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Unretarded London–van der Waals Forces Derived from Classical Electrodynamics with Classical Electromagnetic Zero-Point Radiation

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The unretarded London–van der Waals force between a neutral polarizable particle and a conducting wall, and between two neutral polarizable particles are derived from classical electromagnetism under the assumption that the universe contains fluctuating classical electromagnetic radiation with a Lorentz-invariant spectrum, (classical electromagnetic zero-point radiation). The classical derivations correspond in detail to the familiar charge-fluctuation arguments which have been advanced previously as qualitative semiclassical descriptions of quantum calculations of these forces.

I. INTRODUCTION

In 1926, London[1] evaluated the interaction energy between two neutral polarizable atoms using fourth-order quantum perturbation theory, and he found a potential function varying as R^{-6}, where R is the separation between the particles. The heuristic description[2] of this interaction follows a semiclassical analysis. Fluctuations in the charge distribution within one polarizable particle produce an electric dipole whose field then induces a dipole in the other polarizable particle. The electrostatic interaction energy between these two dipoles corresponds to the potential in London's quantum calculation. Although this essentially classical description is very appealing, its applicability is customarily diminished[2] by the emphasis that the interaction is basically a quantum phenomenon. It is the purpose of this paper to show that the heuristic description here can be used to give a purely classical calculation of the van der Waals force which is in agreement with London's quantum result.

The new calculation takes on interest beyond the confines of dispersion forces because it forms another step in a general program in theoretical physics. It has been suggested recently[3–7] that much of what is presently regarded as quantum physics may be accounted for in terms of classical electrodynamics in which we allow the possibility of temperature-independent fluctuating classical electromagnetic radiation. The spectrum of this classical zero-point radiation, $\frac{1}{2}\hbar\omega$ per normal mode, can be derived[3] (up to a multiplicative constant) from the requirement of Lorentz invariance. The theory of classical electromagnetic interactions including this classical zero-point radiation has been made the basis for classical derivations of the blackbody radiation spectrum,[3,4] of photon statistics,[4] of the third law of thermodynamics,[5] of rotator specific heats,[6] and of asymptotic retarded dispersion forces.[7] Here we perform the necessary calculations to show that the unretarded London forces are also predicted by the theory. In another publication, we will calculate the full fourth-order (Casimir-Polder)[8] force between two neutral polarizable particles, and then show the equivalence to all orders of perturbation theory between the quantum electrodynamic and classical electrodynamic calculation of dispersion forces.

II. SINGLE DIPOLE OSCILLATOR IN CLASSICAL ZERO-POINT RADIATION

A. Physical Description

The following calculations for unretarded van der Waals forces will be more easily understood if we first consider a single classical dipole oscillator in free space. According to traditional classical theory, such an oscillator will be at rest at the equilibrium position. However, in the view which we are exploring, there is no such thing as "empty space" because the universe contains fluctuating classical electromagnetic radiation with a Lorentz-invariant spectrum. Thus the oscillator will not remain at rest, but due to the random impulses from the random electromagnetic field, will oscillate about its equilibrium position. Marshall[9] has considered this situation in detail and has found that the classical oscillator closely resemble a quantum oscillator in its ground state. Here we will consider only those aspects of use in our later calculations.

We consider a one-dimensional harmonic oscillator of mass m, charge e, and natural frequency ω_0. The equation of motion for the oscillator is then

$$m\ddot{x} = -m\omega_0^2 x + eE_x + \tfrac{2}{3}(e^2/c^3)\dddot{x} . \qquad (1)$$

This is simply Newton's second law including the elastic restoring force $-m\omega_0^2 x$, a force eE_x due to the random radiation field, and a radiation damping force $\frac{2}{3}(e^2/c^3)\dddot{x}$. It is sometimes convenient to rewrite this equation in terms of the electric dipole moment

$$p = ex \ , \tag{2}$$

$$\ddot{p} - \Gamma \dddot{p} + \omega_0^2 p = \tfrac{3}{2}\Gamma c^3 E_x \ , \tag{3}$$

where the damping coefficient Γ is

$$\Gamma = \tfrac{2}{3} e^2/mc^3 \ . \tag{4}$$

The fluctuating classical radiation in our theory[3,4] has an electric field

$$\vec{E}(\vec{x},\ t) = \mathrm{Re} \sum_{\lambda=1}^{2} \int d^3k\, \hat{\epsilon}(\vec{k},\ \lambda) \mathfrak{h}(\omega_{\vec{k}}) \times \exp\{i[\vec{k}\cdot\vec{x} - \omega t + \theta(\vec{k},\ \lambda)]\} \ , \tag{5}$$

where

$$\omega = ck \ , \tag{6}$$

$$\pi^2 \mathfrak{h}^2(\omega) = \tfrac{1}{2}\hbar\omega \ , \tag{7}$$

$\hat{\epsilon}(\vec{k},\ \lambda)$ is a polarization vector, and we are employing the random-phase description of Planck[10] and of Einstein and Hopf.[11] The spectrum of the radiation here is Lorentz invariant and normalized to $\tfrac{1}{2}\hbar\omega$ per normal mode.

B. Mathematical Calculation

Introducing a Fourier decomposition for $p(t)$ which formally follows that for $\vec{E}$,

$$p(t) = \mathrm{Re} \sum_{\lambda=1}^{2} \int d^3k\, p(\omega,\ t) \ , \tag{8}$$

where $p(\omega,\ t)$ has time dependence $e^{-i\omega t}$, Eq. (3) becomes

$$Cp = \beta E_x \ , \tag{9}$$

where

$$C = -\omega^2 - i\Gamma\omega^3 + \omega_0^2 \ , \tag{10}$$

$$\beta = \tfrac{3}{2}\Gamma c^3 = e^2/m \ , \tag{11}$$

and

$$E_x = \hat{\epsilon}_x(\vec{k},\ \lambda)\mathfrak{h}(\omega)\exp\{i[\vec{k}\cdot\vec{x} - \omega t + \theta(\vec{k},\ \lambda)]\} . \tag{12}$$

The solution follows as

$$p = \beta E_x/C \ . \tag{13}$$

The energy of the oscillator is that associated with the particle kinetic energy and the potential energy of the elastic restoring force,

$$\mathcal{E} = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\omega_0^2 x^2 \ . \tag{14}$$

We are interested in the average energy of the fluctuating system, so that we write

$$\mathcal{E} = \langle \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\omega_0^2 x^2 \rangle = \langle (1/2\beta)(\dot{p}^2 + \omega_0^2 p^2) \rangle \ . \tag{15}$$

In terms of the Fourier transform, the mean-square value of the dipole moment p is,

$$\langle p^2 \rangle = \sum_{\lambda=1}^{2} \sum_{\lambda'=1}^{2} \int d^3k \int d^3k' \langle \mathrm{Re}p(\omega,\ t)\mathrm{Re}p(\omega',\ t) \rangle \ . \tag{16}$$

Introducing the solution (13), we have

$$\langle \mathrm{Re}p(\omega,\ t)\mathrm{Re}p(\omega',\ t) \rangle = \langle \mathrm{Re}(\beta E/C)\mathrm{Re}(\beta E'/C') \rangle \ , \tag{17}$$

where the prime indicates dependence on ω'.

The averaging in our random-phase description involves

$$\langle \exp\{i[\vec{k}\cdot\vec{x} - \omega t + \theta(\vec{k},\ \lambda)]\} \times \exp\{-[\vec{k}'\cdot\vec{x} - \omega' t + \theta(\vec{k}',\ \lambda')]\} \rangle = \delta_{\lambda\lambda'}\delta^3(\vec{k} - \vec{k}') \ , \tag{18}$$

$$\langle \exp\{i[\vec{k}\cdot\vec{x} - \omega t + \theta(\vec{k},\ \lambda)]\} \times \exp\{i[\vec{k}'\cdot\vec{x} - \omega' t + \theta(\vec{k}',\ \lambda')]\} \rangle = 0 \ . \tag{19}$$

Thus we have

$$\langle \mathrm{Re}p(\omega,\ t)\mathrm{Re}p(\omega',\ t) \rangle = \tfrac{1}{2}(\beta^2/|C|^2)\epsilon_x^2 \mathfrak{h}^2(\omega)\delta_{\lambda\lambda'}\delta^3(\vec{k} - \vec{k}') \ . \tag{20}$$

Noting equations (10), (11), and (16) we have

$$\langle p^2 \rangle = \sum_{\lambda=1}^{2} \int d^3k \frac{1}{2}\left(\frac{e^2}{m}\right)^2 \epsilon_x^2 \frac{\hbar\omega}{2\pi^2} \frac{1}{|-\omega^2 - i\Gamma\omega^3 + \omega_0^2|^2} . \tag{21}$$

Summing over polarizations

$$\sum_{\lambda=1}^{2} \epsilon_x^2(\vec{k},\ \lambda) = 1 - k_x^2/k^2 \ , \tag{22}$$

and carrying out the angular integration, this becomes

$$\langle p^2 \rangle = \int_{\omega=0}^{\infty} d\omega \frac{\omega^2}{c^3}\left(4\pi\frac{2}{3}\right)\left(\frac{e^2}{m}\right)^2 \frac{\hbar\omega}{4\pi^2} \times \frac{1}{(-\omega^2 + \omega_0^2)^2 + (\Gamma\omega^3)^2} \ . \tag{23}$$

The integral in (23) is convergent at both ends, $\omega \to 0$ and $\omega \to \infty$, and the integrand is sharply peaked at $\omega = \omega_0$. Following a traditional procedure for approximating integrals of this form, we change the variable of integration from ω to $x = \omega - \omega_0$, replace all terms in ω not involving $\omega - \omega_0$ by ω_0, and extend the lower limit of integration to $x \to -\infty$. Changing variables once more to $z = 2x/\omega_0^2\Gamma$, the integral takes the form

$$\int_{-\infty}^{\infty} dz \frac{1}{z^2 + 1} = \pi \ . \tag{24}$$

C. Results for Single Oscillator

The result for the mean square dipole moment is

$$\langle p^2\rangle = \tfrac{1}{2}\hbar e^2/\omega_0 m \ . \tag{25}$$

The value for $\langle \dot{p}^2\rangle$ may be obtained in analogous fashion and merely involves two more powers of ω^2 which become ω_0^2 because of the sharply peaked integrand

$$\langle \dot{p}^2\rangle = \tfrac{1}{2}\hbar\omega_0 e^2/m \ . \tag{26}$$

The energy is

$$\mathcal{E} = \frac{1}{2\beta}\langle \dot{p}^2\rangle + \frac{\omega_0^2}{2\beta}\langle p^2\rangle = \tfrac{1}{2}\hbar\omega_0 \ . \tag{27}$$

The average energy of the classical oscillator emersed in zero-point radiation is just that of a quantum oscillator with the same natural frequency. The values for the mean-square displacement and momentum also agree

$$\langle x^2\rangle = (1/e^2)\langle p^2\rangle = \hbar/2m\omega_0 \ , \tag{28}$$

$$\langle (m\dot{x})^2\rangle = (m^2/e^2)\langle \dot{p}^2\rangle = \tfrac{1}{2}m\hbar\omega_0 \ . \tag{29}$$

III. UNRETARDED FORCE BETWEEN A NEUTRAL POLARIZABLE PARTICLE AND A CONDUCTING WALL

A. Physical Situation

We consider a neutral polarizable particle located a distance R from a perfectly conducting wall. The particle is pictured as a three-dimensional harmonic oscillator of natural frequency ω_0 and damping constant Γ. When the particle is polarized by the fluctuating zero-point radiation present, it becomes a source of electromagnetic fields. These in turn produce charge separations in the conducting wall, and hence fields back at the position of the polarizable particle which attract the particle toward the wall.

If we write out the explicit differential equation for the oscillator near the conducting wall, and then calculate the kinetic and elastic potential energy, we find that this mechanical energy is

$$E_{\mathrm{KE}} + E_{\mathrm{PE}} = \tfrac{1}{4}\hbar\omega' + \tfrac{1}{4}\hbar\omega_0^2/\omega' \ ,$$

where ω' is the new resonant frequency of the oscillator in the presence of the wall. To order e^2/m, the change in the kinetic energy cancels the change in potential energy so that the effective change in energy is that associated with the electromagnetic field. The calculations alluded to here will be presented in detail in a subsequent paper.

The energy associated with the electrostatic attraction of a dipole $\vec{p}$ to a conducting wall is half the energy of attraction[12] to the image dipole $\vec{p}\,'$,

$$U(R) = \frac{1}{2}\,\frac{\vec{p}\cdot\vec{p}\,' - 3(\hat{\kappa}\cdot\vec{p})(\hat{\kappa}\cdot\vec{p}\,')}{(2R)^3} \ , \tag{30}$$

and we have taken the z axis perpendicular to the wall. The factor of $\frac{1}{2}$ occurs because the work done in bringing the dipole in from spatial infinity involves forces on the dipole, but not on the image dipole which also moves in from spatial infinity. It is the average energy which is of interest to us in the attraction of the polarizable particle to its image,

$$U(R) = \left\langle \frac{p_x p_x' + p_y p_y' - 2p_z p_z'}{16R^3}\right\rangle \ . \tag{31}$$

The components of the image dipole are easily found from the theory of electrostatic images

$$p_x' = -p_x\,, \quad p_y' = -p_y\,, \quad p_z' = p_z\,, \tag{32}$$

giving

$$U(R) = -\frac{\langle p_x^2\rangle + \langle p_y^2\rangle + 2\langle p_z^2\rangle}{16R^3} \ . \tag{33}$$

Now the components p_x, p_y, and p_z of the vector dipole $\vec{p}$ may be regarded as independent one-dimensional oscillators of the type considered in Sec. II. Hence, in the lowest approximation, all we need do is substitute for each of $\langle p_x^2\rangle$, $\langle p_y^2\rangle$, and $\langle p_z^2\rangle$ the value of $\langle p^2\rangle$ in (20).

B. Results for Particle-Wall Potential

With the substitution $\langle p^2\rangle = \frac{1}{2}\hbar e^2/\omega_0 m$, the potential is

$$U(R) = -e^2\hbar/8m\omega_0 R^3 = -\omega_0\hbar\alpha/8R^3 \ , \tag{34}$$

where

$$\alpha = e^2/m\omega_0^2 \tag{35}$$

is the static polarizability corresponding to the physical oscillator of Sec. II. The result in (34) is just that obtained by Casimir and Polder[8] from second-order quantum perturbation theory when we make the natural identification of the oscillator frequency ω_0 as

$$\omega_0\hbar = \mathcal{E}_1 - \mathcal{E}_0 \ , \tag{36}$$

where $\mathcal{E}_1$ is the energy of the first excited P state and $\mathcal{E}_0$ is the energy of the ground state for the atom treated by Casimir and Polder.

IV. UNRETARDED FORCE BETWEEN TWO NEUTRAL POLARIZABLE PARTICLES

A. Need for a More Sophisticated Analysis

The force between two identical neutral polarizable particles is obtained in a fashion analogous to that given above for a particle and a conducting wall. However, while the particle-wall force was of order $e^2/\hbar c$ (second order), the lowest nonvanishing contribution to the particle-particle force is of order $(e^2/\hbar c)^2$ (fourth order).

The change in the order of the interaction necessitates a more sophisticated treatment of the interaction. In the previous calculation we ignored any change in the mechanical energy of the oscillator caused by the presence of the conducting wall, and we computed only the change in electrostatic energy associated with the attraction to the wall. However, when two polarizable particles are near each other, each produces electrostatic fields which modify the mechanical energy of the other dipole to the same order $(e^2/\hbar c)^2$ as the electrostatic energy of interaction. An accurate calculation of the position-dependent energy of the system must include both the mechanical and electrostatic energies.

B. Physical Situation Involving Two Polarizable Particles

When two polarizable particles are separated by a distance R, the classical electromagnetic zero-point radiation causes a fluctuating polarization of each particle, and, in turn, the induced dipoles produce electrostatic fields which influence the polarization of the other particle. Thus Newton's second law gives two coupled equations for the dipoles,

$$m\ddot{\vec{x}}_A = -m\omega_0^2\vec{x}_A + e\vec{E}_B + e\vec{E}(\vec{x}_A,\ t) + \tfrac{2}{3}(e^2/c^3)\dddot{\vec{x}}_A\ , \tag{37}$$

$$m\ddot{\vec{x}}_B = -m\omega_0^2\vec{x}_B + e\vec{E}_A + e\vec{E}(\vec{x}_B,\ t) + \tfrac{2}{3}(e^2/c^3)\dddot{\vec{x}}_B\ , \tag{38}$$

where $\vec{E}(\vec{x}_A,\ t)$, $\vec{E}(\vec{x}_B,\ t)$ are the fields due to the zero-point radiation (5) at the positions of particles A and B, respectively, and $\vec{E}_A$, $\vec{E}_B$ are the electric fields due to the induced dipoles acting on the other particle. Rewriting the equations in terms of the dipoles $\vec{p}_A = e\vec{x}_A$, $\vec{p}_B = e\vec{x}_B$, and introducing the explicit form of the electrostatic fields, we have for the Fourier decomposition

$$-\omega^2\vec{p}_A = -\omega_0^2\vec{p}_A + \tfrac{3}{2}\Gamma c^3\vec{E}(\vec{x}_A,\ t) + i\Gamma\omega^3\vec{p}_A + \tfrac{3}{2}\Gamma\omega^3\left\{\frac{(\hat{\kappa}\times\vec{p}_B)\times\hat{\kappa}}{kR} + [3\hat{\kappa}(\hat{\kappa}\cdot\vec{p}_B) - \vec{p}_B]\times\left(\frac{1}{(kR)^3} - \frac{i}{(kR)^2}\right)\right\}\exp(ikR), \tag{39}$$

$$-\omega^2\vec{p}_B = -\omega_0^2\vec{p}_B + \tfrac{3}{2}\Gamma c^3\vec{E}(\vec{x}_B,\ t) + i\Gamma\omega^3\vec{p}_B + \tfrac{3}{2}\Gamma\omega^3\left\{\frac{(\hat{\kappa}\times\vec{p}_A)\times\hat{\kappa}}{kR} + [3\hat{\kappa}(\hat{\kappa}\cdot\vec{p}_A) - \vec{p}_A]\times\left(\frac{1}{(kR)^3} - \frac{i}{(kR)^2}\right)\right\}\exp(ikR), \tag{40}$$

where we shall assume that particle B lies to the right of particle A, a distance R along the z axis.

C. Mathematical Analysis

The equations (39) and (40) give a system of six coupled differential equations for p_{Ax}, p_{Ay}, p_{Az}, p_{Bx}, p_{By}, and p_{Bz}. However, the coupling is only between corresponding components of the two particles—between p_{Ax} and p_{Bx}, p_{Ay} and p_{By}, and p_{Az} and p_{Bz}. Thus for this paper we will not introduce a matrix notation, but will consider three pairs of equations of the form

$$-\omega^2 p_A - i\Gamma\omega^3 p_A + \omega_0^2 p_A + \eta p_B = \tfrac{3}{2}\Gamma c^3 E(\vec{x}_A,\ t)\ , \tag{41}$$

$$-\omega^2 p_B - i\Gamma\omega^3 p_B + \omega_0^2 p_B + \eta p_A = \tfrac{3}{2}\Gamma c^3 E(\vec{x}_B,\ t)\ , \tag{42}$$

where it is understood that p_A, p_B, η, $E(\vec{x}_A,\ t)$, and $E(\vec{x}_B,\ t)$ all stand for the x component, or all for the y component, or all for the z component. When the z component is involved, then

$$\eta_z = -\tfrac{3}{2}\Gamma\omega^3 2\left(\frac{1}{(kR)^3} - \frac{i}{(kR)^2}\right)\exp(ikR); \tag{43}$$

when the x or y component is involved,

$$\eta_x = \eta_y = -\tfrac{3}{2}\Gamma\omega^3\left(\frac{1}{kR} + \frac{i}{(kR)^2} - \frac{1}{(kR)^3}\right)\exp(ikR)\ . \tag{44}$$

Choosing one pair of these equations, we have

$$Cp_A + \eta p_B = \beta E_A\ , \tag{45}$$

$$\eta p_A + Cp_B = \beta E_B\ , \tag{46}$$

where

$$C = -\omega^2 - i\Gamma\omega^3 + \omega_0^2, \tag{47}$$

$$\beta = \tfrac{3}{2}\Gamma c^3 = e^2/m\ . \tag{48}$$

Also, we introduce the symbols

$$\lambda_- = C - \eta\ , \tag{49}$$

$$\lambda_+ = C + \eta\ . \tag{50}$$

The solutions to Eqs. (45) and (46) are

$$p_A = \beta(CE_A - \eta E_B)/(C^2 - \eta^2)\ , \tag{51}$$

$$p_B = \beta(CE_B - \eta E_A)/(C^2 - \eta^2)\ . \tag{52}$$

It is the energy of the system which is of interest to us

$$\mathcal{E} = \langle\tfrac{1}{2}m(\dot{x}_A^2 + \dot{x}_B^2) + \tfrac{1}{2}m\omega_0^2(x_A^2 + x_B^2) + up_Ap_B\rangle\ , \tag{53}$$

with

$$u = m\mathrm{Re}\eta/e^2, \tag{54}$$

including the kinetic energy of the oscillators, the elastic potential energy, and the electrostatic energy of interaction when $k_0R \ll 1$ so that u becomes to lowest order in R

$$u_z = \frac{m}{e^2}\mathrm{Re}\eta_z \to -\frac{2}{R^3}\ , \qquad u_x = u_y = \frac{m}{e^2}\mathrm{Re}\,\eta_x \to \frac{1}{R^3}\ .$$

The expression can be written in terms of the dipole moment $p=ex$,

$$\mathcal{E}=\frac{1}{2\beta}\left(\langle\dot{p}_A^2\rangle+\langle\dot{p}_B^2\rangle\right)+\frac{\omega_0^2}{2\beta}\left(\langle p_A^2\rangle+\langle p_B^2\rangle\right)+u\ \langle p_A p_B\rangle . \qquad (55)$$

The evaluation of the energy proceeds as in Sec. II. Introducing the Fourier decomposition (8) with the solutions (51) and (52), we carry out the averages as in (18) and (19), finding expressions analogous to (20).

Thus here

$$\langle \mathrm{Re}\,p_A(\omega,\ t)\,\mathrm{Re}\,p_A(\omega',\ t)\rangle = \frac{\beta^2\epsilon^2\mathfrak{h}^2}{2|\lambda_-|^2|\lambda_+|^2}\,\delta_{\lambda\lambda'}\,\delta^3(\vec{k}-\vec{k}')\times\left(|C|^2+|\eta|^2-C\eta^* e^{-i\vec{k}\cdot(\vec{x}_A-\vec{x}_B)}-C^*\eta\, e^{i\vec{k}\cdot(\vec{x}_A-\vec{x}_B)}\right), \qquad (56)$$

$$\langle \mathrm{Re}\,p_A(\omega,\ t)\,\mathrm{Re}\,p_B(\omega',\ t)\rangle = \frac{\beta^2\epsilon^2\mathfrak{h}^2}{4|\lambda_-|^2|\lambda_+|^2}\,\delta_{\lambda\lambda'}\,\delta^3(\vec{k}-\vec{k}')\times\left[-2(C\eta^*+C^*\eta)+(|C|^2+|\eta|^2)\times\cos(\vec{k}\cdot(\vec{x}_A-\vec{x}_B))\right], \qquad (57)$$

where it is understood that we are involved with only a single component when writing p_A, p_B, and ϵ^2. The expression for $\langle \mathrm{Re}\,p_B(\omega,\ t)\,\mathrm{Re}\,p_B(\omega',\ t)\rangle$ is obtained by interchanging the subscripts A and B in (56).

Introducing the averages (56) and (57) into the expression for the energy, while noting that

$$2(|C|^2+|\eta|^2)=|\lambda_-|^2+|\lambda_+|^2, \qquad (58)$$

$$-2(C\eta^*+C^*\eta)=|\lambda_-|^2-|\lambda_+|^2, \qquad (59)$$

and defining

$$\omega_-^2=\omega_0^2-\mathrm{Re}\,\eta, \qquad (60)$$

$$\omega_+^2=\omega_0^2+\mathrm{Re}\,\eta, \qquad (61)$$

we find

$$\mathcal{E}=\mathcal{E}_-+\mathcal{E}_+ ,$$

where

$$\mathcal{E}_-=\sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta\epsilon^2\mathfrak{h}^2}{4}\left(\frac{\omega^2+\omega_-^2}{|\lambda_-|^2}\right)\times\left[1-\cos(\vec{k}\cdot(\vec{x}_A-\vec{x}_B))\right] \qquad (62)$$

and $\mathcal{E}_+$ is analogous to (62) with ω_- and λ_- replaced by ω_+ and λ_+ and with a plus sign appearing in the last bracket. The sum over polarizations can be carried out as in (22), polar coordinates introduced, and some symbols eliminated from (48) and (49), giving

$$\mathcal{E}_-=\int_{\omega=0}^{\infty}\omega^2 d\omega\,\frac{3\hbar\omega\Gamma}{16\pi^2}\left(\frac{\omega^2+\omega_-^2}{(-\omega^2+\omega_-^2)^2+(\Gamma\omega^3-\mathrm{Im}\,\eta)^2}\right)\times\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{\phi=0}^{2\pi}d\phi\left(1-\frac{k_j^2}{k^2}\right)\times\left[1-\cos(\vec{k}\cdot(\vec{x}_A-\vec{x}_B))\right], \qquad (63)$$

where k_j corresponds to the component under consideration with

$$k_x/k=\sin\theta\cos\phi,\quad k_y/k=\sin\theta\sin\phi,\quad k_z/k=\cos\theta, \qquad (64)$$

$$\cos(\vec{k}\cdot(\vec{x}_A-\vec{x}_B))=\cos(kR\cos\theta) . \qquad (65)$$

Carrying out the angular integrations and then evaluating the integral in ω as described in Sec. II, we find for the z component

$$\mathcal{E}_{z-}=\tfrac{1}{2}\hbar\omega_{z-}, \qquad (66)$$

and for the x and y components

$$\mathcal{E}_{x-}=\mathcal{E}_{y-}=\tfrac{1}{2}\hbar\omega_{x-}, \qquad (67)$$

with $\mathcal{E}_{x+}$, $\mathcal{E}_{y+}$, and $\mathcal{E}_{z+}$ found by replacing the minus signs in (66) and (67) with plus signs. The frequencies ω_{x-}, ω_{y-}, and ω_{z-} correspond to Eq. (60) where the value of η is chosen appropriately for the x, y, or z component.

D. Potential Between Two Polarizable Particles

The results we have obtained are identical with those of nonrelativistic quantum mechanics. The energy of the two polarizable particles is simply

$$\mathcal{E}=\tfrac{1}{2}\hbar\omega_{x-}+\tfrac{1}{2}\hbar\omega_{x+}+\tfrac{1}{2}\hbar\omega_{y-}+\tfrac{1}{2}\hbar\omega_{y+}+\tfrac{1}{2}\hbar\omega_{z-}+\tfrac{1}{2}\hbar\omega_{z+}, \qquad (68)$$

where the frequencies are just the natural frequencies of the classical system in which we omit the zero-point radiation and the radiation damping.[13]

The potential energy associated with the attraction of the polarizable particles is the difference between the system energy at the distance R and at infinite separation

$$U(R)=\mathcal{E}(R)-\mathcal{E}(\infty) . \qquad (69)$$

Now as $R\to\infty$, $\mathrm{Re}\,\eta\to 0$ and each one of the frequencies ω_- and ω_+ becomes ω_0,

$$\mathcal{E}(\infty)=6(\tfrac{1}{2}\hbar\,\omega_0) . \qquad (70)$$

To order $(e^2/\hbar c)^2$, the function $U(R)$ may be found by expanding

$$\omega_-=(\omega_0^2-\mathrm{Re}\,\eta)^{1/2}=\omega_0-\tfrac{1}{2}\mathrm{Re}\,\eta/\omega_0-\tfrac{1}{8}(\mathrm{Re}\,\eta)^2/\omega_0^3+\cdots . \qquad (71)$$

Specifically, from (43) and (44) when $\omega_0 R/c\ll 1$, we have

$$\omega_{x-}=\omega_{y-}=\omega_0-\frac{e^2}{2mR^3}-\frac{1}{8\omega_0^3}\left(\frac{e^2}{mR^3}\right)^2,$$
$$\omega_{x+}=\omega_{y+}=\omega_0+\frac{e^2}{2mR^3}-\frac{1}{8\omega_0^3}\left(\frac{e^2}{mR^3}\right)^2,$$
$$\omega_{z-}=\omega_0+\frac{e^2}{mR^3}-\frac{1}{8\omega_0^3}\left(\frac{2e^2}{mR^3}\right)^2, \quad (72)$$
$$\omega_{z+}=\omega_0-\frac{e^2}{mR^3}-\frac{1}{8\omega_0^3}\left(\frac{2e^2}{mR^3}\right)^2.$$

Combining (68)–(70) and (72), we have

$$U(R)=-\frac{3}{4}\frac{\hbar\omega_0}{R^6}\left(\frac{e^2}{m\omega_0^2}\right)^2=-\frac{3}{4}\frac{\hbar\omega_0\alpha^2}{R^6}, \quad (73)$$

where, as in (35), α is the static polarizability of the atom. Again making the identification

$$\hbar\omega_0=\mathcal{E}_1-\mathcal{E}_0 \quad (74)$$

as the energy difference between the ground state and the lowest P state of the atom, we see that our classical calculation gives the same result as the quantum calculations of London[1] and of Casimir and Polder.[8] It is of interest to remark more generally that there is an equivalence between the quantum and classical calculation for harmonic oscillators interacting with quadratic interactions. It can be shown that quantum mechanics and classical mechanics including classical electromagnetic zero-point radiation give the same values for the expectation values of every observable in the ground state of the system.

V. CLOSING SUMMARY

The unretarded London–van der Waals forces between a polarizable particle and a conducting wall, and between two polarizable particles are calculated here from classical electrodynamics including classical electromagnetic zero-point radiation with a scale $\frac{1}{2}\hbar\omega$ per normal mode imposed upon the Lorentz-invariant power spectrum. In each case, the fluctuations in the zero-point electric field cause a fluctuating polarization of the particles. In the first case, the fluctuating dipole is attracted to its image in the conducting wall. In the second, two polarizable particles are attracted together owing to the further moments induced by the fluctuating dipoles produced by the zero-point radiation. The classical analysis parallels the usual qualitative semiclassical description phrased in terms of spontaneous quantum separations of charge. The classical results obtained agree with the quantum calculations of London and of Casimir and Polder.

[1]F. London, Z. Physik 63, 245 (1926).
[2]See, for example, H. B. G. Casimir, Naturwiss. 54, 435 (1967).
[3]T. H. Boyer, Phys. Rev. 182, 1374 (1969).
[4]T. H. Boyer, Phys. Rev. 186, 1304 (1969).
[5]T. H. Boyer, Phys. Rev. D 1, 1526 (1970).
[6]T. H. Boyer, Phys. Rev. D 1, 2257 (1970).
[7]T. H. Boyer, Phys. Rev. A5, 1799 (1972). The analysis is closely connected with the pioneering semiclassical work of H. B. G. Casimir, Proc. Koninkl. Ned. Akad. Wetenschap 51, 793 (1948); J. Chim. Phys. 46, 407 (1949). See also T. H. Boyer, Phys. Rev. 174, 1631 (1968); 180, 19 (1969); 185, 2039 (1969); Ann. Phys. 56, 474 (1970).
[8]H. B. G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948).
[9]T. W. Marshall, Proc. Roy. Soc. (London) A276, 475 (1963).
[10]M. Planck, *Theory of Heat Radiation* (Dover, New York, 1959).
[11]A. Einstein and L. Hopf, Ann. Physik 33, 1105 (1910).
[12]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 102.
[13]At this point it would be possible to take advantage of work by M. J. Renne which goes from the zero-point energy expressions over to expressions for the van der Waals forces. I wish to thank Dr. Renne for sending me a prepublication copy of his quantum calculations for van der Waals forces using the Drude-Lorentz model for an atom, Physica 53, 193 (1971).