timized while taken alone. Will this result remain valid to all orders? While a formal analysis proves to be rather difficult, a numerical test is possible for both liquid He^4 and the charged Bose gas. One could first carry out a paired-phonon analysis using the Jastrow function alone, and then study the effect of including three-particle factors such as that in Eq. (79). The energy thus obtained is expected to be *lower* than that obtained in (ii).

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Ion Velocity Distribution of a Weakly Ionized Gas in a Uniform Electric Field of Arbitrary Strength*

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The velocity-distribution function of ions in a neutral gas is studied. A uniform electric field of arbitrary strength is assumed and only binary-ion–neutral-particle collisions are considered. Under these conditions part of the Boltzmann-equation collision operator is replaced by a kinetic model which enables the ion velocity distribution to be found in compact analytical form if the mean free time between ions and neutrals is independent of velocity. This velocity distribution exhibits the expected properties of drift, elevated ion temperature (as compared to the neutral gas), and skewness in the field direction. In addition, the velocity distribution obtained agrees with the known distributions in the extreme cases of (a) low fields and arbitrary masses and (b) arbitrary fields but extremely disparate ion and gas masses. Other tests are made for this distribution with satisfactory agreement.

I. INTRODUCTION

Previous research on the effect of an applied electric field on the motion of charged particles colliding elastically with a neutral gas has proceeded along several lines,[1–5] none of which establishes analytically the ion velocity distribution for arbitrary fields and ion-neutral mass ratios. The analytical results thus far obtained for elastic collisions are only applicable for extreme ion-neutral mass ratios.[1–4] Most experiments studying weakly ionized gaseous systems in uniform electric fields are done when the extreme conditions previously mentioned are not applicable.

We shall use the BGK[6] or kinetic-model method which has not apparently been exploited for this problem. We shall in part remedy this situation by first considering this formulation for any elastic ion-neutral potential and next specialize to the case of a velocity-independent mean free time. In this connection a judicious kinetic model will be chosen such that the resulting ion velocity distribution is both accurate and in simple analytic form. The accuracy of this distribution will be tested in two independent ways.

Fortuitously, a velocity-independent mean free time corresponds to an induced-dipole interaction which is the expected dominant long-range interaction for ion-neutral collisions,[3,7] at least for simple molecules at low relative speeds. In addition, for most drift-tube experiments the ion-neutral density ratio is less than 10^{-9} and the ion density is less than $10^7/\mathrm{cm}^3$ so that it is weakly ionized. Thus only binary-ion–neutral-particle collisions may be considered and ion-ion collisions as well as space-charge effects may be ignored.

II. BOLTZMANN EQUATION AND KINETIC-MODEL SOLUTION

Since we are dealing with the binary-collision limit, the use of the Boltzmann equation is appropriate:

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla_{\vec{r}}+\vec{a}\cdot\nabla_{\vec{v}}\right) F_i(\vec{r},\vec{v},t)=J(F_i,F_g)\,, \qquad (1)$$

where $\vec{r}$, $\vec{v}$, and $\vec{a}$ are, respectively, the ion-position, velocity, and electric-field-acceleration vectors. F_i and F_g are the distribution functions of the ions and gas, respectively, and J is the collision operator expressing the rate of change of density in an element of phase space due to collisions both inside and outside that element.

The collision operator in the Boltzmann equation is generally a nonlinear operator on F_i. However, in this case, it is linear since we are neglecting any effect the motion of the ions has on the nuetral gas since the gas is weakly ionized. Dropping the subscript i, we have

$$J(F,F_g)=\int_{\omega_{\vec{\gamma}'}}\int_{\vec{V}}[F(\vec{r},\vec{v}',t)F_g(\vec{r},\vec{V}',t) - F(\vec{r},\vec{v},t)F_g(\vec{r},\vec{V},t)]\,\sigma(\gamma)\gamma\, d\vec{V}d\omega_{\vec{\gamma}'}\,, \qquad (2)$$

where $\vec{V}$ is the gas velocity, γ is their relative speed, and $|\vec{v}-\vec{V}|$, $\vec{v}'$, and $\vec{V}'$ are the ion and gas velocities for scattering from the element of phase space. $d\omega_{\vec{\gamma}'}$ is the solid-angle range about $\vec{\gamma}'$, and σ is the differential-scattering cross section ($\gamma=\gamma'$ for elastic collisions).

The gas density

$$N(\vec{r},t)=\int F_g(\vec{r},\vec{V},t)\,d\vec{V} \qquad (3)$$

is assumed to be constant and in equilibrium, having a Maxwellian velocity distribution $M_g(\vec{V})$.

From the above the Boltzmann equation can be reasonably approximated by[3]

$$a\,\frac{\partial f(\vec{v})}{\partial v_z}=N\iint [f(\vec{v}')M_g(\vec{V}')-f(\vec{v})M_g(\vec{V})] \times \sigma(\gamma)\gamma\, d\omega_{\vec{\gamma}'}\,d\vec{V}\,, \qquad (4)$$

where v_z is the field component of the velocity. This simplification occurs because for most experiments in weakly ionized gaseous systems the number of ion-neutral collisions is believed to be sufficiently large to assume the existence of a steady-state ion velocity distribution. The ion velocity distribution is therefore assumed time independent and so the first term of the left-hand side of the above equation may be deleted. Uniformity of the electric field and small ion-density gradients on the scale of a mean free path give the basis for the assumption of a spatially independent ion-velocity-distribution function. Therefore the second term and two Cartesian components of the third term of the left-hand side of Eq. (4) may also be deleted.

We may now replace Eq. (4) without further approximation by the following:

$$a\frac{\partial f(\vec{v})}{\partial v_z}+\frac{f(\vec{v})}{\tau(v)}=\frac{f^0(\vec{v})}{\tau(v)}\,, \qquad (5)$$

where

$$1/\tau(v)=N\int\int M_g(\vec{V})\gamma\sigma(\gamma,\omega)\,d\vec{V}\,d\omega\,. \qquad (6)$$

The use of such a representation is called the BGK[6] or kinetic-model method. The second term of the collision integral on the right-hand side of Eq. (4), which represents the rate of decrease of density in an element of velocity space, has been equated to the ion-velocity-distribution function evaluated at the element considered, multiplied by the collision frequency between ions and neutrals, $1/\tau(v)$. The remaining collision integral, in Eq. (4), representing the rate of increase of density in the element of velocity space, is equated to the velocity-distribution function of ions entering the element of phase space, again multiplied by the rate of ion-neutral collisions. But the ions entering the element of phase space considered just had a collision and so are not representative of the typical distribution of velocities of the ions because of the applied electric field. The steady-state velocity distribution of these ions just having had a collision is designated as f^0. These arguments use the standard ansatz in the development of the Boltzmann equation that whenever a collision occurs the particle is discontinuously displaced in phase space to a new velocity and that the element of phase space considered is so small that all collisions will result in the ion leaving the element. The reason for such elaboration is to emphasize that Eq. (5) is not approximative but is

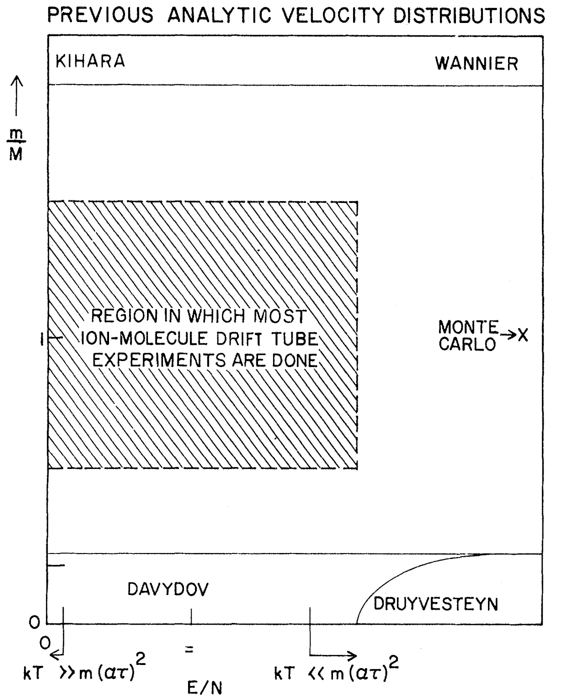

FIG. 1. Space spanned by the known analytical velocity distribution for elastic scattering is shown in the plane of m/M and field strength E/N. This is contrasted with the region where most drift-tube experiments are done. The work of Kihara for large ion-neutral mass ratio is limited to low and medium fields by the assumption that the ion's drift velocity is much smaller than the neutral thermal speed. Wannier's high-mass-ratio study is directly limited to high fields but the convolution theorem may be used to extrapolate to arbitrary fields.

an equivalent way of writing Eq. (4), because of its linearity. The result of writing the collision integral in this form is to convert an integrodifferential equation to a differential equation with another unknown variable f^0, whose physical meaning can be used to permit a relatively easy evaluation of the ion velocity distribution. The resulting velocity distribution is accurate if the correct f^0 is known. If this is unknown, velocity moment and asymptotic limit matching can be done to secure a reasonably accurate f^0 in order to obtain a satisfactory f.

The solution to Eq. (5) can be obtained by standard methods.[8] It is

$$f(\vec{v})=\int_{-\infty}^{v_z}\frac{f^0(\vec{v}'')}{a\tau(v'')}\exp\left(-\int_{v_z''}^{v_z}\frac{dv_z'}{a\tau(v')}\right)dv_z'' . \tag{7}$$

In order to obtain f, $\tau(v)$ must be known as well as an explicit expression for $f^0(\vec{v})$.

III. VELOCITY DISTRIBUTION FOR CONSTANT MEAN FREE TIME

As an example of how the kinetic-model approach can be used to find the velocity distribution, we first treat the case where the mean free time between collisions is independent of the ion velocity, which corresponds to an induced-dipole interaction between ion and neutral particle. An expression for f^0 is chosen which is easy to handle mathematically and which as we will see later, yields a good approximation to the exact f. By noting that the field-independent form of f and f^0 is a Maxwellian, and based on the suspicion that a displaced Maxwellian with temperature parameters is an accurate description of f at low fields, we choose

$$f^0(\vec{v})=(\beta_\perp/\pi)(\beta_\parallel/\pi)^{1/2}\exp[-\beta_\perp v_\rho^2-\beta_\parallel(v_z-v_0)^2] , \tag{8}$$

where v_0, $\beta_\parallel$, and $\beta_\perp$ are parameters to be determined when the moments of the resulting f are matched to the theoretical moments obtained by Wannier[3]; $v_\rho\equiv(v_x^2+v_y^2)^{1/2}$ is the component of the velocity perpendicular to the field. The f obtained from Eq. (7) using the f^0 in Eq. (8), with constant τ, is

$$f(\vec{v})=\frac{\beta_\perp}{2\pi a\tau}\exp\left(-\beta_\perp v_\rho^2-\frac{v_z-v_0}{a\tau}+\frac{1}{\epsilon}\right) \times \operatorname{erfc}\left[\epsilon^{1/2}\left(\frac{1}{\epsilon}-\frac{v_z-v_0}{2a\tau}\right)\right], \tag{9}$$

where

$$\epsilon=4\beta_\parallel(a\tau)^2 , \tag{10}$$

and where erfc is a complementary error function. When the first and second moments in the field direction and the second transverse moment of f are equated to those evaluated by Wannier, the parameters in Eq. (8) are found. They are, for the isotropic scattering case,

$$v_0=\langle v_z\rangle-a\tau , \tag{11}$$

$$1/2\beta_\perp=\langle v_x^2\rangle , \tag{12}$$

$$1/2\beta_\parallel=\langle v_z^2\rangle-\langle v_z\rangle^2-(a\tau)^2 , \tag{13}$$

where the brackets are averages over the velocity distribution f.

The velocity distribution as obtained in Eq. (9) is the first analytical ion velocity distribution found for elastic collisions covering the range of E/N and the ion-neutral mass ratio m/M, where most swarm experiments are done (see Fig. 1), and it approaches all the analytic velocity distributions already obtained for constant mean free time in the extreme cases mentioned previously.

Examination of Eq. (11) shows that the field-component average velocity of f^0, v_0, is less than that of f, $\langle v_z\rangle$, by the quantity $a\tau$ corresponding to the "average" velocity gained in the field direction between collisions.[9] Equation (12) states that the random kinetic energy in the transverse direction of f^0, $(m/4\beta_\perp)$, is equal to that of f where m is the

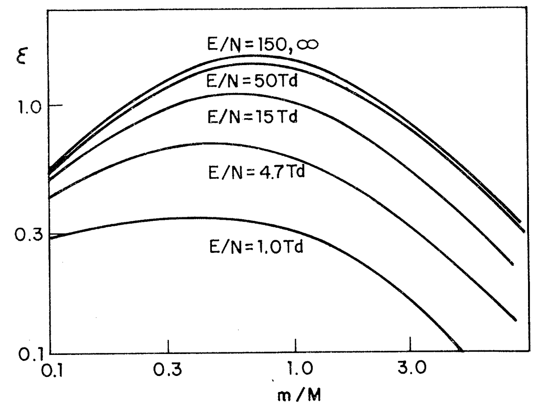

FIG. 2. Skewness parameter ϵ plotted vs m/M for various values of E/N. The neutral-gas temperature and polarizability are held fixed and isotropic scattering is assumed.

ion mass. Equation (13) states that the random energy of f^0 in the longitudinal direction is less than that of f by $\frac{1}{2}m(a\tau)^2$. This is reasonable since f^0 is the steady-state velocity distribution of ions just having had a collision.

Wannier[3] evaluated τ and the velocity averages in Eq. (11)–(13). They are

$$\langle v_z \rangle = (m+M)/\Omega_1 M \,, \tag{14}$$

$$\langle v_z^2 \rangle = \frac{kT}{m} + \frac{(m+M)^3(M\Omega_2 + 4m\Omega_1)}{mM^2\Omega_1^2(3M\Omega_2+4m\Omega_1)} \,, \tag{15}$$

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \frac{kT}{m} + \frac{(m+M)^3\Omega_2}{mM\Omega_1^2(3M\Omega_2+4m\Omega_1)} \,, \tag{16}$$

$$\Omega_l = \frac{N}{a}\int d\vec{V}\, M_g(\vec{V})\gamma \int \sin\chi\, d\chi\, d\phi \times \sigma(\gamma,\chi)(1-\cos^l\chi) = \left\langle \frac{1-\cos^l\chi}{a\tau} \right\rangle \,, \tag{17}$$

where M is the mass of the neutral gas, k is Boltzmann's constant, T is the neutral-gas temperature, χ and ϕ are the polar and azimuthal angles, and Ω_l's are scattering collision integrals.

The scattering integrals Ω_1 and Ω_2 have been evaluated for a polarization interaction by Hassé[10] and Wannier[3] and are found to be

$$\Omega_1 = 1.1052/a\tau_s \,, \tag{18}$$

$$\Omega_2 = 0.772/a\tau_s \,, \tag{19}$$

where τ_s is the mean free time between spiraling collision and is equal to

$$\tau_s = \frac{1}{2\pi e N}\left(\frac{mM}{(m+M)P}\right)^{1/2} \,, \tag{20}$$

where e is the charge of the ion and P its polarizability.

IV. ASYMMETRY OF CONSTANT-MEAN-FREE-TIME VELOCITY DISTRIBUTION

The ϵ in Eq. (10) is a measure of the asymmetry about the mean or skewness of the velocity distribution. The larger the parameter ϵ is, the greater the skewness of the ion velocity distribution. The dependence of ϵ on m/M is shown in Fig. 2, which indicates that ϵ is a maximum when m/M is near unity. From the defining equations of ϵ it can be shown that the functional dependences of ϵ on P, kT, and E/N enter only through the ratio of $(E/N)^2/PkT$. Therefore an increase of E/N by a factor of x is equivalent to a decrease of P or T by a factor of x^2.

The velocity distribution is considerably affected by ϵ as previously stated. This effect is seen in Fig. 3, which is a velocity contour diagram constructed on the plane of the reduced velocities w_x and w_z, while w_y is kept zero. The reduced velocity is defined as $\vec{w} = \vec{v}/a\tau$. Because of the cylindrical symmetry of $f(\vec{v})$ the label of the vertical axis may be either w_x or w_y. Each contour in the figure is a locus of equal values of $f(w_x, 0, w_z)$, whose normalized value is indicated alongside each contour. Four distributions corresponding to four values of E/N are shown. Other relevant parameters used in these figures are $T = 300$ °K and $P = 10^{-24}$ cm^3. The figure for $E/N = 1000$ Td (1 Td $= 10^{-17}$ V cm^2) corresponds to a high-field case, which is defined as a situation where the neutral-gas temperature is negligible in comparison with the ion mean random energy. When this condition is met, the shape of the ion velocity distribution in the reduced velocity space is independent of E/N. Indeed, this is why reduced velocities are introduced. For example, in the case considered here a value of E/N of 50 Td yields the same shaped velocity distribution as $E/N = 1000$ Td to within plotting accuracy. The figure for $E/N = 2$ Td represents a low-field case. The figure for $E/N = 5$ and 10 Td represents transition regions of E/N between the high-field and low-field cases. The value of ϵ for isotropic scattering corresponding to $E/N = 1000$, 10, 5, and 2 Td is approximately $\frac{4}{3}$, 1, $\frac{2}{3}$, and $\frac{1}{3}$, respectively.

V. COMPARISON WITH RELATED WORK OF WANNIER

For constant mean free time, Wannier[3] has found the moments of the velocity distribution for arbitrary field strength in terms of scattering integrals (of which the first two have been evaluated[3,10]). He has also done a Monte Carlo calculation of the velocity distribution for the special case of high field, unity ion-neutral mass ratio, and isotropic ion scattering in the center-of-mass frame (along with constant mean free time). We will make use of this work to check on the quality

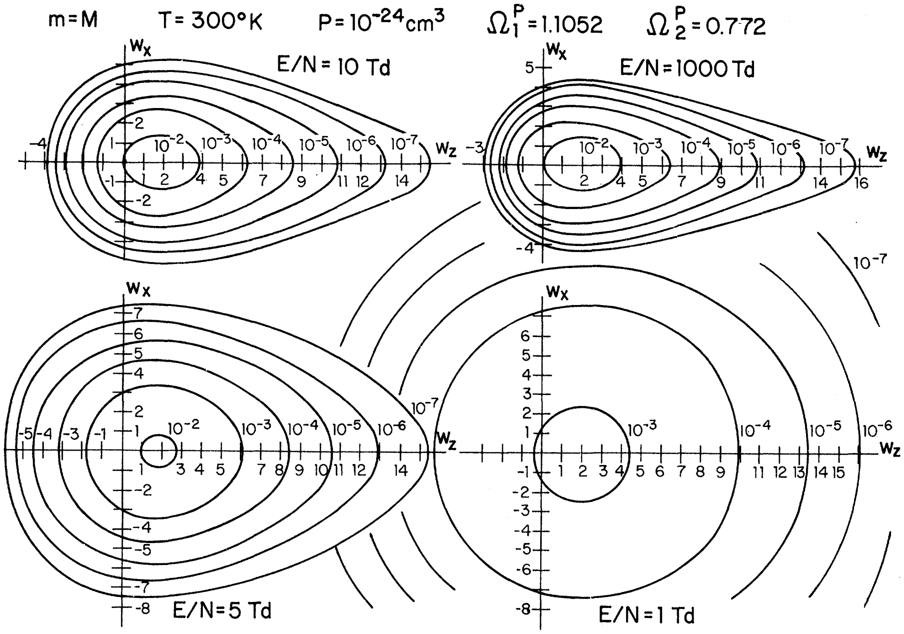

FIG. 3. Velocity-distribution contour diagrams are plotted for various values of E/N for the case of equal ion and gas mass. The neutral-gas temperature and polarizability are kept fixed and polarization angular scattering is considered.

of the kinetic-model solution for the velocity-distribution function.

A. Comparison of $\langle v_z^3 \rangle$

One way of evaluating the accuracy of the velocity distribution f is to compare its moments with those calculated by Wannier. It is known that two velocity distributions are equal if all their moments are equal.[11] We have forcibly matched the first two velocity moments. The assumption of cylindrical symmetry about the field has guaranteed that the value of $\langle v_x^{2n+1} v_y^m v_z^p\rangle$ and $\langle v_x^n v_y^{2m+1} v_z^p\rangle$ will be zero and equal to those of Wannier. n, m, p are non-negative integers. The high-v_z moments are also equal, as will be discussed in the Sec. VI. Therefore we will compare the lowest unmatched moment $\langle v_z^3\rangle$ to get an idea of the accuracy of the intermediate moments in the field direction.

The moments of the kinetic model f are

$$\langle v_\rho^n\rangle = (\pi/4\beta_\parallel\beta_\perp^n)^{1/2}\,\Gamma(1+\tfrac{1}{2}n)(a\tau)^n \qquad (21)$$

and

$$\langle v_z^n\rangle = (-1)^n \exp\left(\frac{v_0}{a\tau}+\frac{1}{\epsilon}\right)\lim_{b\to 1}\frac{d^nQ(b)}{db^n}\,, \qquad (22)$$

where Γ is a Γ function and $Q(b)$ is a generating function defined by

$$Q(b) = \frac{1}{b}\exp\left[\frac{b^2}{\epsilon} - b\left(\frac{v_0}{a\tau}+\frac{2}{\epsilon}\right)\right]. \qquad (23)$$

Therefore,

$$\langle v_z^3\rangle = \langle v_z\rangle^3 + 3\langle v_z\rangle + 6\langle v_z\rangle/\epsilon + 2\ . \qquad (24)$$

The $\langle v_z^3\rangle$ established by Wannier for high fields is

$$\langle v_z^3\rangle_w = \frac{a\tau}{5}\left(\frac{9\langle v_z^2\rangle - 3\langle v^2\rangle}{\langle 1-I_{33}\rangle} + \frac{4\langle v_z^2\rangle + 5\langle v^2\rangle}{\langle 1-I_{31}\rangle}\right), \qquad (25)$$

where

$$\langle 1-I_{31}\rangle = \frac{(M^3+3m^2M)\Omega_1 + 2mM^2\Omega_2}{(m+M)^3} \qquad (26)$$

and

$$\langle 1-I_{33}\rangle = \frac{(6m^2M-3M^3)\Omega_1 + 9mM^2\Omega_2 + 5M^3\Omega_3}{2(m+M)^3}\,. \qquad (27)$$

Since the polarization scattering integral Ω_3^p has not yet been calculated, the isotropic scattering integral Ω_3^I, whose value is unity, is used.

A comparison of $\langle v_z^3\rangle$ of Eq. (24) with that of Wannier is made. The comparison is shown in Fig. 4. In this figure $\langle v_z^3\rangle/(a\tau)^3$ is plotted against m/M. There is also a mass-ratio dependence hidden in the scaling factor $a\tau$, so that the shape of the curves should not be used to draw other inferences. The ordinate in Fig. 4 is chosen so that in this high-field case the curves are independent of E/N. The kinetic model f has a third field moment within 10% of that of Wannier.

B. Comparison of General Features of Distributions in Velocity Contour Diagram

The Monte Carlo study of Wannier was done for the special case of high fields, unity mass ratio, isotropic center-of-mass scattering, as well as constant mean free time, etc. Hence the analogous case is taken for the kinetic model. The contour

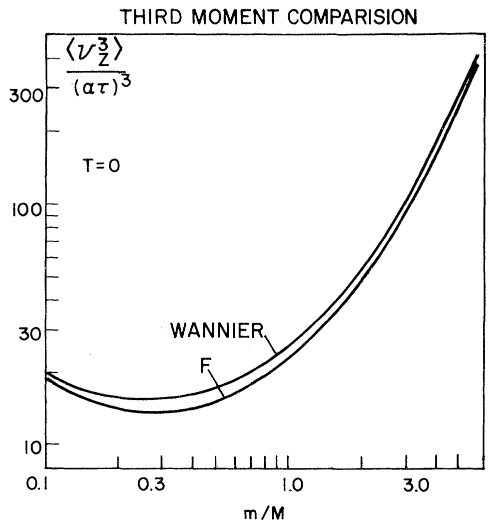

FIG. 4. Third moment $\langle v_z^2\rangle$ of the kinetic model is compared with that found by Wannier. Reduced third moments $\langle v_z^3\rangle/(a\tau)^3$ are plotted vs m/M. High fields and isotropic scattering are considered.

diagrams shown in Fig. 5 are of the same nature as those exhibited in Fig. 3. As before, the values of $f(\vec{w})$ of the contours are spaced logarithmically. Therefore, if neighboring contours are separated by the same distance, $f(w)$ is undergoing an e^{-w} dropoff going from the point of the higher value of f. If such contours get closer together, then $f(\vec{w})$ falls off faster than e^{-w}.

Major similarities and differences of these two distributions are as follows: Since the first two velocity moments are forcibly matched, there exists an over-all resemblance between the two distributions, that is, most of the particles are situated at about the same velocity space. Beyond that, one should note that the population density and the skewness of the contours at high w_z are very similar. In fact, at values of w_z greater than 3.5, the contours of the two distributions almost coincide with each other. This implies that the e^{-w_z} dependence of f at large w_z, as is predicted by Eq. (9) calculated from the kinetic model, is essentially correct. This also implies that all high moments of the two distributions are substantially equal. These additional similarities are not unexpected, for so long as f^0 does not overprescribe the population density at high w_z, the density of f at high w_z is primarily controlled by the field effect between collisions and the rough correctness of f^0 at low w_z. The prescription by f^0 for population density at low w_z is always fairly good, on account of the moment matching at the low moments, and, as was stated earlier, the effect of the electric field between collisions is correctly taken care of by the field term in Eq. (5). One major difference between the two distributions occurs at negative w_z, where the Monte Carlo distribution has a much sharper falloff in population density in the $-w_z$ direction. At $w_z = -0.5$, the population density is already as low as 0.001 while the corresponding value for the kinetic model is

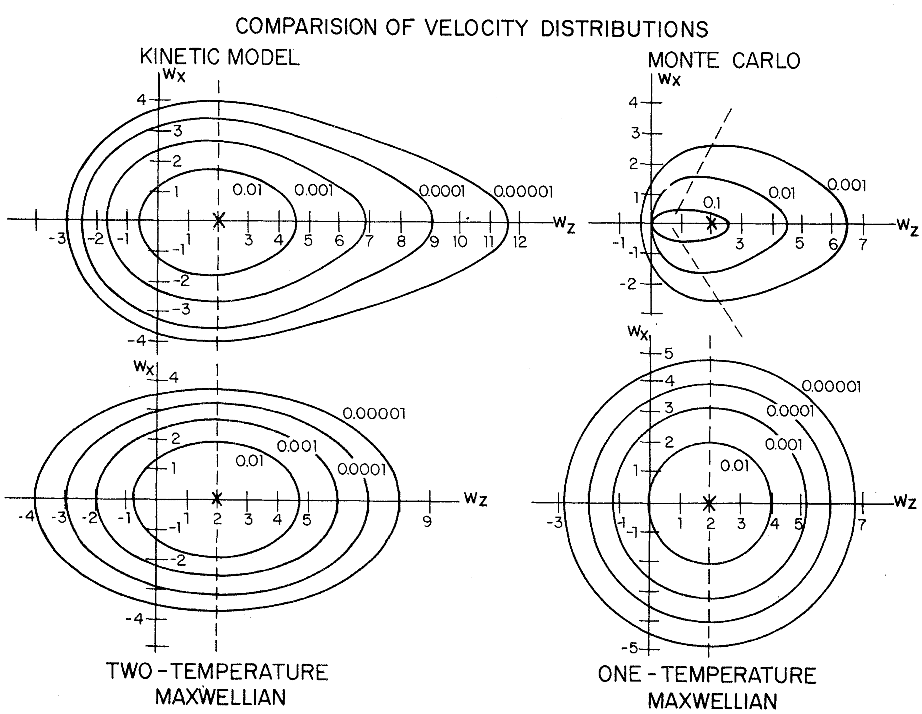

FIG. 5. Velocity contour diagrams are shown for four different velocity distributions: the kinetic model, the Monte Carlo calculations, and one- and two-temperature displaced Maxwellians. High fields, unity ion-neutral mass ratios, and isotropic scattering are assumed.

still 0.01. This relative overabundance of population in the negative w_z region for the kinetic model is made up by the relative depletion of population around its innermost contour. It is estimated that approximately 20% of the total population is included within the 0.01 contour of the Monte Carlo calculation, while only 15% resides within the corresponding contour calculated from the kinetic model. Another difference exists in the locus of relative maximum of population density along the w_z direction in planes of constant w_x. They are indicated by dotted lines. The Monte Carlo calculation shows that there is a coupling between w_x and w_z such that the location of the relative maximum is dependent on the value of w_x of the plane. The corresponding locus for the kinetic model is a straight line perpendicular to the w_z axis. This difference arises from the assumption of the product separation $f(w_x, w_\rho)=g_1(w_z)g_2(w_\rho)$ in the kinetic-model calculation.

For comparison purposes, both the one- and the two-temperature displaced Maxwellians of appropriate moments are also shown in Fig. 5.

VI. SUMMARY

The velocity distribution obtained in Sec. III for constant mean free time has the following characteristics: It agrees with the known analytic velocity distributions in the asymptotic cases of extreme ion-neutral mass ratios at arbitrary fields and arbitrary ion-neutral mass ratios at low fields. It has the correct w_z dependence at large w_z since the electric field is correctly accounted for. It is in substantial agreement with a previous Monte Carlo study, particularly at large w_z.

Further, many moments of two distributions are equal, as was discussed in Sec. V. In light of this agreement, we feel that the kinetic-model approach is highly satisfactory for studying the ion velocity distribution of a weakly ionized gaseous system under a uniform electric field of arbitrary strength.

The method used herein has reduced the problem of finding an analytical velocity distribution to that of solving a simple integral, shown in Eq. (7). This advantage is intrinsic from the use of the BGK approach. From the example of finding a velocity distribution for the constant-mean-free-time interaction, we are led to conclude that one can find a velocity distribution of reasonable accuracy with relatively simple f^0 constructed with straightforward physical intuition. At low fields, f is strongly dependent on f^0, and deviates only slightly from it. But for such situations, physical intuition suggests the description of f^0 by a displaced asymmetric Maxwellian is fairly reasonable. At high fields, intuition starts to fail. But for these cases, a significant portion of the velocity distribution becomes strongly dependent upon the field term which is, as we said before, correctly accounted for. Consequently, the velocity distribution, particularly at large w_z, is still reasonably accurate. The weakness of the method is that unless the f^0 has w_x, w_y, and w_z coupled, the field does not act on the velocity distribution in the w_x and w_y direction at all. Therefore, the distribution in such directions is always only as accurate as one assumed for f^0.

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