(\vec{r}\,')\rangle \langle E(\vec{r}\,')\rangle d\vec{r}\,' = \delta(\vec{r}) \,. \quad (7)$$

This result is the *equivalent* of that which could be obtained by applying the "honest" method introduced by Keller[18] to the plasma problem. Proceeding by the methods of Frisch,[20] Keller,[18] and Tatarskii,[5] it follows that the solution of Eq. (7) may be written as

$$\langle E(\vec{r})\rangle \simeq -(1/4\pi r)\, e^{ik_{eff} r} \,. \quad (8)$$

In the calculation of the effective wave number k_{eff}, Keller, Frisch, and Tatarskii have all taken an exponential form for the space-correlation function of the random variable. While we note that this may not be reasonable, given the physics of the plasma turbulence,[7] we will nevertheless use their result as this does not effect the principal purpose of this paper. Hence, for the low-frequency limit,

$$k_{eff} \simeq \bar{k}_0 + 8\pi^2 r_0^2 \langle n_e^2\rangle \bar{k}_0^{-1} l_0^2 + i16\pi^2 \langle n_e^2\rangle r_e^2 l_0^2 \quad \text{for } \bar{k}_0 l_0 \ll 1 \,, \quad (9)$$

where

$$\langle n_e(\vec{r}_1) n_e(\vec{r}_2)\rangle = \langle n_e^2\rangle \exp(-|\vec{r}_2-\vec{r}_1|/l_0) \,.$$

In the high-frequency limit, we obtain

$$k_{eff} \simeq \bar{k}_0 + i(1/2l_0) \,. \quad (10)$$

The imaginary part of Eqs. (9) and (10) result in an attenuation of the field as it propagates into the plasma. This is similar in form to the results of the transport theory, as we will note later in this paper.

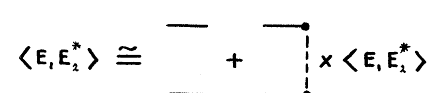

FIG. 9. Bilocal or first-order smoothing approximation for the statistical expectation of the space-correlation function of the electric field intensity.

SPACE-CORRELATION FUNCTION OF TOTAL ELECTRIC FIELD

The bilocal or first-order smoothing approximation for the space covariance of the field intensity may be obtained from Fig. 9. The first few terms are shown in Fig. 11. It is self-evident that this is the ladder approximation. Writing the equation in Fig. 9 formally, we have

$$\langle E_1 E_2^* \rangle \cong \langle E_1 \rangle \langle E_2 \rangle + 16\pi^2 r_e^2 \int \int_V \langle E(\vec{r}_1 - \vec{r}_1') \rangle \times \langle E(\vec{r}_2 - \vec{r}_2') \rangle \langle E(\vec{r}_1') E(\vec{r}_2') \rangle \langle n_e(\vec{r}_1') n_e(\vec{r}_2') \rangle \, d\vec{r}_1' \, d\vec{r}_2' \quad , \tag{11}$$

where $\langle E \rangle$ may be obtained from the results of the first-order smoothing approximation of the Dyson equation. The geometry is illustrated by Fig. 12.

Having written the integral equation for the bilocal approximation for the field covariance, it is now possible to compare the results of the smoothing theory to the results of the stochastic transport theory originally proposed in 1958.[2-4]

COMPARISON OF RESULTS OF SMOOTHING THEORY TO RESULTS OF STOCHASTIC TRANSPORT THEORY

In picking a particular starting point, we have chosen a form of the stochastic transport theory consistent with the conditions used in the discussion of the bilocal approximation. These conditions are statistical stationarity of the plasma turbulence, as E would be time dependent (other than $i\omega t$) if this were not the case, and a well-behaved total cross section. These conditions have been discussed in detail previously.[3] The stochastic transport equation may be written as

$$\vec{v} \cdot \nabla \langle f_e \rangle = \int \langle \sigma_e(\vec{r}, \vec{k}) f_e(\vec{r}, \vec{k}') \rangle \, d\Omega' - Q_s \langle f_e \rangle \quad . \tag{12}$$

In this equation, f_e is the photon density function for scattering by the plasma turbulence, σ_e is the wave-scattering cross section per unit volume per unit solid angle of the plasma turbulence, and Q_s is the total cross section for scattering. We have modified our earlier result in one respect by including σ_e within the ensemble average in the integral, *in order to stress that they are not necessarily statistically independent quantities.* If the first Born approximation is used to find the cross section σ_e, then we have

$$\langle \sigma_e f_e \rangle \cong \sigma_e \langle f_e \rangle \quad , \tag{13}$$

which is the previous result,[3] in some cases written simply as $\sigma_e f_e$.[4] It should also be stressed that the density function f_e is related to the *field* intensity by way of the integral[3]

$$\langle \vec{P} \rangle = \hbar \omega_0 \int \vec{v} \langle f_e \rangle \, dV_{k'} \quad , \tag{14}$$

where $\langle P \rangle$ is the expectation of the Poynting vector of the total field.

In order to compare the results of the transport theory to the results of the bilocal approximation for the field intensity, it is convenient to consider a simple geometry. Let the source be directional with a radiation beam width equal to $\frac{1}{2}\beta$, as illustrated in Fig. 13.

The solution of Eq. (12) consistent with the approximation of Eq. (13) is

$$f(R) \cong f(0) \frac{e^{-Q_s R}}{16\pi^2 R^2} + \int_0^R dr' \frac{e^{-Q_s (R-r')}}{16\pi^2 (R-r')^2} \times \int \sigma_e(r', K) f_e(r', k') \, d\Omega' \quad , \tag{15}$$

where the angle of integration in ϕ is bounded by α, as a function of r, such that

$$\tan\alpha = \tfrac{1}{2}\beta R/(R-r) \quad . \tag{16}$$

While β must be small, this is often the case in applications of interest.

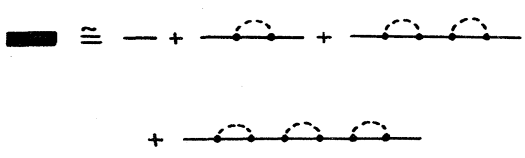

FIG. 10. First four terms of the bilocal or first-order smoothing approximation for the mean value of the electric field intensity.

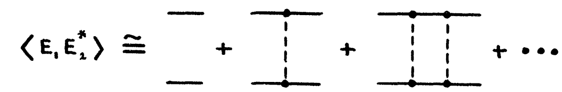

FIG. 11. First three terms of the bilocal or first-order smoothing approximation for the mean value of the space-correlation function of the electric field intensity.

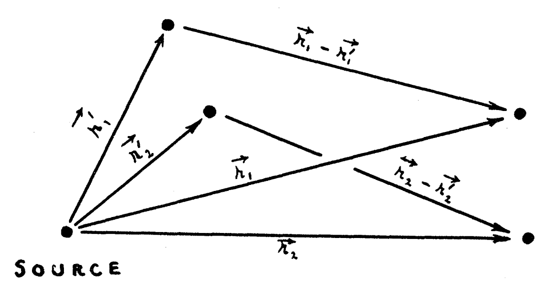

FIG. 12. Geometry for the bilocal or first-order smoothing approximation for the mean value of the space-correlation function of the electric field intensity.

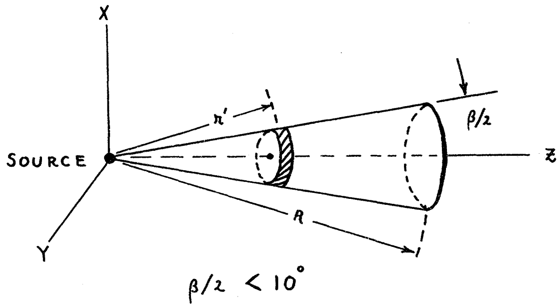

FIG. 13. Geometry for the solution of the stochastic transport theory.

The transport theory, Eq. (15), may now be compared directly to the bilocal result by substituting for E in Eq. (11), from Eqs. (8)–(10). Then taking $r_1 = r_2$, we have

$$\langle |E(R)|^2\rangle$$
$$\cong \langle |E_0|^2\rangle \frac{e^{-2k_2R}}{16\pi^2R^2} + 16\pi^2 r_e^2 \iint_V d\vec{r}_1{}' d\vec{r}_2{}' \frac{e^{-k_2|\vec{R}-\vec{r}_1{}'|}}{4\pi|\vec{R}-\vec{r}_1{}'|}$$
$$\times e^{ik_1|\vec{R}-\vec{r}_1{}'|} e^{ik_1|\vec{R}-\vec{r}_2{}'|} \frac{e^{-k_2|\vec{R}-\vec{r}_2{}'|}}{4\pi|\vec{R}-\vec{r}_2{}'|} \langle E(\vec{r}_1{}')E(\vec{r}_2{}')\rangle$$
$$\times \langle n_e(\vec{r}_1{}')n_e(\vec{r}_2{}')\rangle \ , \quad (17)$$

where k_1 is the real part and k_2 the imaginary part of k_{eff}. *Physically,* these results tell us that the ensemble average of the photon density function or the mean-square field intensity consists of a *coherent* and *incoherent* contribution, the first and second terms on the right-hand side of each equation.

The *coherent* terms may be compared directly. Taking k_2 from Eqs. (9) and (10) for the bilocal approximation, we have

$$2k_2 = 32\pi^2 r_e^2 \langle n_e^2\rangle l_0^3 \quad \text{for } \bar{k}_0 l_0 \ll 1 \ ,$$
$$2k_2 = 1/l_0 \quad \text{for } \bar{k}_0 l_0 \gg 1 \ . \quad (18)$$

These may be compared with the results of the transport theory by using Q_s from Ref. 3 or by starting with some of the material in a later paper.[26] [In this paper the result is obtained in the high-frequency limit, see Eq. (3. 5) of Ref. 26. The result for the low-frequency limit may be obtained by using Eqs. (2. 1) and (3. 3) of Ref. 26]. In so doing it should be noted that

$$\langle \Delta\epsilon^2\rangle = 16\pi^2 r_e^2 \langle n_e^2\rangle (\bar{k}_0)^{-4} \quad (19)$$

for the case of a turbulent plasma. Hence, I obtain the result

$$Q_s = 32\pi^2 r_e^2 \langle n_e^2\rangle l_0^3 \quad \text{for } \bar{k}_0 l_0 \ll 1 \ ,$$
$$Q_s = 2 r_e^2 \langle n_e^2\rangle l_0 (\bar{\lambda}_0)^{-2} \quad \text{for } \bar{k}_0 l_0 \gg 1 \ . \quad (20)$$

It is evident that the coherent terms agree in the *low-frequency* limit, but are quite different in the *high-frequency* limit. I am inclined to dismiss the bilocal result in the high-frequency limit since an attenuation factor of $1/l_0$ is excessive and would lead to a very rapid attenuation of the field. This has not been observed.[9] Furthermore, it should be noted that Tatarskii and Gertsenshtein in their paper specifically restricted their result for the bilocal approximation to the low-frequency limit [see Eq. (29) of Ref. 5].

An argument in favor of the validity of the transport theory result in the high-frequency limit follows directly from a criterion previously reported by the author (see Ref. 26), where it was noted that the mean-square value of the change in phase introduced by the dielectric noise (which in the present case is due to the turbulent plasma) is related to the total scattering cross section Q_s by

$$\langle \alpha^2\rangle \cong 4 Q_s L \quad \text{for } \bar{k}_0 l_0 \gg 1$$

from which it follows that $\langle \alpha^2\rangle \cong \pi$ at a distance of one mean free path in the plasma. This, of course, is that required for the destructive interference of the wave. In addition, we may note that this criterion also exhibits the proper frequency dependence.

The *incoherent* components of the field [second term on the right-hand side of Eqs. (15) and (17)] are somewhat different. Frisch[20] has observed that Eq. (11) or (17) for the bilocal approximation may be reduced to a transport equation for the mean intensity such as that we introduced (see Ref. 20, p. 145). We note, however, that the transport equation is not phenomenological but can indeed be derived from Maxwell's equations.[3]

We shall take a somewhat different approach. Taking the usual definition for the space spectrum of the plasma turbulence

$$S(\vec{K}) = \int \frac{\langle n_e(\vec{r}_1)n_e(\vec{r}_2)\rangle}{\langle n_e^2\rangle} e^{i\vec{K}\cdot(\vec{r}_2-\vec{r}_1)} d(\vec{r}_2-\vec{r}_1) \quad (21)$$

and noting that the cross section σ_e is related to $S(K)$ by

$$\sigma_e = r_e^2 \langle n_e^2\rangle S(\vec{K}) \quad (22)$$

when Eq. (13) is assumed valid (first Born approximation for σ_e) it follows that the bilocal approximation can be simplified in the *low-frequency limit* and written as

$$\langle |E(R)|^2\rangle \cong \langle |E_0|^2\rangle \frac{e^{-2k_2R}}{16\pi^2R^2}$$
$$+ \int dr' \frac{e^{-2k_2(R-r')}}{16\pi^2(R-r')^2} \langle |E(r')|^2\rangle \sigma_{e_{1f}}(K) \ , \quad (23)$$

where $\sigma_{e_{1f}}$ is the cross section in the low-frequency limit. While this result is similar in form to the transport result, it should be stressed that the equivalence is limited to the low-frequency limit. The two results differ substantially in the high-frequency limit. This is consistent with the previous result for the *coherent* contributions.

Barring further experimental evidence to the contrary, it would appear reasonable to conclude that the transport theory yields a reasonable result for both the low- and high-frequency limit, while the first-order smoothing approximation for the Dyson and Bethe-Salpeter equations is only correct in the low-frequency limit. As noted previously, the inadequacy of the smoothing approximation is probably due to the fact that only a small number of the many multiple-scattering terms of the Born series are included. This observation naturally leads us to a discussion of the secular terms in the Born series.

ON QUESTION OF SECULAR TERMS IN BORN SERIES

First of all, it is of importance to note that secular terms do indeed exist in the Born series for the case of an *unbounded* space completely filled with a statistically homogeneous turbulent plasma. The proof for this is similar to that given by Frisch (Ref. 20, p. 146) and will not be repeated here. However, it is also of importance to realize that the existence of such terms does not in itself prove that the *complete* series will diverge. There may, in fact, be some bound on the range r, given $\langle n_e^2 \rangle$, where the complete series is absolutely convergent. This point merits further study since the smoothing approximation is probably invalid in the high-frequency limit.

While the question of secularity is theoretically interesting it may in reality be mathematically "academic." In actual practice the turbulent plasmas of interest are small in volume. While multiple-scattering effects can be of importance, secular terms are well behaved for all observers located in the *far field* of the plasma scattering.

A further observation may also be of value. While the mathematical proof for the existence of secular terms is straightforward, Frisch[20] has not provided us with a physical explanation for the phenomena. I should like to suggest the following: Consider an infinite space filled with a turbulent plasma. Let a point source of photons be located at r' and an observer at r. Let us assume, as in the theory, that both the source and observer have omnidirectional transducers. The photons reaching the observer arrive from the forward scattering of nearby turbulons and from the backscattering of distant turbulons, as illustrated by Fig. 14. I should like to suggest that the secular effect is due to backscattering from the more distant parts of the plasma where the volume, and consequently the number density of the turbulons, increases at the same rate at which the field is attenuated by

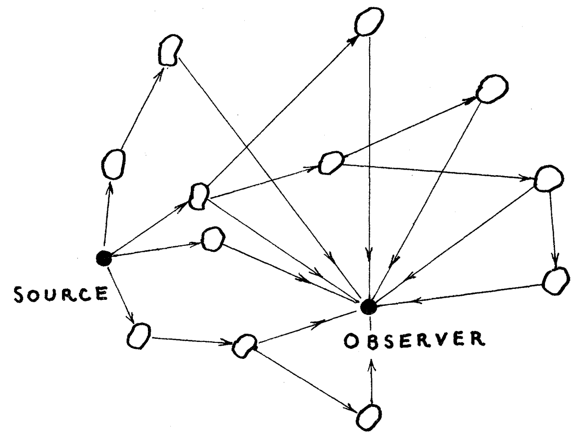

FIG. 14. Geometry to illustrate the question of secularity.

spherical spreading, $1/r^2$. A close examination of Frisch's result[20] yields a contribution from the double pole of

$$\bar{k}_0^4 \langle \Delta\epsilon^2 \rangle \frac{1}{4\pi r} \frac{d}{dK} \left(\frac{K e^{iKr}}{(\bar{k}_0 + iL^{-1})^2 - K^2} \right)_{K=\bar{k}_0} , \qquad (24)$$

producing the secular term

$$\langle \Delta\epsilon^2 \rangle \frac{e^{i\bar{k}_0 r}}{4\pi} \frac{i\bar{k}_0^6}{(\bar{k}_0 + iL^{-1})^2 - \bar{k}_0^2} \qquad (25)$$

which, apart from a phase dependence, is independent of r. This is in keeping with the physical explanation for this effect. It follows that the complete series can only converge if the phase and amplitude of the individual secular terms are such that they cancel each other in some fashion. This merits further study.

CONCLUSION

It has, of late, become fashionable to use the equivalent of the Dyson and Bethe-Salpeter equations, together with a first-order smoothing approximation to calculate the amplitude, phase, and intensity of an electromagnetic wave scattered by a turbulent plasma. This method was first introduced by Tatarskii[5] and has recently been justified by Frisch[20] on the premise that the complete Born series contains secular terms in an unbounded homogeneous space. I have readdressed myself to this problem and have concluded that some interesting differences exist in the multiple-scattering region between the results of the first-order smoothing theory and the results of the transport theory. In particular, it has been possible to show that the smoothing approximation for the Dyson and Bethe-Salpeter equations and the transport theory yield similar results in the *low-frequency limit*, provided a first Born approximation is used to construct the cross section in the transport theory. This approximation for the stochastic transport theory has been defined as the first stochastic approximation, in that the cross section and the field are assumed to be statistically independent in the ensemble average of the transport equation.

The two methods yield substantially different results in the *high-frequency limit*. It is evident that these differences are probably due to the approximation inherent in the smoothing result for the Dyson and Bethe-Salpeter equations, as many of the multiple-scattering terms in the Born series are neglected. Barring further experimental evidence to the contrary, it would appear reasonable to conclude that the first stochastic approximation for the transport theory yields a reasonable result for both the low- and high-frequency limits, and is therefore probably valid at all frequencies.

We have also concluded that the secular terms that exist in the ordinary perturbation series as expressed by the Born series may be fictional.

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Spatial Period of Bend Oscillations in the Dielectric Electrohydrodynamical Instability of a Nematic Liquid Crystal*

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We observed the spatial period of "chevrons" in an ac-excited sample of room-temperature nematic methoxy benzylidene butyl aniline. The frequency dependence of the period is explained by the increase in the dielectric relaxation rate due to ionic diffusion currents, with a normal diffusion constant.

Under the influence of ac electric fields, nematic liquid crystals (NLC) undergo electrohydrodynamical (EH) instabilities. These can be reasonably well explained by the Carr-Helfrich mechanism.[1] A quantitative one-dimensional model is now available[2] which describes the instability in terms of oscillations of space charges and angular deformations of the NLC, parametrically coupled through the applied electric field; this model has been quantitatively tested by threshold measurements.[3]

One of the more striking properties of the EH in-