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Asymptotic Eigenvalue Distribution for the Wave Equation in a Cylinder of Arbitrary Cross Section

H. P. Baltes
Institut für Theoretische Physik (Kondensierte Materie), Freie Universität Berlin, FB 20, 1 Berlin 33, W.-Germany
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The asymptotic distribution of the eigenvalues k_n of the scalar wave equation $\Delta u + k^2 u = 0$ is calculated for a three-dimensional finite domain of general cylindrical shape with Dirichlet and Neumann boundary conditions. In the limit of large k, the number $\bar{N}(k)$ of modes not exceeding k, smoothed in order to eliminate its fluctuating part, is determined. Four terms of the expansion of $\bar{N}(k)$ are obtained. The boundary effects for the thermal phonon radiation of thin films are discussed as an application. The respective results for the electromagnetic vector waves in a lossless closed cavity (blackbody radiation) are presented as well. Here the constant term in the expansion is independent of the shape of the domain and does not account for corners or connectivity. E- and H-type resonances have to be observed separately in order to yield the complete shape dependence of $\bar{N}(k)$. The edge- and curvature-dependent corrections of the Planck, Wien, and Stefan–Boltzmann radiation formulas are given.

I. INTRODUCTION

In this article we investigate refinements of the smoothed asymptotic distribution of the eigenvalues k_n of the Laplace operator for the following boundary-value problems: (a) the scalar wave equation

$$(\Delta + k^2)u = 0 \tag{1.1}$$

in a finite domain $G \subset R^3$ with Dirichlet or Neumann boundary conditions

$$u = 0 \tag{1.2}$$

or

$$\frac{\partial}{\partial n} u = 0 \tag{1.3}$$

on the closed boundary surface ∂G; and (b) the electromagnetic vector wave equation

$$(\Delta + k^2)\begin{Bmatrix} \vec{\mathrm{E}} \\ \vec{\mathrm{H}} \end{Bmatrix} = 0 , \tag{1.4}$$

with the divergence conditions

$$\vec{\nabla} \cdot \vec{\mathrm{E}} = 0 , \quad \vec{\nabla} \cdot \vec{\mathrm{H}} = 0 \tag{1.5}$$

in the interior of a cavity covering the domain G, with the boundary conditions

$$\vec{\mathrm{n}} \times \vec{\mathrm{E}} = 0 , \quad \vec{\mathrm{n}} \cdot \vec{\mathrm{H}} = 0 \tag{1.6}$$

on the enclosing wall ∂G. For our calculations, G is a cylinder with arbitrary cross section specified below.

We describe the eigenvalue distribution by the "mode number"

$$N(k) = \sum_{k_n < k} 1 + \sum_{k_n = k} \tfrac{1}{2} , \tag{1.7}$$

denoting the number of eigenvalues not exceeding k. In the limit $k \to \infty$, the asymptotic behavior for both cases (a) and (b) is known as[1–3]

$$N(k) = N_0(k) + \mathcal{O}(k^2 \ln k) , \tag{1.8}$$

where the logarithmic factor can be removed for polyhedral domains[4] in the case (a) and where N_0 denotes Weyl's volume term.[5]

The determination of $N(k)$ for cases (a) and (b) constitutes an old problem of mathematical physics[5] with implications on many branches of modern physics as discussed recently by Hilf,[6] Balian and Bloch,[7] and Baltes.[8] For most applications in physics, it is appropriate to study the *smoothed* mode number $\bar{N}(k)$, where the fluctuating part has been eliminated by one of the averaging procedures reviewed in Refs. 7 and 8. This leads to the refined expansion

$$\bar{N}(k) = N_0(k) + N_1(k) + N_2(k) + N_3 , \tag{1.9}$$

with the surface term N_1 proportional to the area S of ∂G and the second power of the wave number k for the scalar problem,[9,10] and vanishing for the electromagnetic problem.[8,11–13] The second-order linear correction in k, accounting for the curvature and for the edges of ∂G, is known for smooth boundary surfaces[6,7,10] and for the particular case of the parallelepiped-shaped domain.[6,14] N_3 is known only for the cube-shaped domain, where it reads $\mp\frac{1}{8}$ for the scalar problems[10] and $\frac{1}{2}$ for the electromagnetic case.[8]

It is the aim of this article to study the edge and corner contributions to the average asymptotic mode number $\bar{N}(k)$ occurring for piecewise smooth domain boundaries ∂G. For this purpose we consider the following domain G.

Let G be a general prism or cylinder of volume V, surface area S, and length L. Assume that the cross section g of G is a simply connected two-dimensional domain of area σ with a piecewise smooth boundary curve ∂g of length $\gamma = \sum_i \gamma_i$, where γ_i denotes the length of the ith smooth arc of ∂g. The angle of the jth corner of g is denoted by α_j. G is sufficiently general for the study of the edge and corner contributions hitherto unknown, but on the other hand, G is still sufficiently simple, allowing us to determine N_2 and N_3 using the summation procedure described in Sec. II. N_2 and N_3 for the scalar problem [case (a)] are derived in Sec. III, where a discussion of Fedosov's results[4] is included. Section IV describes boundary effects in the long-wave acoustic emission of thin metallic films. In Sec. V, we investigate N_2 and N_3 for the cavity radiation [case (b)] and discuss the corresponding corrections of the Planck, Wien, and Stefan–Boltzmann radiation formulas. The comparison with computer results for finite k is included. An extension to multiply connected cylinder cross sections is suggested in Sec. VI.

II. SUMMATION PROCEDURE FOR SCALAR PROBLEMS

For the eigenvalue problem [case (a)], the density of states $D = dN/dk$ can be written in terms of the time-independent Green's function $\mathcal{G}(\vec{r}, \vec{r}'; E)$, reminiscent of quantum field theory.[15] The domain G described above allows the separation of the variables such that

$$\mathcal{G}(\vec{r}, \vec{r}'; E) = \sum_m \mathcal{G}^{\mathrm{II}}(\vec{\rho}, \vec{\rho}'; E - E_m) Z_m(z) Z_m(z'), \tag{2.1}$$

where

$$\mathcal{G}^{\mathrm{II}}(\vec{\rho}, \vec{\rho}'; E - E_m) = \sum_\nu \frac{P_\nu(\vec{\rho}) P_\nu(\vec{\rho}')}{E - E_\nu - E_m + i\delta} \tag{2.2}$$

is the Green's function for the respective two-dimensional problem on the domain g, with $E = k^2$,

$$E_n = E_\nu + E_m = k_\nu^2 + k_m^2 = k_\nu^2 + (\pi m/L)^2 = k_n^2, \tag{2.3}$$

and

$$\vec{\rho} = (x, y, z = \mathrm{const}) = (x, y) \epsilon g .$$

We observe that

$$m = m_D = 1, 2, 3, \ldots \tag{2.4}$$

for the Dirichlet and

$$m = m_N = 0, 1, 2, 3, \ldots \tag{2.5}$$

for the Neumann problem. The functions $P_\nu(\vec{\rho})$ and $Z_m(z)$ are orthonormal systems on g and $[0, L]$, respectively. As a consequence

$$\overline{N}_{D(N)}(k) = \sum_{m=1(0)}^{\max} \overline{N}^{\mathrm{II}}_{D(N)}(\tilde{k}_m), \tag{2.6}$$

with

$$\tilde{k}_m = [k^2 - (m\pi/L)^2]^{1/2} . \tag{2.7}$$

The conditions (2.4) and (2.5) are accounted for. The summation has to be extended over all $\tilde{k}_m \geq 0$, i.e., $m \leq kL/\pi$.

The averaged mode numbers corresponding to the D and N problems for the domain g read[10,16,17]

$$\overline{N}^{\mathrm{II}}_{D(N)}(k) = \frac{\sigma k^2}{4\pi} \mp \frac{\gamma k}{4\pi} + C_1 + C_2, \tag{2.8}$$

with the curvature term

$$C_1 = \sum_i (1/12\pi) \int_{\gamma_i} \kappa(\gamma_i)\, d\gamma_i , \tag{2.9}$$

where $\kappa(\gamma_i)$ denotes the curvature of the arc γ_i, and the corner term

$$C_2 = \sum_j c(\alpha_j) = \sum_j \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} . \tag{2.10}$$

Here $c(\alpha)$ is the "corner number."[16] That is, we have $C_1 = \frac{1}{6}$, $C_2 = 0$ for a circle, and $C_1 = 0$, $C_2 = 4c(\frac{1}{2}\pi) = \frac{1}{4}$ for a rectangle. The average remainder is $\overline{\mathcal{O}}(k^{-r})$ with arbitrary real $r > 0$ for polygonal domains, but only $\overline{\mathcal{O}}(k^{-1}\ln k)$ for domains showing a curved piece γ_i.[10]

III. EDGE AND CORNER TERMS FOR SCALAR DIRICHLET AND NEUMANN PROBLEMS

The summation [Eqs. (2.6) and (2.7)] is carried through by converting the sum into an integral. This procedure can be made rigorous, because we sum over a one-dimensional set of equidistant points. Thus we obtain

$$\overline{N}_{D(N)}(k) = (L/\pi) \int_0^k dk_z\, \overline{N}^{\mathrm{II}}_{D(N)}\left((k^2 - k_Z^2)^{1/2}\right) \mp \tfrac{1}{2} \overline{N}^{\mathrm{II}}_{D(N)}(k) + \overline{\mathcal{O}}(k^{-r}), \tag{3.1}$$

where the density

$$\overline{D}^{\mathrm{I}}(k_Z) = L/\pi + \overline{\mathcal{O}}(k^{-r}) \tag{3.2}$$

of the $(k_z)_m = k_m$ and the conditions (2.4) and (2.5) at the lower boundary $k_z = 0$ are accounted for. Inserting (2.8) and using

$$\int_0^k (k^2 - x^2)^{1/2}\, dx = \tfrac{1}{4}\pi k^2 , \tag{3.3}$$

we find

$$\overline{N}_{D(N)}(k) = \frac{Vk^3}{6\pi^2} \mp \frac{Sk^2}{16\pi} + \left(\frac{C_1 L}{\pi} + \frac{C_2 L}{\pi} + \frac{\gamma}{8\pi}\right) k \mp \tfrac{1}{2}(C_1 + C_2) . \tag{3.4}$$

The remainder is $\overline{\mathcal{O}}(k^{-r})$ for polygonal cross sections g (i.e., $C_1 = 0$) and $\overline{\mathcal{O}}((\ln k)^2)$ for curved $\partial g (C_1 \neq 0)$. The first two terms in (3.4) are the well-known volume and surface contributions.

Let us discuss the term $N_2(k)$ which is linear in k. It consists of the following three contributions:

(i) The expression

$$\pi^{-1} C_1 L k = (12\pi^2)^{-1} k \sum_i \int_{\gamma_i} \kappa(\gamma_i)\, d(L\gamma_i) \tag{3.5}$$

can be written

$$(12\pi^2)^{-1} k \sum_i \int_{S_i^{\parallel}} dS_i^{\parallel} \kappa(S_i^{\parallel}) , \qquad (3.6)$$

where $S_i^{\parallel}$ denotes the ith smooth component of the area

$$S_{\parallel} = \gamma L = \sum_i \gamma_i L$$

of the cylinder and $\kappa(S_i^{\parallel})$ is the curvature of $S_i^{\parallel}$. Thus (3.6) is identified as the curvature term accounting for the curved smooth pieces of G, and it has the same form as the well-known curvature term for smooth boundaries ∂G.[6,7,10]

(ii) $\pi^{-1} C_2 L k$ obviously accounts for the edges parallel to the z direction with length L, angles α_j, and curvature zero. This term has to be compared with Fedosov's results for m-dimensional polyhedra.[4] Putting $m=3$, Fedosov's integral formula for $k \to \infty$, implies an edge term showing the *opposite sign* of the above correct edge contribution in $\overline{N}(k)$.

(iii) $(8\pi)^{-1}\gamma k$ stems from the edges at the top and the bottom of the general cylinder with angle $\frac{1}{2}\pi$, length $\gamma = \sum_i \gamma_i$, and curvature $\kappa(\gamma)$.

We observe that (ii) reads as $(1/\pi)\sum_j c(\alpha_j) L k$ and (iii) can be written as $(1/\pi) 2c(\frac{1}{2}\pi)k$. Hence we conjecture that a general edge of angle α and arbitrary finite curvature contributes

$$\pi^{-1} c(\alpha) k = (\pi^2 - \alpha^2)(24\pi^2\alpha)^{-1} k \qquad (3.7)$$

per unit length.

Strictly speaking, the discussion of the constant term N_3 is meaningful for polygonal cross sections only. Here the term reads as

$$N_3 = \mp \tfrac{1}{48} \sum_j \pi/\alpha_j - \alpha_j/\pi \qquad (3.8)$$

and obviously stems from the corners with angles $\Theta_1 = \frac{1}{2}\pi$ and $\Theta_2 = \alpha_j$ at the top and the bottom of G.

For curved parts of ∂g, N_3 shows a contribution due to the curvature $\kappa(\gamma_i)$ of the edges of length γ_i and angle $\frac{1}{2}\pi$. This term, however, is less important than the remainder $\overline{o}((\ln k)^2)$. From the results known for the circle,[16] we have the impression that the $(\ln k)^2$ cannot be removed in general.

Let us consider two simple domain shapes. For the *circular cylinder* with radius R, we find from (3.4)

$$N_2 + N_3 = (\tfrac{1}{4}R + L/6\pi)k \mp \tfrac{1}{12} , \qquad (3.9)$$

whereas for the *parallelepiped* (3.4) yields

$$N_2 + N_3 = (4\pi)^{-1}(A_1 + A_2 + A_3)k \mp \tfrac{1}{8} , \qquad (3.10)$$

with A_i denoting the edge lengths. This result comprehends the one for the cube-shaped domain (1.10) due to Brownell.[10] The edge contributions occurring in (3.4) cannot be obtained from the extrapolation of the smooth boundary formula.[6,7,10]

IV. ACOUSTIC PHONON EMISSION OF THIN FILMS

Recently Herth and Weis[18–20] studied the thermal phonon radiation of thin metallic films deposited on a dielectric crystal. The thickness L of these films is below 1000 Å (typically between 100 and 300 Å) and hence is not very large compared to the wavelengths of the low-frequency phonons (10^{11}–10^{12} $\sec^{-1}$) prevailing at low temperatures. Consequently, the use of the "infinite-thickness" eigenvalue distribution $N_0(k)$ adopted by Weis[18] is not justified and the corrections (3.4) of Weyl's formula are relevant. In particular the spectral and total thermal phonon radiation deviates from the ideal "blackbody" behavior, even in the case of perfect acoustic match as a consequence of size and shape effects. We mention that size and shape corrections of Weyl's formula are known in technical acoustics for the case of a parallelepiped-shaped domain.[21]

For the study of the long-wavelength corrections it is sufficient to consider the regime $k \ll k_{\max}$ and hence to adopt the Debye model as it was done by Herth and Weis.

The vibrations are described by $\Delta\vec{u} + k^2\vec{u} = 0$ with $k = \omega c_t$ and $k = \omega c_l$ for the transverse ($\vec{\nabla}\cdot\vec{u} = 0$) and for the longitudinal ($\vec{k}\times\vec{u} = 0$) waves, respectively. The mode numbers are $N_t = 2N_{\text{scalar}}$ and $N_l = N_{\text{scalar}}$, where N_{scalar} is the distribution studied in Sec. III. For simplicity let us assume Dirichlet or Neumann conditions for $\vec{u}$. With $L \ll \gamma$, formula (3.4) leads to the following *relative surface corrections* for the spectral density $\overline{D}(\lambda^{-1})$, with $\lambda = 2\pi/k$. Thus we have

$$\frac{\overline{D}_1(\lambda^{-1})}{\overline{D}_0(\lambda^{-1})} = \mp \frac{1}{8}\left(\frac{2}{L} + \frac{\gamma}{\sigma}\right)\lambda \approx \mp \frac{1}{4}\frac{\lambda}{L} \qquad (4.1)$$

because $V = \sigma L$ and $S = 2\sigma + L\gamma$, where σ denotes the area and γ the circumference of the thin film. The relative second-order correction reads

$$\frac{\overline{D}_2(\lambda^{-1})}{\overline{D}_0(\lambda^{-1})} = \frac{C_1 L + C_2 L + \frac{1}{8}\gamma}{2\pi\sigma L}\lambda^2 \approx \frac{\gamma\lambda^2}{16\pi\sigma L} . \qquad (4.2)$$

Inserting the data $L = 500$ Å, $\gamma = 4$ mm, and $\sigma = 1$ mm^2 for the constantan film studied in Ref. 19 and considering the longitudinal modes with $c = 5.24$ km/sec, we obtain the relative corrections

$$\mp 2.62\times10^{10}\nu^{-1}(\text{Hz}^{-1}) \text{ and } 4.15\times10^{16}\nu^{-2}(\text{Hz}^{-2}) .$$

Hence the first-order correction amounts to approximately 26% for 10^{11} Hz and 2.6% for 10^{12} Hz, whereas the second-order correction is only 4×10^{-4} and 4×10^{-6}% for the same phonon frequencies and can be neglected. For the thinner and smaller crystals reported in Ref. 20, the corrections are accordingly larger.

The formulas describing the spectral and total phonon emission have to be corrected accordingly.

For an isotropic medium this yields, e.g., for the longitudinal branch

$$\rho_l(\omega, T) = (e^{\hbar\omega/k_BT} - 1)^{-1} \times\left(\frac{\hbar\omega^3}{2\pi^2}\,\frac{1}{c_l^2}\,\frac{dk}{d\omega(k,l)} \mp \frac{S}{V}\,\frac{\hbar\omega^2}{8\pi}\,\frac{1}{c_l}\,\frac{dk}{d\omega(k,l)}\right), \tag{4.3}$$

if $\omega \ll \omega_{\text{Debye}}$. The peak of the spectral radiant power is shifted accordingly. For small temperatures T compared to the Debye temperature $\Theta = \hbar\omega_{\max}/k_B$, the frequency integration can be extended to infinity and the Stefan–Boltzmann equation for phonons is obtained. Assuming ideal acoustic impedance match and zero temperature for the transmitting dielectric medium, we find the refined formula

$$P = P_0\left(1 \mp \frac{c_l^{-1} + 2c_t^{-1}}{c_l^{-2} + 2c_t^{-2}}\cdot\frac{\hbar}{k_BT}\,\frac{S}{V}\,\frac{15}{2\pi^3}\,\zeta(3)\right), \qquad P_0 = \frac{\pi^2 k_B^4}{120\hbar^2}\,(c_l^{-2} + 2c_t^{-2})T^4 \tag{4.4}$$

describing the total phonon-radiation flux per unit contact area. ζ denotes the Riemann ζ function with $\zeta(3) = 1.202\ldots$. For a constantan film of thickness $L \approx 2V/S$ with $c_t = 2.46$ km/sec and c_l as given above, we obtain $P = P_0\,[1 \mp 122/T\,(\text{K})\,L(\text{Å})]$. Hence, for $L = 500$ Å and $T = 10$ K the correction is 2.44%. The correction is approximately proportional to c_t because $c_l \gg c_t$ and therefore is much smaller for "softer" materials like lead.

The author is well aware of the fact that the boundary conditions considered above are unrealistic, as phonons can be reflected both specularly and diffusely at the contact area and furthermore, can be refracted or undergo a conversion of polarization. Such effects can reduce or enhance the surface corrections described above, in particular, because the two thin-film faces are under different physical conditions.

V. ELECTROMAGNETIC WAVES IN FINITE CAVITIES

The procedure developed in the Secs. II and III has to be modified accordingly for the case of the electromagnetic boundary-value problem [Eqs. (1.4)–(1.6)]. In a cavity with a general cylindrical shape, E- and H-type (TM and TE) modes can be studied separately.[22] The peculiar boundary condition (1.6) leads to the following structure of the spectra. For the E-type resonances, the eigenvalues are

$$k_E^2 = k_{\nu,D}^2 + (m\pi/L)^2\,, \tag{5.1}$$

where $k_{\nu,D}^2$ belongs to the two-dimensional Dirichlet problem, but where $m=0$ is not forbidden[22] as it was in the case of the three-dimensional Dirichlet problem (see 2.4). For this reason we obtain

$$\overline{N}_E = (L/\pi)\int_0^k dk_z\,\overline{N}_D^{\text{II}}((k^2 - k_z^2)^{1/2}) + \tfrac{1}{2}\overline{N}_D^{\text{II}} = \overline{N}_D + \overline{N}_D^{\text{II}}\,, \tag{5.2}$$

instead of (3.1). From (5.2) we conclude that

$$\overline{N}_E = \frac{Vk^3}{6\pi^2} + \frac{1}{16\pi}\,(S_\perp - S_\parallel)k^2 + \frac{1}{8\pi}\,[8L(C_1 + C_2) - \gamma]k + \tfrac{1}{2}(C_1 + C_2)\,, \tag{5.3}$$

with $S_\perp = 2\sigma$ and $S_\parallel = \gamma L$. The remainder is the same as the one found for the scalar problems in Sec. III. The analogous, though more complicated, considerations for the H-type resonances yield

$$\overline{N}_H = (L/\pi)\int_0^k dk_Z\,[\overline{N}_N^{\text{II}}((k^2 - k_Z^2)^{1/2}) - 1] - \tfrac{1}{2}[\overline{N}_N^{\text{II}}(k) - 1] = \overline{N}_N - \overline{N}_N^{\text{II}} - \pi^{-1}Lk + \tfrac{1}{2}\,, \tag{5.4}$$

because

$$k_H^2 = k_{\nu,N}^2 + (m\pi/L)^2\,, \tag{5.5}$$

where $m=0$ is allowed, but where $k_{\nu,N}^2 = 0$ leads to zero-amplitude fields[22] and has to be eliminated, giving rise to the term $-\pi^{-1}Lk$. The term $\frac{1}{2}$ results from the discussion of the point $\vec{k} = (0,0,0)$ involved in the terms in different ways. As a consequence we find

$$\overline{N}_H = \frac{Vk^3}{6\pi^2} + \frac{1}{16\pi}\,(S_\parallel - S_\perp)k^2 + \frac{1}{8\pi}\,[8L(C_1 + C_2 - 1) - \gamma]k - \tfrac{1}{2}(C_1 + C_2) + \tfrac{1}{2}\,. \tag{5.6}$$

We observe that the surface terms are identical with those derived previously using a different technique.[13] In particular, the surface terms in Eqs. (5.3) and (5.6) cancel if the total mode number $\overline{N} = \overline{N}_E + \overline{N}_H$ is calculated. Thus we gave a further, simple proof for the well-known vanishing of the surface term N_1.[11–13] The total mode number reads

$$\overline{N} = N_0 + [\pi^{-1}L(2C_1 + 2C_2 - 1) - (4\pi)^{-1}\gamma]k + \tfrac{1}{2} \tag{5.7}$$

and comprehends the new result that the corner- and edge-curvature contributions $\pm\frac{1}{2}(C_1 + C_2)$ of Eqs. (5.3) and (5.6) cancel as well. Hence *the constant term* $N_3 = \frac{1}{2}$ *does not depend on the shape of the cavity*. The vanishing of the surface and the corner terms is peculiar to the electromagnetic problem and does not occur in the scalar case studied in Sec. III. The term $-\pi^{-1}Lk$ in Eq. (5.7) is a further peculiarity of the electromagnetic problem and is due to the cancellation of certain modes with zero amplitude. The term involving C_1 accounts for the curvature of the smooth pieces of the surface parallel to z and agrees with the linear term derived by Balian and Bloch[12] for cavities with a smooth boundary.

The only surviving shape-dependent correction is

$$N_2 = \pi^{-1}(2C_1L + 2C_2L - \tfrac{1}{4}\gamma - L)k , \qquad (5.8)$$

with C_1 and C_2 given above [Eqs. (2.9) and (2.10)]. The first three terms account for curved surfaces and edges. We observe that the contributions per unit length of a $\frac{1}{2}\pi$ edge parallel to z and of a $\frac{1}{2}\pi$ edge orthogonal to z are equal, but have opposite sign, i.e., $\pm k/8\pi$. Therefore a general conjecture concerning the contribution of arbitrary edges is not possible here. Let us consider a few special cavity geometries in order to compare (5.7) with known computer results for finite k.[8] For the parallelepiped, we obtain

$$\overline{N} - N_0 = -(A_1 + A_2 + A_3)(k/2\pi) + \tfrac{1}{2} + \overline{\mathcal{O}}(k^{-r}) \qquad (5.9)$$

in agreement with Ref. 8. The result for the *circular cylinder* with radius R is

$$\overline{N} - N_0 = -(\tfrac{4}{3}L + \pi R)(k/2\pi) + \tfrac{1}{2} + \overline{\mathcal{O}}((\ln k)^2) \qquad (5.10)$$

and is compatible with the $(1.4 \pm 0.2)L$ found by computation.[8] For sectoral cavities with sectoral angle Φ, length L, and radius R the second-order correction reads

$$N_2 = -\{R(1 + \tfrac{1}{2}\Phi) + L[\tfrac{3}{2} + \tfrac{1}{6}(\Phi/\pi - \pi/\Phi)]\}(k/2\pi) . \qquad (5.11)$$

This is compatible with the computer results[8] obtained for $\Phi = \pi$, $\frac{1}{2}\pi$, and $\frac{1}{4}\pi$, within the accuracy achieved by the computer program. We mention that the term (5.11) becomes positive for extremely small sectoral angles Φ only, i.e., for

$$\Phi \lesssim (\tfrac{1}{3}\pi)(3 + 2R/L)^{-1} . \qquad (5.12)$$

The mode density $D = dN/dk$ as obtained from (5.7) leads to shape and size correction of Planck's radiation formula. The accordingly refined Wien and Stefan–Boltzmann radiation formulas read as follows:

$$\frac{h\nu_{\max}}{k_B T} = 2.822 + 0.0094\,\frac{\Lambda}{V}\left(\frac{hc}{k_B T}\right)^2 , \qquad (5.13)$$

$$E(T) = \frac{4\sigma_0}{c} V T^4 - \frac{(\pi^2/6)\Lambda(k_B T)^2}{hc} + \frac{k_B T}{2} , \qquad (5.14)$$

where σ_0 denotes the Stefan–Boltzmann constant and c denotes the velocity of light. The length Λ is given by

$$\Lambda = \tfrac{1}{2}\gamma + (2 - 4C_1 - 4C_2)L . \qquad (5.15)$$

We notice that the last term $\frac{1}{2}k_B T$ in (5.14) is independent from the shape of the cavity. For the cube-shaped cavity with edge length A, we find $\Lambda = 3A$, and our formula (5.14) is consistent with the result derived by Case and Chiu[11] for this special case.

The difference between the E- and H-type mode numbers [Eqs. (5.3) and (5.6)] reveals an anisotropy of the radiation field and leads to corrections of the temporal coherence function.[8,13] The difference reads

$$\tilde{N} = \overline{N}_E - \overline{N}_H = (8\pi)^{-1}(S_\perp - S_\parallel)k^2 + Lk/\pi + C_1 + C_2 - \tfrac{1}{2} . \qquad (5.16)$$

We notice that the edge and curvature terms cancel in (5.16) whereas the k^2 and the constant terms are shape dependent now. That is, $\overline{N}$ comprehends the volume as well as the edges and curvatures of the cavity, whereas $\tilde{N}$ implies the surface and the corners. In order to obtain the full information it is therefore not sufficient to measure the total spectral intensity, but the "polarization" is compulsory as well. Thus we may say that it is easier to "hear the shape of an organ pipe" than to "see the shape of a blackbody."

VI. MULTIPLY CONNECTED CROSS SECTIONS

For a cross section g with smooth convex boundary ∂g showing p holes the constant term in $N^{\text{II}}_{D(N)}$ reads[23] $\frac{1}{6}(1-p)$. The according terms for the three-dimensional cylinder with holes parallel to the z direction read as

$$\overline{N}_{D(N)} - N_0 - N_1 = \left(\frac{(1-p)L}{6\pi} + \frac{k}{8\pi}\right)k \mp \frac{(1-p)}{12} \qquad (6.1)$$

for the scalar problems and

$$\overline{N} - N_0 = \left(\frac{(1-p)L}{3\pi} - \frac{L}{\pi} - \frac{\gamma}{4\pi}\right)k + \tfrac{1}{2} \qquad (6.2)$$

for the electromagnetic problem, where γ denotes the length of the total boundary ∂g of g. We notice that the connectivity does not appear in the constant term of (6.2). It appears however, in the constant term of the mode number difference

$$\tilde{N} - \tilde{N}_1 = Lk/\pi + \tfrac{1}{6}(1-p) - \tfrac{1}{2} . \qquad (6.3)$$

For a circular cylinder with a concentric hole of radius $\tilde{R}$, Eqs. (6.1) and (6.2) become

$$\tfrac{1}{4}(R + \tilde{R})k \quad \text{and} \quad -[L/\pi + \tfrac{1}{2}(R + \tilde{R})]k + \tfrac{1}{2} .$$

Apparently, the contribution of the surface curvature vanishes.

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[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953), Vol. I, Chap. 6.
[2] C. Clark, SIAM Rev. 9, 627 (1967).
[3] H. Niemeyer, Arch. Rat. Mech. Anal. 7, 412 (1961).
[4] B. V. Fedesov, Sov. Math. Doklady 5, 988 (1964).
[5] H. Weyl, Math. Ann. 71, 441 (1911); and J. Reine Angew. Math. 141, 163 (1912); 143, 177 (1913).
[6] E. Hilf, dissertation (Universität Frankfurt, 1967)

(unpublished); Z. Naturforsch. 25a, 1190 (1970).
[7]R. Balian and C. Bloch, Ann. Phys. (N.Y.) 60, 401 (1970).
[8]H. P. Baltes, R. Muri, and F. K. Kneubühl, *Submillimeter Waves* (Press of Polytechnic Institute of Brooklyn, New York, 1970), p. 667; and H. P. Baltes, Helv. Phys. Acta (to be published).
[9]A. Pleijel, Arkiv f. Matematik 2, 553 (1954).
[10]F. H. Brownell, J. Math. Mech. 6, 119 (1957).
[11]K. M. Case and S. C. Chiu, Phys. Rev. A 1, 1170 (1970).
[12]R. Balian and C. Bloch, Ann. Phys. (N.Y.) 64, 271 (1971).
[13]H. P. Baltes and F. K. Kneubühl, Opt. Commun. 4, 9 (1971).
[14]D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).
[15]A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinsky, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, London, 1963); W. A. Harrison, *Solid State Theory* (McGraw-Hill, New York, 1970), pp. 216–219.
[16]K. Stewardson and R. T. Waechter, Proc. Cambridge Phil. Soc. 69, 353 (1971).
[17]J. P. McKean and I. M. Singer, J. Differential Geom. 1, 43 (1967).
[18]O. Weis, Z. Angew. Phys. 26, 325 (1969).
[19]P. Herth and O. Weis, Acustica 21, 162 (1969).
[20]P. Herth and O. Weis, Z. Angew. Phys. 29, 101 (1970).
[21]I. Malecki, *Physical Foundations of Technical Acoustics* (Pergamon Press, London, 1969), p. 554.
[22]K. Pöschl, *Mathematische Methoden in der Hochfrequenztechnik* (Springer, Berlin, 1956).
[23]M. Kac, Am. Math. Monthly 73/4 (Pt. II), 1 (1969).

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Waves in Inhomogeneous Magnetoplasmas

A. Sivasubramanian* and Ting-wei Tang
Department of Electrical Engineering, University of Massachusetts, Amherst, Massachusetts 01002
(Received 30 March 1972)

The linearized Vlasov equation for a hot inhomogeneous magnetoplasma is solved through the particle-orbit theory using the techniques of Fourier transforms. An analytical integral expression in $\vec{k}$ space is obtained for the current density for waves propagating across the static magnetic field. When inverse Fourier transformed, it gives rise to a differential expression which is accurate up to order $(r_c/\lambda)^2$, $(r_c^2/\lambda L)$, and $(r_c/L)^2$. The integral equation is solved numerically for the problems of Buchsbaum–Hasegawa resonances and the Bernstein modes propagating in an inhomogeneous magnetoplasma.

I. INTRODUCTION

The literature on the kinetic theory of waves in inhomogeneous plasmas is relatively scant in comparison with the existing theories for homogeneous plasmas, and also in comparison with the numerous experiments that have been carried out on inhomogeneous plasmas. As naturally occurring plasmas as well as laboratory plasmas are generally nonuniform, it is a natural curiosity and is of fundamental importance to ask whether one can develop a theory of inhomogeneous plasmas to fit the experimental situations better than that by the theory of homogeneous plasmas. Earlier, Buchsbaum and Hasegawa[1,2] derived an equation for the radial electrostatic modes in the positive column immersed in an axial static magnetic field to account for the observed absorption spectrum. A similar study was followed by Pearson.[3] Pearson developed a set of differential equations for the wave field correct up to the first order in r_c/λ and r_c/L, where r_c is the thermal-electron Larmor radius, λ is the effective wavelength, and L is the scale length of plasma inhomogeneity. More recently, Azevedo and Vianna[4] have used an expansion procedure to obtain similar equations that are valid up to the order of $r_c^2/\lambda L$. In a study on the interaction of quasilongitudinal and quasitransverse waves in an inhomogeneous Vlasov plasma, Hedrick[5] has also derived wave equations for the case of perpendicular wave propagation when the wave vector $\vec{k}$ lies parallel to the direction of inhomogeneity. But, as with Azevedo and Vianna's equations, his are also valid only to order $r_c^2/\lambda L$.

We have systematically developed an integral equation for the wave fields in an unbounded inhomogeneous magnetoactive plasma. Our approach, similar to that of Pearson, is through the particle-orbit theory. However, unlike Pearson, who used a power-series expansion in spatial coordinates before the velocity integrations were carried out, we first make a direct Fourier transform and later expand the result into a power series in wave number k. Such expansion enables us to identify the order of magnitude of each term easily when the equation is inverse Fourier transformed to the coordinate space. A similar approach has been used to study electrostatic modes