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Analysis of Magnetic Field Effects on the Correlation Energy of a Quantum Plasma: Quantum Strong-Field Limit

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The correlation energy of a degenerate electron plasma in an extremely high magnetic field is calculated taking account of the dominant local principal plasmon mode's zero-point energy, as well as the statically shielded electron-electron interaction energy involving the quantum strong-field counterpart of the Debye–Thomas–Fermi static-shielding law. Other plasmon-mode-resonance correlation-energy contributions are estimated in the quantum strong-field limit, and shown to be small for an extremely high magnetic field.

I. INTRODUCTION

One of the most important problems in the historical development of modern many-body theory and its implications concerning collective phenomena was the calculation of correlation energy of an electron-gas quantum plasma.[1–4] The term-by-term divergence of the power series representing expansion of the correlation energy in powers of the electron-electron Coulomb interaction reflected mathematical and physical difficulties in treating collective self-consistent aspects of the quantum plasma. This power series, which can also be described as an expansion in ring diagrams and also as an expansion in powers of the free-electron polarizability, suffered term-by-term divergencies at low wave number due to the long-range character of the Coulomb interaction. The resolution of this difficulty was achieved by summing all powers of the free-electron polarizability ("sum-on-ring diagrams") into a closed-form expression for the inverse dielectric function whose collective-mode-plasmon pole and self-consistent static-shielding behavior absorb the low-wave-number divergences of the power series into a convergent result for the correlation energy (e.g., zero-point plasmon energy).

Our object in this work is to analyze the effects of a high magnetic field on the correlation energy of a quantum plasma. In particular we shall carry out this calculation in detail in the quantum strong-field limit for a degenerate plasma subject to an extremely high magnetic field, so that only the lowest Landau eigenstate is populated. Earlier works[5,6] on the problem of evaluating magnetic-field effects on correlation energy have not explored this physically interesting regime of extremely high magnetic field strength. Our own earlier estimates[7] indicate that this regime should be of interest, and we have already made a preliminary report[8] of the physical and mathematical approximation scheme which permits explicit calculation of the correlation energy in an extremely high magnetic field. This calculation is carried out in detail below in a manner which takes careful account of the collective and self-consistent aspects of quantum plasma behavior, which played so important a role in the zero-field counterpart of this problem. The resulting correlation energy in the quantum strong-field limit $\hbar\omega_c > \zeta \sim E_F$ is found to depend on the parameter $r = \hbar\omega_p^2/4\zeta\omega_c$ according to

$$E_{\text{corr}}/V = -\tfrac{1}{8}Br\ln r \quad \text{for } r<1,$$

where V is the volume. We have also taken $\omega_p/\omega_c < 1$. All of these conditions $\zeta/\hbar\omega_c < 1$, $\omega_p/\omega_c < 1$, and $r<1$ can be realized in appropriate semiconductors such as indium antimonide in magnetic fields as low as $\sim 200\times10^3$ G with mobile carrier densities of $\sim 10^{17}$. (The small effective mass of indium antimonide, $m = 0.01m_e \sim 10^{-29}$ g, is important in obtaining such favorable numbers for experimental realization.) Thus, the conditions and approximations employed here are valid for indium antimonide in presently available steady magnetic fields: Moreover, our results suggest that experiments involving correlation energy and its derivatives, particularly dc magnetic susceptibility, can be carried out and should exhibit rather interesting logarithmic behavior.

II. APPROXIMATION SCHEME AND ITS SIGNIFICANCE

It is well known[9] that the correlation energy may be expressed in terms of a coupling constant integral over the spectral weight of the inverse dielectric function of the quantum plasma (integrated also over frequency and wave vector). Within the framework of the random-phase approximation ("sum-on-ring diagrams"), the correlation energy may be conveniently rewritten as[9]

$$E_{\rm corr} = \frac{\hbar V}{4\pi}\int \frac{d\vec{p}}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega\, \{\ln[1+\alpha(\vec{p}\omega)] - \alpha(\vec{p},\omega)\}, \tag{1}$$

where $\alpha(\vec{p},\omega) = 4\pi\alpha_0(\vec{p}, i\omega) =$ free-electron polarizability with $\omega \to i\omega$. The analysis of magnetic field effects on $E_{\rm corr}$ devolves upon the evaluation of the integrals involved in (1), when the correct magnetic-field-dependent free-electron polarizability is introduced for $4\pi\alpha_0(\vec{p},\omega)$: This is readily obtained from our earlier work[10] by making the identification

$$4\pi\alpha_0(\vec{p},\omega) = -(4\pi e^2/p^2)\,{\rm Im} I(\vec{p}, \omega + i\epsilon) \tag{1'}$$

in our earlier notation. [The notation of Refs. 10–12 will be maintained here: ω is the frequency; $\vec{p} = (\bar{p}, p_z)$, the wave vector; magnetic field $\vec{H}$ is in the z direction; ω_p is the plasma frequency; ω_c is the cyclotron frequency; p_F is the Fermi wave number; p_D is the Debye–Thomas–Fermi wave number in quantum strong-field limit; ρ is the density; ζ is the chemical potential; and m is the electron mass.] The resulting expression may be used in all regimes of magnetic field strength (weak, intermediate, and strong), and we have already employed it in estimating magnetic field effects on correlation energy and magnetic susceptibility.[7] In considering the quantum strong-field limit (for a degenerate plasma with all electrons in the lowest Landau eigenstate with spins antiparallel to the magnetic field), it is particularly convenient to employ another evaluation of Eq. (1′), which we have used in exploring the plasmon resonance spectrum[11] and self-consistent static screening[12] for $\hbar\omega_c > \zeta$. Expressed in terms of $\alpha(\vec{p},\omega)$, this quantum strong-field-limit result is given by

$$\alpha(\vec{p}\omega) = -\frac{m\omega_p^2}{2\hbar p^2}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2} e^{-\hbar\bar{p}^2/2m\omega_c}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n \times \ln\left|\frac{\omega^2 + [n\omega_c + \hbar p_z^2/2m - (2p_z^2\zeta/m)^{1/2}]^2}{\omega^2 + [n\omega_c + \hbar p_z^2\ 2m + (2p_z^2\zeta/m)^{1/2}]^2}\right| , \tag{2}$$

where $\omega_p^2 = 4\pi e^2\rho/m$ and $\rho = \zeta^{1/2}\omega_c(2m)^{3/2}/(2\pi\hbar)^2$ in the quantum strong-field limit,[11] which is characterized by $\hbar\omega_c > \zeta \sim E_F$.

The collective self-consistent quantum plasma phenomena embodied in the frequency-dependent structure of $\alpha(\vec{p},\omega)$, as given by Eq. (2), include[11] two local principal plasmon modes, one undamped "Bernstein"-type plasmon resonance near each higher multiple of the cyclotron frequency $n\omega_c$ ($n \geq 2$) for propagation nearly perpendicular to the magnetic field, and one undamped "quantum"-type plasmon resonance near each higher multiple of the cyclotron frequency $n\omega_c$ ($n \geq 2$) for propagation off the perpendicular direction. In fact each such undamped "quantum"-type plasmon resonance is accompanied by another damped one near $n\omega_c$, which is of somewhat lesser significance because of its natural damping and thus will be ignored here, although the comments we shall make about the undamped quantum resonances are generally applicable to the damped ones as well. It is useful to note that "sound"-type plasmon resonances[13,14] for propagation parallel to the magnetic field are not active in the case of the quantum strong-field limit, under consideration here. The static limit of $\alpha(\vec{p}, \omega = 0)$, furthermore, embodies[12] the low-wave-number quantum strong-field limit counterpart of Debye–Thomas–Fermi shielding, and the quantum strong-field counterpart of Friedel–Kohn "wiggle" shielding contributions as well.

All of the collective self-consistent phenomena indicated above have been investigated in detail in the references cited,[15] and we shall draw freely on the results of these studies to evaluate the relative importance of roles played by these phenomena in the calculation of the correlation energy. Our discussion of this is supplemented by further quantitative description provided in Appendix A. In considering plasmon contributions to the correlation energy, we have already noted that the sound-type plasmon resonances are not active in the quantum strong-field limit. The active plasmon resonances include undamped Bernstein-type plasmon resonances and undamped quantum-type plasmon resonances (also damped ones), as well as the two local principal plasmon modes. Now, the excitation amplitudes associated with undamped Bernstein-type plasmon resonances are small, vanishing like powers of wave number; moreover, Bernstein-type plasmon resonances only exist in a relatively small angular interval about the perpendicular propagation direction and thus occupy a correspondingly small region of phase space. For these reasons the contributions to correlation energy from Bernstein-type plasmon resonances will be small in comparison with the contributions of the local principal plasmon modes, whose excitation amplitudes are finite in the local limit. Similar consideration of the undamped quantum-type plasmon resonances (also damped ones) reveals that the associated excitation amplitudes are very weak, vanishing like $e^{-1/w}$, where w is the wave-number parameter, for the strongest one; despite the fact that quantum-type plasmon resonances occupy a larger region of phase space than do the Bernstein-type plasmon resonances, the extreme weakness of their excitation amplitudes dictates that the contributions to correlation energy associated with quantum-type plasmon resonances will be small in comparison with the contributions of the local principal plasmon modes. Considering finally the two local principal plasmon modes in an extreme high magnetic field limit $\omega_c > \omega_p$, one is located at $\Omega \sim \omega_p \sin\theta$ and the other is located at $\Omega' \sim \omega_c$. The ratio of their excitation amplitudes Z is given by $Z(\Omega')/Z(\Omega) \sim \omega_p/\omega_c < 1$, and it is therefore evident that the local plasmon mode at $\Omega \sim \omega_p \sin\theta$ makes the dominant contribution to the correlation energy. The other local plasmon mode at $\Omega' \sim \omega_c$ makes a smaller contribution which is larger in turn than the other plasmon-resonance contributions discussed above. Our basic approximation must therefore incorporate the $\Omega \sim \omega_p \sin\theta$ plasmon-mode contribution to the correlation energy. A quantitative description of the smaller correlation-energy contributions from the $\Omega' \sim \omega_c$ plasmon mode and from the other plasmon resonances as well is developed in Appendix A to firmly establish the smallness of these contributions; moreover, this development in Appendix A provides estimates of the associated correlation-energy corrections to our basic approximation scheme.

Focusing attention on the roles of static-shielding phenomena[12] in the calculation of correlation energy, it should be pointed out that the quantum strong-field counterpart of Friedel–Kohn "wiggle" shielding is highly anisotropic, leading to a long-range wiggle-shielding contribution only for a very narrow range of angles about the direction parallel to the magnetic field: Off the parallel direction, this Friedel–Kohn shielding contribution has an extremely fast exponential falloff in comparison with the quantum strong-field counterpart of the Debye–Thomas–Fermi shielding contribution, so that the latter is dominant. Thus our basic approximation must correctly incorporate the quantum strong-field counterpart of Debye–Thomas–Fermi shielding into the calculation of correlation energy, while Friedel–Kohn wiggle shielding may be neglected.

In summary, our basic approximation scheme for handling the exceedingly complicated structure of $\alpha(\vec{p}, \omega)$ as given by (2) must correctly describe the local principal plasmon mode at $\Omega \sim \omega_p \sin\theta$ and the quantum strong-field counterpart of Debye–Thomas–Fermi shielding. The $n = 0$ term of $\alpha(\vec{p}, \omega)$ does incorporate these features, and we therefore drop all higher n terms from $\alpha(\vec{p}, \omega)$. One might argue the reasonableness of this approximation on the basis of the prelog factor $(\hbar\bar{p}^2/2m\omega_c)^n$ being small in a low-wave-number analysis, but in fact the $n = 1$ term, thus identified, actually yields a contribution of the same order in $\bar{p}^2$ as the $n = 0$ term provides in p_z^2: These two terms correspond to the $\Omega' \sim \omega_c$ and $\Omega \sim \omega_p \sin\theta$ local principal plasmon modes for $n = 1$ and 0, respectively. Thus the distinction cannot be drawn on the basis of the order in $(\hbar\bar{p}^2/2m\omega_c)^n$ alone, but must in fact include considerations such as we have discussed above to eliminate the $n = 1$ term on the basis of lower excitation amplitude for its associated mode (in comparison with that of the $n = 0$ term). Furthermore, the argument for dropping higher $n \geq 2$ terms from $\alpha(\vec{p}, \omega)$ on the basis of the smallness of $(\hbar\bar{p}^2/2m\omega_c)^n$ really should be augmented by further consideration of the behavior of the term as a whole in the neighborhood of the branch point of the log factor involved, since violent perturbations take place in this region. In fact, it is just these violent perturbations which are responsible for inducing the additional plasmon resonances into the spectrum, with the nth log singularity giving rise to the additional plasmon resonances near $n\omega_c$, and our discussion above serves to systematically eliminate these terms on the basis of the smallness of their associated plasmon contributions to the correlation energy. Moreover, we have also verified that their static-shielding contributions, which are predominantly of a Friedel–Kohn wiggle type, are also negligible in comparison with the quantum strong-field counterpart of Debye–Thomas–Fermi shielding terms incorporated in the structure of the $n = 0$ term. In accordance with these arguments we drop from $\alpha(\vec{p}, \omega)$ all terms $n \geq 1$, leaving only the $n = 0$ term:

$$\alpha(\vec{p}, \omega) = -\frac{m\omega_p^2}{2\hbar p^2}\left(\frac{m}{2p_z^2\zeta}\right)^{1/2} e^{-\hbar\bar{p}^2/2m\omega_c} \times \ln\left|\frac{\omega^2 + [\hbar p_z^2/2m - (2p_z^2\zeta/m)^{1/2}]^2}{\omega^2 + [\hbar p_z^2/2m + (2p_z^2\zeta/m)^{1/2}]^2}\right| . \quad (3)$$

This may be further approximated in the low-wave-number regime by expanding to the lowest order in

$$\frac{\pm(\hbar p_z^2/m)(2p_z^2\zeta/m)^{1/2}}{\omega^2 + 2p_z^2\zeta/m} ,$$

with the result [the Debye–Thomas–Fermi wave number p_D of the quantum strong-field limit[12] is given by $p_D^2 = m\omega_p^2/2\zeta$; we also introduce the notation $p_H^2 = 2m\omega_c/\hbar = p_F^2(\hbar\omega_c/\zeta)$]

$$\alpha(\vec{p}, \omega) = \frac{p_D^2}{p^2} e^{-\bar{p}^2/p_H^2} \frac{1}{1 + \omega^2/(2p_z^2\zeta/m)} . \quad (4)$$

One can readily verify that the $p \to 0$ limit of this incorporates the $\Omega = \omega_p \sin\theta$ local principal plasmon structure,[11] since

$$\alpha(\vec{p} = 0, \omega) = \frac{\omega_p^2}{\omega^2}\frac{p_z^2}{p^2} = \frac{\omega_p^2 \sin^2\theta}{\omega^2} .$$

Moreover, the static limit is given by

$$\alpha(\vec{p}, \omega = 0) = \frac{p_D^2}{p^2} e^{-\bar{p}^2/p_H^2} \cong \frac{p_D^2}{p^2}\left(1 - \frac{\bar{p}^2}{p_H^2}\right) \cong \frac{p_D^2}{p^2} - \frac{\bar{p}^2}{p^2}\frac{p_D^2}{p_H^2} ,$$

which should be compared with the corresponding structure of $\alpha(\vec{p}, \omega = 0)$ known to be responsible for the full quantum analog of the Debye–Thomas–

Fermi static-shielding law in the quantum strong-field limit within the framework of the random-phase approximation. The latter is given by[12]

$$\alpha(\vec{p}, \omega=0) \rightarrow \frac{p_D{}^2}{p^2} - \frac{\bar{p}^2}{p^2}\frac{\omega_p^2}{\omega_c^2}\left(\frac{\hbar\omega_c}{4\zeta} - 1\right) + \frac{p_z{}^2}{p^2}\left(\frac{1}{12}\frac{p_D{}^2}{p_F{}^2}\right) ,$$

which agrees well with our present static limit in extremely high magnetic fields. Thus Eq. (4) does indeed incorporate the requisite features of the quantum strong-field counterpart of Debye–Thomas–Fermi shielding, as well as the $\Omega \sim \omega_p \sin\theta$ dominant local principal plasmon mode. We have already made a preliminary report[8] of this approximation scheme and its physical significance.

III. CALCULATION OF CORRELATION ENERGY IN EXTREME HIGH MAGNETIC FIELD LIMIT

The correlation energy [Eq. (1)] may be rewritten using an integration by parts,

$$E_{\text{corr}} = \frac{\hbar V}{4\pi}\int \frac{d\vec{p}}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega\, \omega\, \frac{\alpha(\vec{p}, \omega)\, d\alpha(\vec{p}, \omega)/d\omega}{1+\alpha(\vec{p}, \omega)} . \tag{5}$$

The collective self-consistent aspects of quantum plasma behavior are embodied in the zeros of the denominator $[1+\alpha(\vec{p}, \omega)]^{-1}$ of the integrand, so special care must be taken to treat this denominator in closed form in the integration, avoiding any reference to expansion of it in powers of $\alpha(\vec{p}, \omega)$ (which would lead to a divergent series). It is for this reason that we have developed in Sec. II a relatively simple but physically significant form for $\alpha(\vec{p}, \omega)$, which is analytically tractable in a closed-form evaluation of the correlation-energy integral involving the denominator $[1+\alpha(\vec{p}, \omega)]^{-1}$.

Introducing a change of the frequency variable

$$u = \omega/(2p_z{}^2\zeta/m)^{1/2} , \tag{6}$$

we have

$$\alpha(\vec{p}, \omega) = (p_D{}^2/p^2)\, e^{-\bar{p}^2/p_H{}^2}\, \overline{X}(u) ,$$

where $\overline{X}(u) = (1+u^2)^{-1}$. Then the correlation-energy integral takes the form

$$E_{\text{corr}} = \frac{\hbar V}{4\pi}\frac{2}{(2\pi)^2}\left(\frac{2\zeta}{m}\right)^{1/2} p_D{}^4 \int_0^{\infty} d\bar{p}\,\bar{p}\, e^{-2\bar{p}^2/p_H{}^2} \times \int_{-\infty}^{\infty} d\nu\, \nu \overline{X}(\nu)\frac{d\overline{X}(\nu)}{d\nu}\int_0^{\infty} dp_z p_z \times \frac{1}{(\bar{p}^2+p_z{}^2)[\bar{p}^2+p_z{}^2+p_D{}^2 e^{-\bar{p}^2/p_H{}^2}\overline{X}(\nu)]} . \tag{7}$$

The p_z integral is elementary, yielding

$$\int_0^{\infty} dp_z \cdots = \frac{1}{2p_D{}^2 e^{-\bar{p}^2/p_H{}^2}\overline{X}(\nu)} \times \ln\left(1+\frac{p_D{}^2}{\bar{p}^2}\, e^{-\bar{p}^2/p_H{}^2}\overline{X}(\nu)\right) . \tag{8}$$

The ensuing ν integral can be carried out in closed form [Ref. 16(a)] with the result

$$\int_{-\infty}^{\infty} d\nu \cdots = -\frac{\pi e^{+\bar{p}^2/p_H{}^2}}{p_D{}^2}\left\{\ln\left[1+\left(1+\frac{p_D{}^2}{\bar{p}^2}\, e^{-\bar{p}^2/p_H{}^2}\right)^{1/2}\right] + \left[1+\left(1+\frac{p_D{}^2}{\bar{p}^2}\, e^{-\bar{p}^2/p_H{}^2}\right)^{1/2}\right]^{-1} - (\ln 2+\tfrac{1}{2})\right\} , \tag{9}$$

and redefining the final $\bar{p}$-integration variable according to $z = \bar{p}^2/p_H{}^2$, we have the correlation energy given by

$$E_{\text{corr}}/V = B[F(r) - (\ln 2 + \tfrac{1}{2})], \tag{10}$$

where

$$r = \frac{p_D{}^2}{p_H{}^2} = \frac{\hbar\omega_p}{4\zeta}\frac{\omega_p}{\omega_c} = \frac{p_D{}^2}{p_F{}^2}\frac{\zeta}{\hbar\omega_c} , \tag{11}$$

$$B = \frac{-\hbar p_H{}^2 p_D{}^2}{4(2\pi)^2}\left(\frac{2\zeta}{m}\right)^{1/2} = -\frac{2e^2}{(2\pi)^3}\left(\frac{m\omega_c}{\hbar}\right)^2 , \tag{12}$$

and the final z integral is given by

$$F(r) = \int_0^{\infty} dz\, e^{-z}\left\{\ln\left[1+\left(1+\frac{re^{-z}}{z}\right)^{1/2}\right] + [1+(1+re^{-z}/z)^{1/2}]^{-1}\right\} . \tag{13}$$

It should be noted that a series expansion in powers of the electron-electron Coulomb interaction (or equivalently, in powers of the polarizability) would correspond here to the expansion of the correlation-energy integrand in powers of $(re^{-z}/z)^n$, which would induce spurious term-by-term divergencies at the lower limit of the z integral ($p^2 \rightarrow 0$). In recognition of this fact we seek an alternative evaluation procedure, avoiding expansion in powers of r^n. Of course, one may carry out the integration of $F(r)$ numerically, and such results are presented in Table I and Fig. 1. However, it is desirable to treat the interesting regime $r \ll 1$ (high density) analytically to gain insight into the functional dependence of E_{corr} on r. The over-all convergence of $F(r)$ at the upper limit is guaranteed by the first e^{-z} factor in the integrand, and this in turn dictates that the predominant contributions to $F(r)$ come from the region $0 \leq z \leq 1$. Now, the smallness of r renders $F(r)$ relatively insensitive to the detailed behavior of the second exponential e^{-z} occurring in the combination $(1+re^{-z}/z)^{1/2}$, and since the pre-

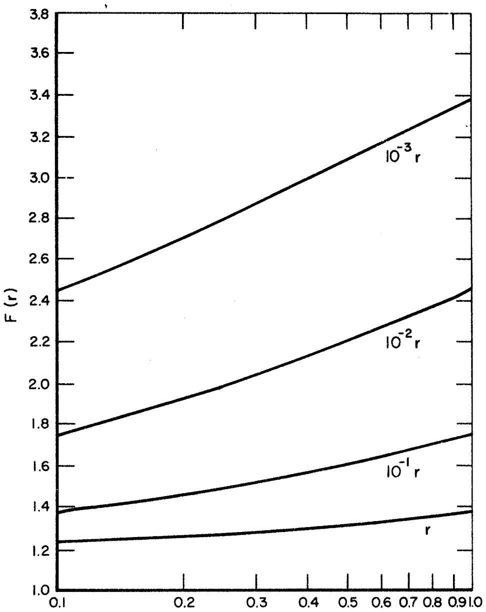

FIG. 1. Plot of the numerical-integration results for $F(r)$.

dominantly important region is $0 \leqslant z \leqslant 1$, we replace this second exponential by unity, setting

$$(1 + re^{-z}/z)^{1/2} \to (1 + r/z)^{1/2} \quad . \tag{14}$$

This replacement may be regarded as the leading term of an expansion of the integrand involving $(1 + re^{-y}/z)^{1/2}$ in powers of y, subsequently setting $y = z$. The leading term is evaluated in detail here, and the next term is evaluated in Appendix B. In accordance with the qualitative argument above, the leading term is found to dominate the behavior of $F(r)$ for $r \ll 1$.

The evaluation of the leading term

$$F(r) = \int_0^\infty dz\, e^{-z} \left\{ \ln\left[1 + \left(1 + \frac{r}{z}\right)^{1/2}\right] + \frac{1}{1 + (1 + r/z)^{1/2}} \right\} \tag{15}$$

can be carried out in closed form. We have

$$\begin{aligned}
\int_0^\infty dz\, e^{-z} \ln\left[1 + \left(1 + \frac{r}{z}\right)^{1/2}\right]
&= \int_0^\infty dz\, e^{-z} \ln\left(\frac{r}{z}\right)^{1/2} + \int_0^\infty dz\, e^{-z} \ln\left(\frac{z^{1/2} + (z+r)^{1/2}}{r^{1/2}}\right) \\
&= \tfrac{1}{2} \ln r \int_0^\infty dz\, e^{-z} - \tfrac{1}{2} \int_0^\infty dz\, e^{-z} \ln z + \int_0^\infty dz\, e^{-z} \ln\left(\frac{z^{1/2} + (z+r)^{1/2}}{r^{1/2}}\right) \\
&= \tfrac{1}{2} \ln r + \tfrac{1}{2} c + \tfrac{1}{2} e^{r/2} K_0(r/2) \quad ,
\end{aligned} \tag{16}$$

where we have employed Ref. 16(b), and c is the Euler–Mascheroni constant $c = 0.5772\ldots$. Furthermore, we have

$$\begin{aligned}
\int_0^\infty dz\, e^{-z} \frac{1}{1 + (1 + r/z)^{1/2}}
&= \frac{1}{r} \int_0^\infty dz\, e^{-z} z \left[(1 + r/z)^{1/2} - 1\right] \\
&= \frac{1}{r} \int_0^\infty dz\, e^{-z} [z(z+r)]^{1/2} - \frac{1}{r} \int_0^\infty dz\, e^{-z} z \\
&= \frac{1}{2} e^{r/2} K_1\left(\frac{r}{2}\right) - \frac{1}{r} \quad ,
\end{aligned} \tag{17}$$

where we again employed Ref. 16(b) (p. 138, No. 13). Thus the leading term yields

$$\frac{E_{\text{corr}}}{V} = B \left\{ \tfrac{1}{2} \ln r + \tfrac{1}{2} c + \tfrac{1}{2} e^{r/2} \left[K_0\left(\frac{r}{2}\right) + K_1\left(\frac{r}{2}\right) \right] - \frac{1}{r} - (\ln 2 + \tfrac{1}{2}) \right\} \quad . \tag{18}$$

In keeping with the commitment of $r \ll 1$, here, we introduce appropriate expansions for $K_0(r/2)$ and $K_1(r/2)$ [16(c)] and finally obtain

$$E_{\text{corr}}/V = B\left[-\tfrac{1}{8} r \ln r + O(r) + O(r^2) \ln r\right] \quad . \tag{19}$$

The terms of order $O(r)$ and $O(r^2)\ln r$ are negligible compared to the principal term $-\frac{1}{8} r \ln r$ for $r \ll 1$. The first correction term to this procedure is evaluated in Appendix B and is also negligible for $r \ll 1$. The fact that this result is not analytic in the electron-electron Coulomb-interaction coupling strength is not surprising in view of corresponding results in the zero-field case.

IV. CONCLUSIONS

The quantum strong-field-limit analysis of electron correlation energy undertaken here for ex-

TABLE I. Numerical-integration results for $F(r)$.

r	$F(r)$	r	$F(r)$	r	$F(r)$
0	1.1931	9	1.730	90	2.409
0.2	1.2635	10	1.755	100	2.446
0.4	1.3005	12	1.798	150	2.597
0.6	1.3297	14	1.840	200	2.708
0.8	1.3543	16	1.871	250	2.796
1.0	1.3759	18	1.902	300	2.869
1.5	1.4211	20	1.931	350	2.933
2.0	1.4583	25	1.994	400	2.988
2.5	1.4980	30	2.047	500	3.0815
3.0	1.5185	35	2.094	600	3.159
3.5	1.5438	40	2.136	700	3.225
4.0	1.5669	45	2.173	800	3.283
5.0	1.608	50	2.207	900	3.335
6.0	1.644	60	2.268	1000	3.382
7.0	1.675	70	2.320	1200	3.462
8.0	1.704	80	2.367		

tremely high magnetic-field strength (in all senses) yields the result

$$E_{\text{corr}}/V = -\tfrac{1}{8} Br \ln r \tag{20}$$

for $r < 1$, where

$$r = \frac{p_D{}^2}{p_H{}^2} = \frac{\hbar\omega_p}{4\zeta}\,\frac{\omega_p}{\omega_c} = \frac{p_D{}^2}{p_F{}^2}\,\frac{\zeta}{\hbar\omega_c}\,, \tag{21}$$

$$B = -\frac{\hbar p_H{}^2 p_D{}^2}{4(2\pi)^2}\left(\frac{2\zeta}{m}\right)^{1/2} = -\frac{2e^2}{(2\pi)^3}\left(\frac{m\omega_c}{\hbar}\right)^2 . \tag{22}$$

The nonanalyticity of this result as a function of the strength of the electron-electron Coulomb interaction can be attributed to essentially the same considerations which lead to such nonanalyticity in the zero-field counterpart of this calculation.

The approximation scheme employed here takes good account of zero-point oscillator energy associated with the dominant local principal plasmon mode $\Omega \sim \omega_p \sin\theta$ as well as statically shielded electron-electron interaction energy involving the quantum strong-field counterpart of the Debye–Thomas–Fermi static-shielding law. The correlation-energy contributions associated with the other local plasmon mode $\Omega' \sim \omega_c$ and with nonlocal Bernstein-type and nonlocal quantum-type plasmon resonances are estimated and shown to be small in Appendix A, and are neglected in our basic approximation scheme. Even within this simplified framework the high-field correlation-energy integral is formidable, and we have employed an evaluation procedure appropriate to small r obtaining $E_{\text{corr}} \sim r \ln r$: Corrections to this procedure are evaluated in Appendix B and are shown to be small for $r < 1$.

It is of interest to compare the high-field parameter r with the corresponding zero-field parameter $r_s = r_0/a_0$, where $r_0 = (\tfrac{3}{4}\pi\rho)^{1/3} \sim$ interelectron spacing, and $a_0 = \hbar^2/me^2 \sim$ Bohr radius. Introducing the *zero-field* ($H = 0$) expressions for the (degenerate) Debye–Thomas–Fermi wave number $p_D(H=0)$ and for the Fermi wave number $p_F(H=0)$, we have[1,3]

$$r_s = \frac{\pi}{4}\left(\frac{9\pi}{4}\right)^{1/3} \frac{[p_D(H=0)]^2}{[p_F(H=0)]^2} \qquad (r_s < 1)\,,$$

which may be compared with our high-field parameter r as given by Eq. (21):

$$r = \frac{p_D{}^2}{p_F{}^2}\,\frac{\zeta}{\hbar\omega_c} \qquad (r < 1)\,.$$

As indicated in the Introduction the result for correlation energy in an extremely high magnetic field obtained here should be valid for semiconductors such as indium antimonide in magnetic fields as low as $\sim 200\times 10^3$ G with mobile carrier densities of $\sim 10^{17}$. Although correlation energy itself is not directly observable, it would be of interest to observe the corresponding correlation contribution to the dc magnetic susceptibility, which is readily obtained from our result. An evaluation of the dc magnetic susceptibility in extremely high magnetic field, including exchange contributions as well as correlation contributions, is in preparation.

APPENDIX A: ESTIMATES OF CORRELATION-ENERGY CONTRIBUTIONS FROM PLASMON MODE $\Omega' \sim \omega_c$ AND FROM OTHER PLASMON RESONANCES IN THE QUANTUM STRONG-FIELD LIMIT

We shall assess the relative importance of plasmon contributions to correlation energy in two ways. The first measure of the relative importance of a plasmon mode resonance $\Omega(\vec{p})$ to be considered is given by the relative excitation amplitude[10] $Z(\Omega(\vec{p}))$ which determines the plasmon's contribution to the logarithm of the grand partition function W according to the relation[7]

$$\frac{W - W(0)}{V} \cong \frac{\hbar\beta}{2}\int \frac{d\vec{p}}{(2\pi)^3}\int_0^1 \frac{dK}{K}\, Z(\Omega(\vec{p}))\,. \tag{A1}$$

[Note that K measures Coulomb coupling strength here with $K = 1$ for full coupling and $K = 0$ for zero coupling; $W(0)$ is the zero-coupling limit of W.] The correlation-energy contribution associated with a given plasmon mode resonance $\Omega(\vec{p})$ may be obtained from this by differentiation with respect to β, and thus $Z(\Omega(\vec{p}))$ clearly measures the relative importance of $\Omega(\vec{p})$ in contributing to correlation energy. The qualitative discussion in Sec. II of this paper is based on this fact, and we shall supply further details on $Z(\Omega(\vec{p}))$ here drawing freely on the results of our earlier analysis of the plasmon spectrum in the quantum strong-field limit[15] (with $\omega_c > \omega_p$):

(a) for the local principal plasmon mode $\Omega \sim \omega_p \sin\theta$,

$$Z(\Omega) \cong \tfrac{1}{2}\,\omega_p \sin\theta\,; \tag{A2a}$$

(b) for the local principal plasmon mode $\Omega' \sim \omega_c$,

$$Z(\Omega') \cong \frac{\omega_p^2}{2\omega_c}\cos^2\theta\,; \tag{A2b}$$

(c) for the quantum-type plasmon resonance near $n\omega_c$ ($n \geq 2$),

$$Z(\Omega_{(n\omega_c)}) \cong \frac{(2\hbar p^2 p_z{}^2 \zeta/m^2\omega_p^2)(n\omega_c + \hbar p_z{}^2/2m)\, n!\,(2m\omega_c/\hbar \bar{p}^2)^n e^{\hbar\bar{p}^2/2m\omega_c}}{\Omega_{(n\omega_c)} \sinh^2\{[2C_n(\vec{p})]^{-1}[1 - \omega_p^2 \sin^2\theta/(n\omega_c)^2 - \omega_p^2\cos^2\theta/(n\omega_c)^2 - \omega_c^2]\}}\;; \tag{A2c}$$

(d) for the Bernstein-type plasmon resonance near $n\omega_c$ $(n \geq 2)$,

$$Z(\Omega_{(n)}) \cong \left(\frac{(n^2-1)\omega_c^2}{n^2\omega_c^2 - (\omega_c^2+\omega_p^2)}\right)^2 \frac{\bar{p}^{2n-2}}{2n\omega_c}\,\omega_p^2\lambda_n\,. \quad \text{(A2d)}$$

Here,

$$C_n(\vec{p}) = \frac{m\omega_p^2}{2\hbar p^2}\left(\frac{m}{2p_z{}^2\zeta}\right)^{1/2}\frac{1}{n!}\left(\frac{\hbar\bar{p}^2}{2m\omega_c}\right)^n e^{-\hbar\bar{p}^2/2m\omega_c}$$

$$= \frac{m\omega_p^2}{2\hbar p^2}\left(\frac{m}{2p_z{}^2\zeta}\right)^{1/2}\frac{1}{n!}\left(\frac{\bar{p}}{p_H}\right)^{2n} e^{-\bar{p}^2/p_H{}^2} \quad \text{(A3)}$$

and

$$\lambda_n = \frac{4n\sqrt{2}}{\pi n!}\,\frac{e^2 m^{3/2}\zeta^{1/2}}{\hbar^3}\,\frac{\omega_c^2}{\omega_p^2}\,\frac{1}{(p_H)^{2n}}\,e^{-\bar{p}^2/p_H{}^2}. \quad \text{(A4)}$$

The low-wave-number behavior of these relative excitation amplitudes is roughly given by (dropping numerics, etc.)

$$Z(\Omega) \sim \omega_p\,; \quad \text{(A2a}')$$

$$Z(\Omega') \sim \omega_p^2/\omega_c \sim (\omega_p/\omega_c)\,Z(\Omega) < Z(\Omega)\,; \quad \text{(A2b}')$$

$$Z(\Omega_{(n\omega_c)}) \sim \frac{\hbar p^2 p_z{}^2\zeta}{m^2\omega_p^2}\left(\frac{p_H}{\bar{p}}\right)^{2n} e^{-1/C_n(\vec{p})}\,,$$

$$C_n(\vec{p}) \sim \frac{m^{3/2}\omega_p^2}{\hbar\zeta^{1/2}p_H{}^{2n}}\,\frac{\bar{p}^{2n}}{p^2|p_z|}\,; \quad \text{(A2c}')$$

$$Z(\Omega_{(n)}) \sim \frac{e^2 m^{3/2}\zeta^{1/2}\omega_c}{\hbar^3 p_H{}^{2n}}\,\bar{p}^{2n-2}\,. \quad \text{(A2d}')$$

These evaluations of the excitation amplitudes provide detailed support of the qualitative discussion undertaken in Sec. II. Although this discussion is well founded, it has the shortcoming that it considers the relative importance of plasmon-energy contributions on the basis of excitation amplitudes alone, without additionally considering the implications of plasmon damping. Wave-vector (momentum) cutoffs implied by plasmon-mode-resonance natural damping are quantitatively important in the final wave-vector integration for correlation energy, so we shall turn now to a better quantitative description of the plasmon contributions to correlation energy which incorporates such plasmon damping wave-vector cutoffs.

We can obtain quantitatively accurate estimates of plasmon-mode-resonance contributions to correlation energy in the degenerate (zero temperature) limit, employing (A1) in the form

$$\frac{E_{\text{corr}}^{\text{plasmon}}}{V} = \frac{\hbar}{2}\int\frac{d\vec{p}}{(2\pi)^3}\int_0^1\frac{dK}{K}\,Z(\Omega(\vec{p};\,K)) \quad \text{(A5)}$$

and using a well-established procedure[17] for evaluating the coupling-strength (K) integral. The plasmon-dispersion relation for the mode resonance $\Omega(\vec{p};\,K)$ at coupling strength $K(0 \leq K \leq 1)$ has the form

$$1 + 4\pi K\,\alpha_0(\vec{p},\,\Omega(\vec{p};\,K)) = 0, \quad \text{(A6)}$$

and the corresponding excitation amplitude $Z(\Omega(\vec{p};\,K))$ is given by

$$Z(\Omega(\vec{p};\,K)) = \left(4\pi K\,\frac{d\alpha_0(\vec{p},\,\Omega(\vec{p};\,K))}{d\Omega(\vec{p};\,K)}\right)^{-1}. \quad \text{(A7)}$$

Differentiating (A6) with respect to K, one has

$$4\pi\alpha_0(\vec{p},\,\Omega(\vec{p};\,K)) + 4\pi K\,\frac{d\alpha_0(\vec{p},\,\Omega(\vec{p};\,K))}{d\Omega(\vec{p};\,K)} \times \frac{d\Omega(\vec{p};\,K)}{dK} = 0\,,$$

and since the first term is just $-1/K$ by (A6), we obtain

$$Z(\Omega(\vec{p};\,K)) = K\,\frac{d\Omega(\vec{p};\,K)}{dK}\,.$$

The K integrand is thus a total derivative, and this yields the well-known result

$$\frac{E_{\text{corr}}^{\text{plasmon}}}{V} = \frac{\hbar}{2}\int\frac{d\vec{p}}{(2\pi)^3}\,[\Omega(\vec{p}) - \Omega(\vec{p};\,K=0)], \quad \text{(A8)}$$

where $\Omega(\vec{p}) = \Omega(\vec{p};\,K=1)$ is the root of the plasmon-dispersion relation at full-coupling strength, and $\Omega(\vec{p};\,K=0)$ is the zero-coupling limit of this root which is located at the nearest frequency singularity of the free-electron polarizability approached by the root $\Omega(\vec{p};\,K)$ as $K \to 0$. What we have here [Eq. (A8)] is clearly the zero-point plasmon energy less its zero-coupling limit summed over well-defined plasmon states. The wave-vector cutoffs $\vec{p}_c$ arise in consequence of natural damping which blurs the definition of the plasmon state, rendering it meaningless as an elementary excitation. Since the frequency singularities of the free-electron polarizability signal the onset of natural damping in the degenerate case, as well as determining the zero-coupling limit of the plasmon root, we may determine the wave-vector cutoff $\vec{p}_c$ from the condition

$$\Omega(\vec{p}_c) = \Omega(\vec{p}_c;\,K=0)\,. \quad \text{(A9)}$$

We can now estimate the plasmon-mode-resonance correlation-energy contributions using Eqs. (A8) and (A9).[15] We start with the dominant local principal plasmon mode $\Omega \sim \omega_p \sin\theta$ as a base reference, although the correlation-energy contribution of this mode has been included in the approximation scheme employed in the main calculation in the body of this paper. (All results discussed here are for the low-wave-number regime.[15])

a. Dominant local principal plasmon mode Ω. For this mode we have

$$\Omega(\vec{p}) \cong \omega_p\,|\sin\theta|\,, \quad \text{(A10)}$$

and inspecting the polarizability [Eq. (4) with $\omega \to i\omega$] for its frequency singularity, we find

$$\Omega(\vec{p};\ K=0)=p_z\,(2\zeta/m)^{1/2}=p\,|\sin\theta|\,(2\zeta/m)^{1/2}. \tag{A11}$$

The determination of the cutoff $\vec{p}_c$ using (A9) then yields

$$p_c=(m\omega_p^2/2\zeta)^{1/2}=p_D\ . \tag{A12}$$

The correlation-energy contribution of this mode is therefore given by

$$\frac{E_\Omega}{V}=\frac{\hbar}{2}\int_0^{p_D}\int_{-\pi/2}^{\pi/2}\frac{dp\,p^2\,d\theta\cos\theta}{(2\pi)^2}\times\left[\omega_p|\sin\theta|-p\left(\frac{2\zeta}{m}\right)^{1/2}|\sin\theta|\right]$$

$$=\frac{\hbar\omega_p}{24(2\pi)^2}\,p_D{}^3=-\tfrac{1}{6}Br \tag{A13}$$

or (dropping numerics)

$$\frac{E_\Omega}{V}\cong B\times O(r)\ . \tag{A13$'$}$$

b. Local principal plasmon mode Ω'. For this mode we have

$$\Omega'(\vec{p})\cong(\omega_c^2+\omega_p^2\cos^2\theta)^{1/2}\cong\omega_c+\frac{1}{2}\frac{\omega_p^2}{\omega_c}\cos^2\theta \tag{A14}$$

and inspecting the frequency singularity (branch point) of the $n=1$ term of the polarizability [Eq. (2) with $\omega\to i\omega$], we find

$$\Omega'(\vec{p};\ K=0)=\omega_c+p\,|\sin\theta|\,(2\zeta/m)^{1/2}\ . \tag{A15}$$

The determination of the cutoff $\vec{p}_c$ using (A9) then yields

$$p_c=\frac{1}{2}\frac{\omega_p^2}{\omega_c}\left(\frac{m}{2\zeta}\right)^{1/2}\frac{\cos^2\theta}{\sin\theta}\cong O\left(\frac{\omega_p}{\omega_c}\,p_D\right). \tag{A16}$$

Evaluating the correlation-energy contribution of this mode using the integral [Eq. (A8)] (and dropping numerics), we obtain

$$\frac{E_{\Omega'}}{V}\cong O\left(\hbar\,\frac{\omega_p^2}{\omega_c}\,p_c{}^3\right)=O\left(\hbar\,\frac{\omega_p^5}{\omega_c^4}\,p_D{}^3\right) \tag{A17}$$

or (recall $\omega_c>\omega_p$ in our extreme high-field case)

$$\frac{E_{\Omega'}}{V}\cong\left(\frac{\omega_p}{\omega_c}\right)^4 B\,O(r)<\frac{E_\Omega}{V}\cong B\,O(r)\ . \tag{A17$'$}$$

c. Quantum-type plasmon resonances $\Omega_{(n\omega_c)}$. For these resonances we have

$$\Omega_{(n\omega_c)}(\vec{p})^2=\left(n\omega_c+\frac{\hbar p_z{}^2}{2m}\right)^2+\frac{2p_z{}^2\zeta}{m}+\left(\frac{8p_z{}^2\zeta}{m}\right)^{1/2}\left(n\omega_c+\frac{\hbar p_z{}^2}{2m}\right)\coth(L)\ ,$$

where

$$L=\frac{1}{2C_n(\vec{p})}\left(1-\frac{\omega_p^2\sin^2\theta}{(n\omega_c)^2}-\frac{\omega_p^2\cos^2\theta}{(n\omega_c)^2-\omega_c^2}\right),$$

with $C_n(\vec{p})$ given by (A3). Clearly $L\to\infty$ as $p\to 0$, and expanding $\coth(L)$ asymptotically we obtain

$$\Omega_{(n\omega_c)}(\vec{p})\cong\Omega_{\mathrm{BP}n}^{+}+(8p_z{}^2\zeta/m)^{1/2}\,e^{-2L}\ . \tag{A18}$$

Here, $\Omega_{\mathrm{BP}n}^{+}$ is the nearest frequency singularity (branch point) of the polarizability [involving the nth logarithm of Eq. (2) with $\omega\to i\omega$], so that we clearly have $\Omega_{(n\omega_c)}(\vec{p};\ K=0)=\Omega_{\mathrm{BP}n}^{+}$. The determination of the cutoff $\vec{p}_c$ using (A9) then yields an equation of the form [only wave-number dependence is exhibited explicitly here; $D(\theta)>0$, $F(\theta)>0$]

$$p_c\exp\left(\frac{-D(\theta)}{p_c{}^{2n-3}}\,e^{F(\theta)p_c{}^2}\right)\cong 0\ .$$

The low-wave-number solution of this is

$$p_c\cong 0\ , \tag{A19}$$

which implies vanishingly small correlation-energy contributions from the quantum-type plasmon resonances. This is to be expected in view of their exceedingly weak excitation amplitudes (as discussed above), but the result $p_c\cong 0$ is somewhat artificial in that this zero result obtained under low-wave-number approximations would in fact be nonzero if the low-wave-number restriction were to be lifted. An alternative choice $p_c=p_H$ could be taken (respresenting the upper limit of validity of a low-wave-number analysis of quantum resonances), but the resulting contribution to correlation energy of the form

$$\frac{E_{\Omega(n\omega_c)}}{V}=\frac{\hbar}{2}\int_0^{p_H}\int_{-\pi/2}^{\pi/2}\frac{dp\,p^2\,d\theta\cos\theta}{(2\pi)^2}\times\left[\left(\frac{8p_z{}^2\zeta}{m}\right)^{1/2}\exp\left(-\frac{D(\theta)}{p^{2n-3}}\,e^{F(\theta)p^2}\right)\right] \tag{A20}$$

would be very small indeed because of the strong vanishing of the exponential factor $\exp[-D(\theta)/p^{2n-3}]$ for low wave number.

d. Bernstein-type plasmon resonances Ω_n. For these resonances we have

$$\Omega_{(n)}(\bar{p})^2=(n\omega_c)^2+\bar{p}^{2n-2}\,\omega_p^2\lambda_n\left(1-\frac{\omega_p^2}{(n\omega_c)^2-\omega_c^2}\right)^{-1},$$

where λ_n is given by (A4) (note that λ_n is approximately independent of $\bar{p}$ for low $\bar{p}$). Alternatively, we have

$$\Omega_n(\bar{p})\cong n\omega_c+\bar{p}^{2n-2}\frac{\omega_p^2\lambda_n}{2n\omega_c}\left(1-\frac{\omega_p^2}{(n\omega_c)^2-\omega_c^2}\right)^{-1}. \tag{A21}$$

Mathematically, Bernstein-type plasmon resonances have their origin in the fact that the associated frequency singularity of the polarizability [involving the nth logarithm of Eq. (2) with $\omega\to i\omega$] changes from a branch cut near $n\omega_c$ of width $\Delta\omega=2(2p_z{}^2\zeta/m)^{1/2}$ into a simple pole at $n\omega_c$, as propagation approaches the direction perpendicular to the magnetic field ($p_z\to 0$). It is thus clear that

$$\Omega_{(n)}(\bar{p};\ K=0)=n\omega_c\ . \quad \text{(A22)}$$

The low-wave-number determination of the cutoff $\vec{p}_c$ using (A9) then yields

$$p_c \cong 0\ , \quad \text{(A23)}$$

which implies vanishingly small correlation-energy contributions from the Bernstein-type plasmon resonances. Again, this very small result is correct in substance, but somewhat artificial in that the result would not be strictly zero if the low-wave-number restriction in our formulas were to be lifted. It is easy to obtain an upper bound on the correlation-energy contributions of Bernstein modes by taking alternative cutoff choices for $\bar{p}$ and p_z as follows: (i) $\bar{p}_c = p_H$ represents the upper limit of validity of a low-wave-number analysis of Bernstein modes in the quantum strong-field limit. (ii) p_{zc} may be taken from the condition that the width of the logarithmic branch cut $\Delta\omega = 2(2p_z^{\,2}\zeta/m)^{1/2}$ shall be small compared to the shift of the Bernstein mode away from the simple pole at $n\omega_c$, which is the limiting form of the singularity of the polarizability as the branch cut shrinks to zero width for propagation approaching the perpendicular direction. This is to say that the Bernstein mode feels the singularity generating it as a point pole representing a branch cut of essentially zero width. Thus we have $p_z < p_{zc}$, with p_{zc} determined by the condition

$$2\left(\frac{2p_{zc}^{\,2}\zeta}{m}\right)^{1/2} = \bar{p}^{2n-2}\,\frac{\omega_p^2\lambda_n}{2n\omega_c}\left(1-\frac{\omega_p^2}{(n\omega_c)^2-\omega_c^2}\right)^{-1},$$

and for an upper bound we put $\bar{p}\rightarrow p_H$ on the right-hand side obtaining (recall $\omega_p < \omega_c$ in the extreme high-field limit under consideration, and note that we drop numerics here)

$$p_{zc} \cong e^2m^2\omega_c/\hbar^3 p_H^{\,2}\ .$$

In accordance with these cutoffs the upper bound on the correlation-energy contributions of Bernstein modes is given by

$$\frac{E_{\Omega(n)}}{V} \cong \hbar\int_0^{p_{zc}}\int_0^{p_H}\frac{dp_z\,d\bar{p}\,\bar{p}}{(2\pi)^2}\,\bar{p}^{2n-2}\,\frac{\omega_p^2\lambda_n}{2n\omega_c} \times\left(1-\frac{\omega_p^2}{(n\omega_c)^2-\omega_c^2}\right)^{-1} \quad \text{(A24)}$$

following (A8). Further maximizing the upper bound by putting $\bar{p}\rightarrow p_H$ in the integrand and dropping numerics, we obtain

$$\frac{E_{\Omega(n)}}{V} \lesssim \hbar\, p_{zc}^{\,2}\, p_H^{\,2}\left(\frac{\zeta}{m}\right)^{1/2} \cong \frac{\zeta}{\hbar\omega_c}\,Br < B\,O(r)\ , \quad \text{(A24}')$$

so that in the quantum strong-field limit under consideration ($\hbar\omega_c > \zeta$) this exaggerated upper bound is still smaller than the correlation-energy contribution of the dominant local principal plasmon mode Ω. A more careful analysis of the cutoffs and integrals should further reduce the upper bound of Bernstein-mode correlation-energy contributions substantially.

In summary, the results of this Appendix confirm the validity of neglecting the correlation-energy contributions from the local plasmon mode $\Omega' \sim \omega_c$ and from the other quantum and Bernstein types of plasmon resonances in the quantum strong-field limit, thus assuring the accuracy of the approximation scheme employed in the main calculation in the body of this paper.

APPENDIX B: EVALUATION OF THE FIRST-CORRECTION TERM TO $F(r)$

The first correction to the evaluation of $F(r)$ carried out in the body of this paper is obtained by expanding the integrand of $F(r)$ involving $(1+re^{-y}/z)^{1/2}$ to linear order in y and then setting $y=z$. (The zeroth-order term in y was evaluated in the body above, and the first-order term in y is the first correction under consideration here.) Following this procedure the first-correction term is obtained as

$$F^{(1)}(r) = -\frac{r}{2}\int_0^\infty dz\,e^{-z}\,\frac{1}{[1+(1+r/z)^{1/2}]^2} \quad \text{(B1)}$$

$$= -\frac{r}{2}\int_0^\infty dz\,e^{-z}\,\frac{z^2}{r^2}\left[2+\frac{r}{z}-2\left(1+\frac{r}{z}\right)^{1/2}\right]$$

$$= -\frac{r}{2}\left(\frac{2}{r^2}\int_0^\infty dz\,e^{-z}z^2+\frac{1}{r}\int_0^\infty dz\,e^{-z}z - \frac{2}{r^2}\int_0^\infty dz\,e^{-z}z\,[z(z+r)]^{1/2}\right)$$

$$= -\frac{r}{2}\left(\frac{4}{r^2}+\frac{1}{r}-\frac{2}{r^2}\int_0^\infty dz\,e^{-z}z\,[z\,(z+r)]^{1/2}\right). \quad \text{(B1}')$$

The last integral on the right-hand side may be evaluated by noting that [Ref. 16(b), p. 138, No.13]

$$\int_0^\infty dz\,e^{-\alpha z}(z(z+r))^{1/2} = \frac{r}{2\alpha}\,e^{\alpha r/2}K_1\left(\alpha\,\frac{r}{2}\right),$$

and differentiating with respect to α (and subsequently putting $\alpha=1$) [Ref. 16(a), p. 970, No. 12] we obtain

$$-\int_0^\infty dz\,e^{-z}z[z(z+r)]^{1/2} = \frac{r}{2}\left(\frac{r}{2}-2\right)e^{r/2}K_1\!\left(\frac{r}{2}\right)-\frac{r^2}{4}\,e^{r/2}K_0\!\left(\frac{r}{2}\right)\ . \quad \text{(B2)}$$

Thus we have

$$F^{(1)}(r) = -\frac{r}{2}\left\{\frac{4}{r^2}+\frac{1}{r}+\frac{1}{r}\,e^{r/2}\right.$$

$$\times\left[\left(\frac{r}{2}-2\right)K_1\left(\frac{r}{2}\right)-\frac{r}{2}K_0\left(\frac{r}{2}\right)\right]\Bigg\}, \quad (B3)$$

and for $r \ll 1$ we obtain [Ref. 16(c), p. 9, Nos. 37 and 38]

$$F^{(1)}(r) = -\tfrac{1}{16} r^2 \ln r + O(r^3)\ln r + O(r). \quad (B4)$$

This result shows that $F^{(1)}(r)$ is indeed much smaller than $F(r)$ as evaluated in the body of this paper, which supports the validity of our approximation for $r \ll 1$. Although we have calculated $F^{(1)}(r)$ accurately to order $r^2 \ln r$, we have neglected such small terms in the evaluation of $F(r)$ in the body of the paper, so that this result for $F^{(1)}(r)$ really only serves to prove its smallness. Thus, the final result is as stated in the body of the paper,

$$E_{\text{corr}}/V = B\left[-\tfrac{1}{8} r \ln r + O(r) + O(r^2)\ln r\right].$$

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