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PHYSICAL REVIEW A VOLUME 6, NUMBER 1 JULY 1972

# Instability of Electromagnetic Cyclotron Harmonic Waves in Plasmas*

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(Received 2 August 1971)

The stability of electromagnetic waves propagating perpendicular to a uniform and static magnetic field is studied for plasmas with the ring- and loss-cone-type distributions in the perpendicular velocities. Based on the linearized Vlasov-Maxwell equations and the assumption that the ions are infinitely massive, it is found that instability can occur not only with zero frequency but also near the multiples of the electron cyclotron frequency. In general, the higher the average parallel energy, the more the plasma is susceptible to the instability. If the distribution in the parallel velocities is a stationary Maxwellian, instability occurs only for large ratios of parallel thermal pressure to magnetic pressure ($\beta_{\parallel t}$). For counterstreaming plasmas with streaming velocities of the order of the electron thermal speed, the instability can occur for $\beta_{\parallel t}$ values of order unity. A discussion is given on the growth rates of these instabilities.

## I. INTRODUCTION

It is well known that a magnetized plasma can support two independent modes of wave propagation in the direction perpendicular to the magnetic field, i.e., the "ordinary" and the "extraordinary" modes, provided that the velocity distribution function is symmetric in the parallel velocities. In the electrostatic approximation, the extraordinary waves are known as the Bernstein modes.[1] For bi-Maxwellian distributions, the Bernstein modes are stable. Because their dispersion curves exhibit interesting structures at frequencies near the multiples of the cyclotron frequencies $\omega = n\Omega_s$, they are also called the cyclotron harmonic waves. For the "ring-" or loss-cone-type distributions, which have been used as model distributions for plasmas in a mirror magnetic field,[2–5] the Bernstein modes can be unstable when the density exceeds a critical value. These instabilities can occur either with zero frequency ($n = 0$) or near the harmonics of the cyclotron frequencies ($n = 1, 2, 3, \ldots$) and are sometimes referred to as the electrostatic flute modes.[2–4] They belong to the broad class of Harris instabilities, which have been extensively studied during the last decade.[2–11]

During the past several years, the Vlasov theory of the purely electromagnetic ordinary wave has received much attention.[12–19] Since the ordinary-wave-dispersion equation also possesses interesting features at frequencies near $n\Omega_s$,[13,14] it is appropriate to refer to them as the electromagnetic cyclotron harmonic waves (EMCHW), a term already introduced by the present author in Ref. 18. It is found that the average of the square of the parallel velocities $\langle v_\parallel^2 \rangle$, which does not enter in the dispersion relation for the Bernstein modes, plays a crucial role for the ordinary wave. In an electron plasma, if the ratio $\beta_\parallel = 2\langle v_\parallel^2\rangle \omega_p^2/(c^2\Omega^2)$ is sufficiently high, where $\omega_p$ and $\Omega$ are the electron plasma and cyclotron frequencies, respectively, and $c$ is the velocity of light, a zero frequency ($n = 0$ mode) can occur in stationary or counterstreaming plasmas with bi-Maxwellian distributions.[13,16,17] When $\beta_\parallel$ is below the instability threshold, the propagation characteristics are sensitively dependent on $\beta_\parallel$ and the ratio of "temperatures" parallel and perpendicular to the magnetic field.[18]

In a recent letter,[19] attention was drawn to the fact that while the $n > 0$ modes of the EMCHW are stable for bi-Maxwellian distributions, they can become unstable for the ring- and the loss-cone-type distributions. It is the purpose of this paper to give a detailed account of these instabilities. First, the dispersion relation is given in Sec. II. Section III presents the analysis of instability boundaries and some representative results for the case of an electron plasma. A discussion on the growth rate is given in Sec. IV. As in the case of bi-Maxwellian distributions studied previously, we find here that the parameter $\langle v_\parallel^2\rangle$ again plays a dominant role in the present theory. In Sec. V, the paper concludes with some discussion and suggestions for further work.

## II. DISPERSION RELATION

Let us consider small amplitude disturbances in a homogeneous infinite plasma in a static uniform magnetic field $B_0$, described by the linearized Vlasov-Maxwell equations. We shall assume that the unperturbed velocity distribution function of each species $F_{0s}$ depends only upon the parallel and perpendicular components of the particle velocity ($v_\parallel$ and $v_\perp$, where subscripts $\parallel$ and $\perp$ refer to directions with respect to $B_0$) and are even functions of $v_\parallel$. In such a case, the general dispersion equation for waves propagating perpendicular to $B_0$ factors into the "ordinary" and the "extraordinary" modes. The dispersion relation for the ordinary wave, whose electric field vector is parallel to $B_0$, is given by[14]

$$\omega^2 - c^2k^2 = \sum_s \omega_{ps}^2 - \sum_s \omega_{ps}^2\, \Omega_s^2\, 2\pi \times \int_{-\infty}^{\infty} dv_\perp v_\parallel^2 N_s^{-1} \frac{\partial F_{0s}}{\partial v_\perp} \sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2(kv_\perp/\Omega_s)}{\omega^2 - n^2\Omega_s^2}, \quad (1)$$

where

$$\omega_{ps} = (4\pi N_s e_s^2/m_s)^{1/2}$$

and

$$\Omega_s = |e_s| B_0/m_s c .$$

$N_s$ is the equilibrium density for particles of type $s$, and $J_n$ is the Bessel function of the first kind of order $n$.

In this paper, we study the above dispersion relation for equilibrium distribution functions of the form

$$F_{0s}^j(v_\parallel, v_\perp) = N_s f_{0s}(v_\parallel) g_{0s}^j(v_\perp), \quad (2)$$

$$g_{0s}^j(v_\perp) = (\pi\alpha_{\perp s}^2 j!)^{-1}(v_\perp/\alpha_{\perp s})^{2j} e^{-(v_\perp/\alpha_{\perp s})^2}, \qquad j = 1, 2, 3, \ldots \quad (3)$$

where $f_{0s}(v_\parallel)$ is an arbitrary even function of the parallel velocities, normalized to 1. The distribution function in perpendicular velocities $g_{0s}^j(v_\perp)$ was first introduced by Dory, Guest, and Harris.[3] It represents a class of distributions, which reduces to the Maxwellian for $j \to 0$ and to the "ring" when $j \to \infty$, i.e.,

$$g_{0s}^\infty(v_\perp) = (2\pi v_\perp)^{-1}\delta(v_\perp - V_s). \quad (4)$$

For $0 < j < \infty$, they are peaked at the nonzero value $V_s = \alpha_{\perp s} j^{1/2}$. The simplest way to determine the half-widths $\Delta v_{\perp s}$ is through plots of the functions. By this procedure, we find that $\Delta v_{\perp s} \sim 0.58\, \alpha_{\perp s}$ and is independent of $j$.[20] The ratio of peak velocity to half-widths is therefore $V_s/\Delta v_{\perp s} \simeq 1.72 j^{1/2}$. Another quantity of interest is the average of the square of the perpendicular velocities $\langle v_\perp^2\rangle_s$, defined by the integral

$$2\pi \int_0^\infty v_\perp^2 g_{0s}^j(v_\perp) v_\perp dv_\perp .$$

This evaluates to $V_s^2$ for the ring distribution and to $\alpha_{\perp s}^2(j+1)$ for finite integral values of $j$.

The class of distributions defined by Eq. (3) has been used as model distributions for mirror confined plasmas which possess a loss cone.[2-5] They have been studied extensively with regard to the Harris-type electrostatic instabilities.[2-11] Recently, an investigation of their stability against the whistler mode propagating along the magnetic field has also been made.[21] In what follows, we are concerned with the electromagnetic ordinary wave propagating across the field. On substituting Eq. (2) into Eq. (1), there results the dispersion relation

$$\omega^2 - c^2k^2 = \sum_s \omega_{ps}^2 \left[1 + \frac{4\langle v_\parallel^2\rangle_s}{\alpha_{\perp s}^2} \sum_{n=1}^{\infty} \frac{n^2 D_{ns}^j}{(\omega/\Omega)^2 - n^2}\right], \quad (5)$$

where

$$\langle v_\parallel^2\rangle_s = \int_{-\infty}^{\infty} v_\parallel^2 f_{0s}(v_\parallel)\, dv_\parallel, \quad (6)$$

$$D_{ns}^j = -\alpha_{\perp s}^2 \int_0^\infty \frac{\partial g_{0s}^j}{\partial v_\perp} J_n^2(kv_\perp/\Omega_s)\, dv_\perp . \quad (7)$$

We note that if the distribution in the parallel velocities is a stationary Maxwellian, the quantity $2\langle v_\parallel^2\rangle_s = V_{\parallel s}^2$, where $V_{\parallel s}$ is the parallel thermal velocity. For the case in which the particle species are counterstreaming along the field lines, we have

$$f_{0s}(v_\parallel) = (2\pi^{1/2} V_{\parallel s})^{-1}\left(e^{-(v_\parallel - u_s)^2/V_{\parallel s}^2} + e^{-(v_\parallel + u_s)^2/V_{\parallel s}^2}\right), \quad (8)$$

and Eq. (6) evaluates to

$$2\langle v_\parallel^2\rangle_s = V_{\parallel s}^2 + u_s^2, \quad (9)$$

where $u_s$ is the directional velocity.

Consider now the quantities $D_{ns}^j$. If the particles are monoenergetic in the direction perpendicular to the magnetic field, the distribution is given by the ring distribution, which corresponds to the $j \to \infty$ member in the class described by Eq. (3). Substitution of Eq. (4) into (7) yields

$$D_{ns}^\infty = \mu_s \frac{d}{d\mu_s} J_n^2 (2\mu_s)^{1/2}, \quad (10)$$

where

$$\mu_s = k^2 V_s^2/2\Omega_s^2 . \quad (11)$$

The member corresponding to $j = 0$ is the familiar Maxwellian distribution, for which

$$D_{ns}^0 = I_n(\mu_s) e^{-\mu_s}, \quad (12)$$

where

$$\mu_s = k^2 \alpha_{\perp s}^2/2\Omega_s^2 . \quad (13)$$

For other values of $j$, Guest and Dory[4] have shown that the quantities $D^j_{ns}$ can be derived from $D^0_{ns}$ according to the formula

$$D^{l+1}_{ns}=(l+1)^{-1}\left(lD^l_{ns}+\mu_s\frac{d}{d\mu_s}D^l_{ns}\right),\quad l=0,1,2,\ldots \tag{14}$$

where $I_n$ is the modified Bessel function of order $n$. As an example, for $j=1$, we have

$$D^1_{ns}=(-\mu_s-n)I_n(\mu_s)e^{-\mu_s}+\mu_s I_{n-1}(\mu_s)e^{-\mu_s}. \tag{15}$$

Using the identity

$$\sum_{-\infty}^{\infty} J_n^2=1,$$

one can show from the defining Eq. (7) that[4]

$$\sum_{-\infty}^{\infty} D^j_{ns}=0,\quad j=1,2,3,\ \ldots\ . \tag{16}$$

The above identity can be trivially verified for the $j=1$ and $j=\infty$ distributions.

It is instructive to recall that the corresponding dispersion relation for the extraordinary wave under the electrostatic approximation is given by[3]

$$k^2=-\sum_s\frac{2\omega^2_{ps}}{\alpha^2_{\perp s}}\sum_{n=-\infty}^{\infty}\frac{\omega D^j_n}{\omega-n\Omega}\ . \tag{17}$$

Dory *et al.*[3] have analyzed the above equation, which revealed that instability can occur with zero frequency or with frequencies near the cyclotron harmonics. In view of the similarities in mathematical structure of Eqs. (5) and (17), it is not surprising that the ordinary wave will also go unstable under suitable circumstances, as we shall show in detail in Sec. III. It is interesting to note here that $\langle v^2_\parallel\rangle_s$ appears in the ordinary-wave dispersion but plays no role in determining the characteristics of the perpendicularly propagating electrostatic waves.

## III. ANALYSIS OF INSTABILITY BOUNDARIES FOR ELECTRON PLASMAS

In this section, we examine Eq. (5) for the possible occurrence of unstable roots. We limit ourselves to electron dynamics in this paper, i.e., the ions are assumed to be infinitely massive. It is then convenient to rewrite Eq. (5) with the electron terms only in the following nondimensional form:

$$\frac{\omega^2}{\Omega^2}\frac{\Omega^2}{\omega_p^2}-1-\frac{2\mu}{\beta_\parallel}\frac{2\langle v^2_\parallel\rangle}{\alpha^2_\perp}=\frac{2\langle v^2_\parallel\rangle}{\alpha^2_\perp}\left(D^j_0+\frac{2\omega^2}{\Omega^2}\sum_{n=1}^{\infty}\frac{D^j_n}{\omega^2/\Omega^2-n^2}\right). \tag{18}$$

The subscript $e$ has been dropped and all quantities refer to electrons. The quantity $\beta_\parallel$ is the *total* parallel kinetic-energy density (thermal and/or streaming) to magnetic field energy density:

$$\beta_\parallel=\frac{2\langle v^2_\parallel\rangle\omega_p^2}{c^2\Omega^2}=\frac{V^2_\parallel\omega_p^2}{c^2\Omega^2}+\frac{2u^2\omega_p^2}{c^2\Omega^2}=\beta_{\parallel t}+\beta_{\parallel s}\ , \tag{19}$$

where subscripts $t$ and $s$ refer to thermal and streaming, respectively. In arriving at Eq. (18), we have made use of identity (16). Equation (18) holds also for the ring distribution with the substitution of $\alpha_\perp$ by $V$.

Equation (18) admits two types of solutions for $\omega^2$ which correspond to instabilities, namely, $\omega^2$ negative and $\omega^2$ complex. The former is a purely growing mode with zero frequency while the latter is an instability with a real frequency. The possible occurrence of one type or another or both is dependent on the relative signs of $D^j_n$, in a manner to be explained. If $j=0$, the plasma has a bi-Maxwellian distribution. It is then known that for all $n>0$, $D^0_n>0$ and Eq. (18) admits no complex solution for $\omega^2$. When instability arises, it is of the purely growing type.[14] This was the case studied previously.[12–17]

For $j>0$, it is evident from Eqs. (10), (14), and (15) that some of the $D^j_n$ can be negative for certain ranges of $\mu$. In the interest of clarity, our discussion will proceed with the lowest member of the loss-cone family, i.e., $j=1$. The procedure for instability analysis to be developed can then be applied to other values of $j$; in particular some results for plasmas with the ring distribution will be presented.

### A. $j=1$ Distribution

The lowest member of the loss-cone family, i.e., $j=1$, is stable with respect to the electrostatic flute modes ($k_\parallel=0$), since Dory *et al.* have found that these modes are unstable only for distributions with $j>3$, i.e., when the ratio of transverse peak velocity to half-width, $V/\Delta v_\perp$, exceeds 2.97. In this section, we show that the $j=1$ distribution can support growing waves of the ordinary mode. Let us begin with the dispersion relation

$$\frac{\omega^2}{\Omega^2}\frac{\Omega^2}{\omega_p^2}-1-\frac{2\mu}{\beta_\parallel}\frac{2\langle v^2_\parallel\rangle}{\alpha^2_\perp}=\frac{2\langle v^2_\parallel\rangle}{\alpha^2_\perp}\left(D^1_0+\frac{2\omega^2}{\Omega^2}\sum_{n=1}^{\infty}\frac{D^1_n}{\omega^2/\Omega^2-n^2}\right). \tag{20}$$

Since the function $D^1_n$ is fundamental in the analysis of Eq. (20), knowledge of its numerical values is useful, in this and possibly in subsequent investigations. We have therefore tabulated the first seven members ($n=0$ to $n=6$) in Table I for values of $\mu$ between 0 and 15. It turns out that instabilities with frequencies up to the third harmonic of $\Omega$ occur in this range of wave-number space. Moreover, for these $\mu$ values, the infinite series in Eq. (20) converges rapidly, and reasonably accurate results are obtained by taking the first six terms.

TABLE I. Numerical values of $D_0'$, $D_1'$, ..., $D_0'$ for the range $0 \le \mu \le 15.0$.

| $\mu$ | $D_0'$ | $D_1'$ | $D_2'$ | $D_3'$ | $D_4'$ | $D_5'$ | $D_6'$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4 | −0.224 | 0.088 | 0.022 | 0.003 | $1.6\times10^{-4}$ | $8.3\times10^{-6}$ | $3.36\times10^{-7}$ |
| 0.8 | −0.264 | 0.069 | 0.050 | 0.011 | $1.6\times10^{-3}$ | $3.73\times10^{-4}$ | $1.37\times10^{-5}$ |
| 1.2 | −0.245 | 0.031 | 0.063 | 0.023 | 0.005 | $8.16\times10^{-4}$ | $1.0\times10^{-4}$ |
| 1.6 | −0.214 | 0.006 | 0.064 | 0.034 | 0.010 | $2.21\times10^{-3}$ | $3.7\times10^{-4}$ |
| 2.0 | −0.186 | −0.029 | 0.058 | 0.042 | 0.016 | $4.4\times10^{-3}$ | $9.3\times10^{-4}$ |
| 2.4 | −0.163 | −0.045 | 0.048 | 0.047 | 0.022 | $7.3\times10^{-3}$ | $1.84\times10^{-3}$ |
| 2.8 | −0.151 | −0.055 | 0.037 | 0.049 | 0.028 | 0.011 | $3.13\times10^{-3}$ |
| 3.2 | −0.130 | −0.061 | 0.027 | 0.048 | 0.032 | 0.014 | $4.75\times10^{-3}$ |
| 3.6 | −0.121 | −0.064 | 0.018 | 0.046 | 0.035 | 0.017 | $6.66\times10^{-3}$ |
| 4.0 | −0.113 | −0.066 | 0.009 | 0.043 | 0.037 | 0.020 | $8.68\times10^{-3}$ |
| 4.4 | −0.106 | −0.066 | 0.002 | 0.039 | 0.038 | 0.024 | 0.011 |
| 4.8 | −0.101 | −0.067 | −0.003 | 0.034 | 0.038 | 0.026 | 0.013 |
| 5.2 | −0.095 | −0.066 | −0.008 | 0.031 | 0.039 | 0.028 | 0.015 |
| 5.6 | −0.091 | −0.065 | −0.012 | 0.027 | 0.037 | 0.030 | 0.017 |
| 6.0 | −0.088 | −0.064 | −0.016 | 0.024 | 0.036 | 0.030 | 0.019 |
| 6.4 | −0.084 | −0.064 | −0.018 | 0.019 | 0.035 | 0.031 | 0.020 |
| 6.8 | −0.082 | −0.063 | −0.021 | 0.015 | 0.033 | 0.032 | 0.022 |
| 7.2 | −0.079 | −0.061 | −0.023 | 0.014 | 0.031 | 0.031 | 0.023 |
| 7.6 | −0.076 | −0.061 | −0.024 | 0.011 | 0.03 | 0.031 | 0.024 |
| 8.0 | −0.074 | −0.060 | −0.026 | 0.008 | 0.028 | 0.031 | 0.025 |
| 8.4 | −0.072 | −0.059 | −0.027 | 0.006 | 0.026 | 0.030 | 0.026 |
| 8.8 | −0.071 | −0.058 | −0.028 | 0.003 | 0.024 | 0.030 | 0.027 |
| 9.2 | −0.069 | −0.057 | −0.029 | 0.002 | 0.022 | 0.029 | 0.027 |
| 9.6 | −0.067 | −0.056 | −0.030 | 0.0 | 0.020 | 0.029 | 0.026 |
| 10.0 | −0.066 | −0.056 | −0.030 | −0.002 | 0.019 | 0.027 | 0.026 |
| 11.0 | −0.062 | −0.054 | −0.032 | −0.006 | 0.015 | 0.026 | 0.026 |
| 12.0 | −0.060 | −0.052 | −0.032 | −0.009 | 0.012 | 0.023 | 0.025 |
| 13.0 | −0.057 | −0.050 | −0.033 | −0.011 | 0.008 | 0.021 | 0.024 |
| 14.0 | −0.055 | −0.050 | −0.033 | −0.013 | 0.006 | 0.018 | 0.023 |
| 15.0 | −0.053 | −0.048 | −0.033 | −0.014 | 0.004 | 0.016 | 0.022 |

### *1. Zero-Frequency Instability*

Reference to Table I shows that for values of $\mu$ between 0 and 1.6, $D_0^1$ is negative while $D_1^1 - D_1^6$ are positive. A schematic plot of the functions representing the left ($L$)- and right ($R$)-hand sides of Eq. (20) against $\omega^2$ is shown in Fig. 1. The function $R$ is a many-branched curve, with singularities at the cyclotron frequencies. The function $L$ is linear in $\omega^2/\Omega^2$, the slope of which is small if $\Omega^2/\omega_p^2 \ll 1$. It is evident from Fig. 1 that if the two curves intersect at a point corresponding to $\omega^2$ negative, a zero-frequency instability will develop. This will happen if

$$L(0) > R(0),$$

i.e.,

$$1 + \frac{2\mu}{\beta_\perp} < -\frac{2\langle v_\parallel^2\rangle}{\alpha_\perp^2} D_0^1 . \tag{21}$$

Equation (21) is the criterion for the occurrence of a purely growing mode for a plasma with the

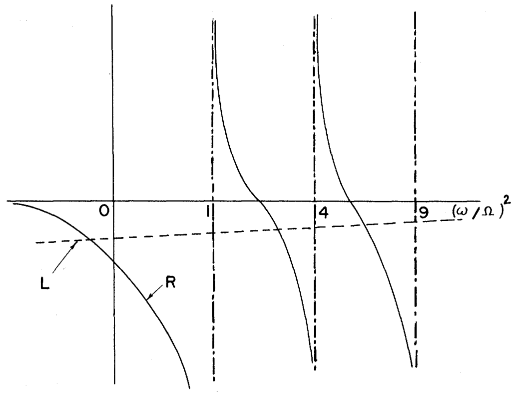


FIG. 1. Schematic plot of the left-hand side (broken line) and the right-hand side (continuous curve) of Eq. (20) as a function of $\omega^2/\Omega^2$ for the case $D_0^1 < 0$, $D_1^1 > 0$, $D_2^1 > 0$. The situation shown corresponds to instability, with the growth rate determined by the point of intersection of the two curves in the negative $\omega^2$ plane.

$j = 1$ loss-cone distribution.[22] In Fig. 2, the functions $L(0)$ and $R(0)$ are plotted against $\mu$, with $2\langle v_\parallel^2\rangle/\alpha_\perp^2$ as parameter for the former and $\beta_\perp$ for the latter. It is seen that for a given $2\langle v_\parallel^2\rangle/\alpha_\perp^2$, there is a minimum value of $\beta_\perp$, and hence $\beta_\parallel$, below which the plasma is stable against the zero-frequency instability. For $\beta_\parallel$ exceeding this value, the straight line $L(0)$ intersects the curve $R(0)$ at two values of $\mu$, denoted by $\mu_{\min}$ and $\mu_{\max}$, between which the instability criterion is satisfied. Thus waves with wave numbers corresponding to $\mu$ between $\mu_{\min}$ and $\mu_{\max}$ are unstable. Some representative values of $(\beta_\parallel)_{\min}$ are shown in Table II. Since in general $2\langle v_\parallel^2\rangle = V_\parallel^2 + 2u^2$, it is instructive

TABLE II. Minimum values of $\beta_\parallel$ required to excite the $n = 0$, 1, and 2 modes as a function of $2\langle v_\parallel^2\rangle/\alpha_\perp^2$ for an electron plasma with a $j = 1$ loss-cone distribution. The values for the $n = 0$ mode are independent of $\Omega/\omega_p$. For the $n > 0$ modes, the values shown are computed for the case $(\Omega/\omega_p) = 0.1$ but they are also good approximations to other values of $(\Omega/\omega_p)$, as long as $(\Omega/\omega_p)^2 \ll 1$.

| | $(\beta_\parallel)_{\min}$ | | |
|---|---|---|---|
| $2\langle v_\parallel^2\rangle/\alpha_\perp^2$ | $n = 0$ | $n = 1$ | $n = 2$ |
| 10 | 6.3 | 11.8 | 28.7 |
| 9 | 6.0 | 12.3 | 30.3 |
| 8 | 8.0 | 12.9 | 31.2 |
| 7 | 10.2 | 13.7 | 32.7 |
| 6 | 14.1 | 14.7 | 35.4 |
| 5 | 21.4 | 16.9 | 39.2 |
| 4 | 100.0 | 21.2 | 49.0 |
| 3 | $\infty$ | 34.3 | 75.9 |

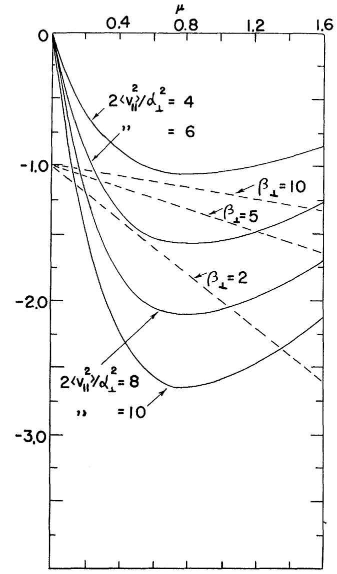


FIG. 2. Functions $L(0) = -(1 + 2\mu/\beta_\perp)$ (continuous curves) and $R(0) = (2\langle v_\parallel^2\rangle/\alpha_\perp^2)D_0^1$ (broken lines) as a function of $\mu$ parametric in $\beta_\perp$ and $2\langle v_\parallel^2\rangle/\alpha_\perp^2$, respectively.

to separate the thermal and streaming contributions. For this purpose, the stability-instability boundaries in the $\beta_{\parallel t}$-vs-$V_\parallel^2/\alpha_\perp^2$ plane are shown in Fig. 3. It appears that in stationary plasmas, the instability occurs only in plasmas with fairly large ratios of thermal to magnetic pressure. In counter-

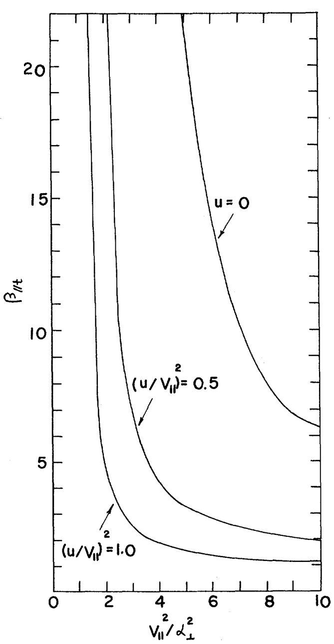


FIG. 3. Stability-instability boundaries in the $\beta_{\parallel t}$-vs-$V_\parallel^2/\alpha_\perp^2$ plane when $(u/V_\parallel)^2 = 0$, 0.5, and 1.0. Unstable regions lie above the curves.

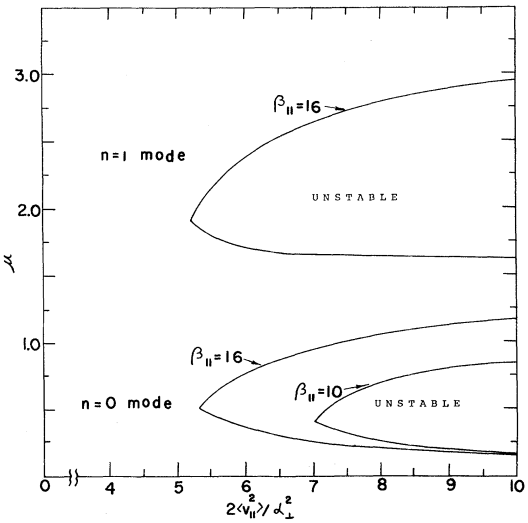


FIG. 4. Schematic plot of the left-hand side (broken line) and the right-hand side (continuous curve) of Eq. (20) as a function of $\omega^2/\Omega^2$ for the case $D_0^1>0$, $D_1^1<0$, $D_2^1>0$. The situation shown corresponds to instability, as the straight line $L$ lies above the curve $R$ everywhere in the interval between $\Omega$ and $2\Omega$.

streaming plasmas, with streaming velocity of the order of $V_{\parallel}$, however, the instability can occur for $\beta_{\parallel t}$ values of the order unity. This feature is similar to the bi-Maxwellian distribution analyzed previously.[16]

In Fig. 4, the ranges of unstable $\mu$'s are shown as a function of $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ for two values of $\beta_{\parallel}$. They increase with increasing $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ and $\beta_{\parallel}$, but lie within the limit 0.2 and 1.6, since only in this range is the condition $D_0^1<0$ and $D_1^1>0$ satisfied.

### 2. *Instability of Cyclotron Harmonic Waves*

Let us now consider the range of $\mu$ for which $D_1^1<0$ but $D_2'>0$, namely, $1.6<\mu<4.6$. The right-hand side of Eq. (18) plotted against $\omega^2/\Omega^2$ now takes the form of the many-branched curve shown in Fig. 5. Suppose originally the parameters are such that the straight line $L$ does intersect $R$ in the frequency interval between $\Omega$ and $2\Omega$. Then there will be two real roots in this frequency range. If, however, the parameters are changed to the extent that $L$ lies above $R$ everywhere in this interval, the two real roots are lost and a pair of complex conjugate ones appear, one of which corresponds to growing waves. A similar argument can be applied to the higher frequencies, with the result that the sufficient criteria for an instability to occur with a frequency near $n\Omega$ are (a) $D_n^1<0$ but $D_{n+1}^1>0$ and (b) $L>R$ for all values of $\omega$ between $n\Omega$ and $(n+1)\Omega$.

Criterion (a) limits the unstable wave numbers to a certain range. Whether criterion (b) is satisfied or not depends on the various parameters and can best be determined numerically. It is found that for a given $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ and $\Omega^2/\omega_p^2$, there is a minimum value of $\beta_{\parallel}$ required for instability. When $\beta_{\parallel}$ exceeds this threshold, there is a range of unstable wave numbers. Table II shows the values of $(\beta_{\parallel})_{\min}$ for $n=1$ and $n=2$ for the case $\Omega/\omega_p=0.1$. These values are rather insensitive to $\Omega/\omega_p$, as long as $(\Omega/\omega_p)^2\ll 1$, as is evident from the left-hand side of Eq. (18). It is seen that the order of magnitude of $\beta_{\parallel}$ required to excite the $n=1$ mode is about the same as that for the zero-frequency mode, while to excite the $n=2$ mode it is much higher. The $n>2$ modes require still larger $\beta_{\parallel}$, and since such values are somewhat unrealistic, we have not listed them in Table II.

Figure 4 shows the range of unstable wave numbers as a function of $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ for $\Omega/\omega_p=0.1$ and $\beta_{\parallel}=16$. When $\beta_{\parallel}$ decreases to 10, the $n=1$ mode becomes stable for plasmas with $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ between 0 and 10.

### B. Ring Distribution

The above procedure for the analysis of stability boundaries can be applied in a straightforward manner to other members of the loss-cone family, i.e., $j=2, 3, \ldots$. The higher members correspond to a larger ratio of the peak velocity $V$ to half-widths $\Delta v_{\perp}$, since this ratio is approximately $1.72j^{1/2}$. In this section, we consider the limiting case when this ratio is infinite, which corresponds to monoenergetic particles in the perpendicular direction. This is described by the ring distribution given by Eq. (4). This distribution has been studied extensively in the literature with regard to electrostatic instabilities.

The ordinary-wave-dispersion relation for the

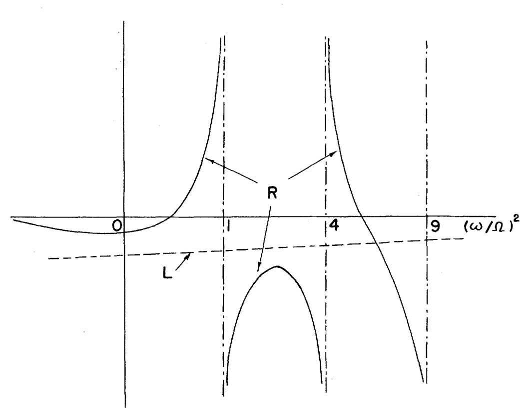


FIG. 5. Range of unstable $\mu$'s as a function of $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ for the $j=1$ distribution with two values of $\beta_{\parallel}$. For $\beta_{\parallel}=16$, both the $n=0$ and the $n=1$ modes are unstable. The range of unstable $\mu$'s increases with increasing $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$. When $\beta_{\parallel}$ decreases to 10, the $n=1$ mode becomes stable for the range of $2\langle v_{\parallel}^2\rangle/\alpha_{\perp}^2$ shown.

ring distribution reads

$$\frac{\omega^2}{\Omega^2}\frac{\Omega^2}{\omega_p^2} - 1 - \frac{2\mu}{\beta_\parallel}\frac{2\langle v_\parallel^2\rangle}{V^2} = \frac{2\langle v_\parallel^2\rangle}{V^2}\left(D_0^\infty + \frac{2\omega^2}{\Omega^2}\sum_{n=1}^{\infty}\frac{D_n^\infty}{\omega^2/\Omega^2 - n^2}\right) . \quad (22)$$

The quantities $D_n^\infty$ are given by Eq. (10). We note in particular that

$D_0^\infty < 0, \quad D_1^\infty > 0 \quad$ for $0 < \mu < 1.69$ ,

$D_1^\infty < 0, \quad D_2^\infty > 0 \quad$ for $1.69 < \mu < 4.65$ ,

$D_2^\infty < 0, \quad D_3^\infty > 0 \quad$ for $4.65 < \mu < 8.85$.

The boundaries of instabilities can again be analyzed as in Sec. III A. For fixed ratios of $2\langle v_\parallel^2\rangle/V^2$ and $\Omega/\omega_p$, there is again a minimum value of $\beta_\parallel$ required for instability, which is different for each harmonic. When $\beta_\parallel$ exceeds the threshold value of a particular harmonic, the unstable wave-number spectrum has an upper and a lower limit. Some representative values of $(\beta_\parallel)_{\min}$ for the zero-frequency instability and the first three harmonics are given in Table III. It is seen that the zero-frequency mode is the most difficult one to excite, while for the first three harmonics, $(\beta_\parallel)_{\min}$ is about the same order. This is in marked contrast with the $j=1$ distribution, for which excitation of the second or higher harmonics requires an unusually large value of $\beta_\parallel$. To make another comparison of the two distributions, let us recall that $V$ and $\alpha_\perp$ can be interpreted as the peak transverse velocities, the former for the ring distribution and the latter for the $j=1$ distribution. Reference to Tables II and III then shows that for the same ratio of $2\langle v_\parallel^2\rangle$ to the square of the peak transverse velocity, instability sets in at a lower value of $\beta_\parallel$ for the ring distribution. Thus a broadening of a sharply peaked distribution has a stabilizing effect, a result similar to the Harris-type electrostatic flute modes.[3] However, there is an essential difference in that the flute-like electrostatic instability is stable when $j < 3$,[3] while the ordinary wave considered here can be unstable for all values of $j$.

TABLE III. Minimum value of $\beta_\parallel$ required to excite $n=0$, 1, 2, and 3 modes as a function of $2\langle v_\parallel^2\rangle/V^2$ for an electron plasma with a ring distribution. The values for the $n=0$ mode are independent of $\Omega/\omega_p$. For the $n>0$ modes, the values shown are computed for the case $(\Omega/\omega_p)=0.1$ but they are also good approximations to other values of $(\Omega/\omega_p)$, as long as $(\Omega/\omega_p)^2 \ll 1$.

| | $(\beta_\parallel)_{\min}$ | | | |
|---|---|---|---|---|
| $2\langle v_\parallel^2\rangle/V^2$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
| 5 | 7.1 | 4.7 | 5.6 | 7.0 |
| 4 | 9.6 | 4.8 | 5.8 | 7.1 |
| 3 | 20.0 | 5.2 | 6.0 | 7.3 |
| 2.33 | $\infty$ | 5.6 | 6.3 | 7.7 |
| 2 | | 6.0 | 6.5 | 8.0 |

TABLE IV. Range of unstable $\mu$'s of the first four modes for an electron plasma with a ring distribution and $\beta_\parallel = 8$, $\Omega/\omega_p = 0.1$.

| | Range of unstable $\mu$'s | | | |
|---|---|---|---|---|
| $2\langle v_\parallel^2\rangle/V^2$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
| 1 | stable | stable | stable | stable |
| 2 | stable | 2.24–3.88 | 5.30–7.77 | stable |
| 3 | stable | 1.97–4.16 | 5.02–8.08 | 9.80–12.45 |
| 4 | stable | 1.88–4.27 | 4.94–8.16 | 9.70–12.60 |
| 5 | 0.43–0.90 | 1.84–4.34 | 4.90–8.25 | 9.60–12.80 |

In Table IV, the ranges of unstable wave numbers of the first four modes are given for the case $\beta_\parallel = 8$ and $\Omega/\omega_p = 0.1$. For $2\langle v_\parallel^2\rangle/V^2 = 1$, the ordinary wave is stable. The first two harmonics become unstable when $2\langle v_\parallel^2\rangle/V^2 = 2$ and when this ratio increases to 5, instability occurs for all four modes.

## IV. DISCUSSION ON GROWTH RATE

### A. Zero-Frequency Instability

When conditions are such that the plasma supports a zero-frequency instability, reference to Fig. 1 shows that the growth rate is determined by the point in the negative $\omega^2$ plane at which the curves representing the left- and right-hand sides of the dispersion relation intersect. This point can be determined numerically. Since the unstable $\mu$'s are of the order unity, the infinite series converges rapidly and little computational difficulty is encountered. Some results on the maximum growth rate are shown in Table V for the $j=1$ loss-cone distribution and the ring distribution. The column $\mu$ corresponds to the value of $\mu$ for which maximum growth occurs for the given set of parameters. It is seen that the growth rates are of the order of the electron cyclotron frequency, which is about the same order of magnitude as those of the anisotropic bi-Maxwellian distribution studied previously.[16]

### B. Cyclotron Harmonic Instabilities

Since the frequencies of the unstable cyclotron harmonic waves involve both real and imaginary parts, the growth rates are no longer determined by the point of intersection of two simple curves. If we denote the real and imaginary parts of $\omega^2$ by $R$ and $I$, respectively, dispersion relation (5) may be written, setting the real and imaginary parts separately to zero and taking into account only

TABLE V. Maximum growth rates of the zero-frequency instability $(\Omega/\omega_p)=0.1$ throughout.

| Ring distribution | | | | $j=1$ distribution | | | |
|---|---|---|---|---|---|---|---|
| $2\langle v_\parallel^2\rangle/V^2$ | $\beta_\parallel$ | $\mu$ | $\gamma_{max}/\omega_p$ | $2\langle v_\parallel^2\rangle/\alpha_\perp^2$ | $\beta_\parallel$ | $\mu$ | $\gamma_{max}/\omega_p$ |
| 10 | 16 | 0.72 | 0.12 | 10 | 16 | 0.6 | 0.084 |
| 10 | 12 | 0.72 | 0.11 | 10 | 12 | 0.4 | 0.063 |
| 10 | 8 | 0.50 | 0.0777 | 10 | 8 | 0.4 | 0.040 |
| 10 | 4 | | 0.00 | 10 | 6 | | 0.00 |
| 5 | 12 | 0.72 | 0.055 | 8 | 12 | 0.4 | 0.045 |
| 7 | 12 | 0.72 | 0.084 | 10 | 12 | 0.4 | 0.064 |
| 9 | 12 | 0.72 | 0.10 | 12 | 12 | 0.6 | 0.077 |

electron dynamics, as two coupled equations

$$R - c^2k^2 = \omega_p^2\left(1+\frac{4\langle v_\parallel^2\rangle}{\alpha_\perp^2}\sum_{n=1}^{\infty}\frac{D_n^j n^2\Omega^2(R-n^2\Omega^2)}{(R-n^2\Omega^2)^2+I^2}\right), \quad (23a)$$

$$1 = -\omega_p^2\left(\frac{4\langle v_\parallel^2\rangle}{\alpha_\perp^2}\sum_{n=1}^{\infty}\frac{D_n^j n^2\Omega^2}{(R-n^2\Omega^2)^2+I^2}\right). \quad (23b)$$

The above system of equations must be solved numerically in order to determine the growth rates accurately. This, however, is a complicated computational problem and will not be undertaken in this paper. The situation is somewhat reminiscent of the Harris instability, for which it is interesting to note that the first few papers on the subject contained no calculation on the growth rates, presumably also because of its complexity. We shall, however, present and discuss an approximate formula which can be derived if we follow a method due to Baldwin, Bernstein, and Weenink.[14] For this purpose, let us go back to Eq. (18). The method of Baldwin *et al.*[14] consists of approximating $\omega^2$ by $m^2\Omega^2$ for frequencies in the vicinity of the $m$th harmonic, which results in the following approximate dispersion relation:

$$(\omega^2-m^2\Omega^2)+(m^2\Omega^2-c^2k^2)-\omega_p^2 = \omega_p^2\,\frac{4\langle v_\parallel^2\rangle\Omega^2}{\alpha_\perp^2}\times\left(\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{n^2D_n^j}{m^2\Omega^2-n^2\Omega^2}+\frac{n^2D_m^j}{\omega^2-m^2\Omega^2}\right). \quad (24)$$

The above equation is a biquadratic in $(\omega^2-m^2\Omega^2)^2$ and can therefore be solved for $\omega^2$. The result is

$$\omega = \Omega\left(m^2-\frac{\omega_p^2}{\Omega^2}F\right)^{1/2}+\frac{\omega_p^2}{2\Omega^2}\,\frac{(F+G)^{1/2}}{[m^2-(\omega_p^2/\Omega^2)F]^{1/2}}, \quad (25)$$

where

$$F=\left[\frac{m^2}{2}\frac{\Omega^2}{\omega_p^2}-\frac{\mu}{\beta_\perp}-0.5-\frac{\langle v_\parallel^2\rangle}{\alpha_\perp^2}\left(2D_m^j+D_0^j+2m^2\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{D_n^j}{m^2-n^2}\right)\right]^2, \quad (26)$$

$$G=4m^2\,\frac{\Omega^2}{\omega_p^2}\,\frac{\langle v_\parallel^2\rangle}{\alpha_\perp^2}\,D_m^j. \quad (27)$$

Based on Eq. (25), a pair of complex conjugate roots for $\omega$ appears, one of which corresponds to an instability, when $F+G<0$. The imaginary part of the frequency is given by

$$\omega_i \simeq \frac{\omega_p^2}{2\Omega}\,\frac{(F+G)^{1/2}}{[m^2-(\omega_p^2/\Omega^2)F^{1/2}]^{1/2}}. \quad (28)$$

It follows from Eq. (28) that the maximum growth rate for the $m$th harmonic, denoted by $\gamma_m$, occurs approximately when $F=0$ and is

$$\gamma_m \simeq \frac{\omega_p^2}{2m\Omega}G^{1/2} = \frac{\langle v_\parallel^2\rangle}{\alpha_\perp^2}(D_m^j)^{1/2}\omega_p. \quad (29)$$

Equation (29) shows explicitly that the growth rate increases with $\omega_p$ and the ratio $\langle v_\parallel^2\rangle/\alpha_\perp^2$, if the parameters are such that $F=0$. However, the range of validity of Eqs. (28) and (29) is not at all clear. For some parameters, the growth rates based on them can be so large compared to the cyclotron frequency that their validity is suspect. For example, consider the first harmonic $(m=1)$ in a plasma with a ring distribution in the perpendicular velocities. For $\beta_\perp=5$, $(\Omega/\omega_p)^2=0.002$, $2\langle v_\parallel^2\rangle/V^2=3.17$, the quantity $F$ vanishes at $\mu=2.88$ and according to Eq. (29), $\gamma_1\simeq 0.65\,\omega_p$. Such a large growth rate seems to be questionable on two accounts. First, it is much larger than those of the zero-frequency instability, which are typically of the order of the cyclotron frequency. Second, a large imaginary part of $\omega^2$ may not be consistent with the approximation upon which Eq. (24) is based, which assumes $\omega\simeq m\Omega$. Consequently, the correctness of Eq. (29) is most likely limited to parameters which result in $\gamma_m \ll m\Omega$, i.e., near marginal instability. In order to establish the range of validity of Eq. (29), and to determine the maximum growth rate accurately, especially for parameters for which (29) yields $\gamma_m \gtrsim m\Omega$ (such as the example given), the system of Eqs. (23a) and (23b) must be solved numerically. This procedure involves extensive calculations, the results of which we hope to be able to report subsequently.

## V. DISCUSSION

In this paper, we have shown that in stationary or counterstreaming plasmas whose distributions in the prependicular velocities are of the ring- or the loss-cone-type, the electromagnetic ordinary mode propagating perpendicular to the magnetic field can become unstable with either zero frequency or near the cyclotron harmonics. Detailed analysis of the instability boundaries have been carried out for electron plasmas with the lowest $(j=1)$ and the highest $(j=\infty)$ members of the loss-cone distributions. The results show that the higher the average parallel energy, the more is the plasma susceptible to the instability. If the distribution in the parallel velocities is a stationary Maxwellian, instability occurs only for large ratios of parallel thermal pressure to magnetic pressure ($\beta_{\parallel t}$). For counter-streaming plasmas with streaming velocities of the order of the parallel electron thermal speed, the instability can occur for $\beta_{\parallel t}$ values of order unity. Representative values of the growth rates of the zero-frequency mode have been determined numerically. For the cyclotron harmonics, an approximate formula for the growth rate, valid near marginal instability, has been derived. The general problem of obtaining the growth rates of the cyclotron instabilities has been formulated in terms of two coupled equations, each of which contains an infinite series. The solution of these equations, however, awaits further investigation.

The results on the stability analysis of the $j=1$ and the $j=\infty$ distributions indicate that a broadening of a sharply peaked distribution in the transverse velocities tends to stabilize the electromagnetic cyclotron harmonic instability. Since the parameter $j$ is directly related to the ratio of peak transverse velocity to half-width, a detailed analysis of distributions with intermediate values of $j$ may be desirable. However, it is expected that the results would fall in between the $j=1$ and $j=\infty$ cases presented in this paper.

Finally, it is evident from the dispersion relation (5) that if ion dynamics is included, there is the additional possibility that waves around the harmonics of the ion cyclotron frequency may become unstable. In the study of the bi-Maxwellian distribution, it was found that the presence of ion streaming enables the zero-frequency instability to occur in plasmas with very low values of $\beta_{\parallel t}$.[17] It is reasonable to expect that a similar situation would occur for the instability at the cyclotron harmonics. A detailed analysis incorporating ion dynamics is currently in progress.

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*Work supported by the National Science Foundation under Grant No. Gk-4592.

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