clearly demonstrates the effects of the time-dependent dissipation and mode coupling coefficients on the instantaneous time scale for the argument of the functions $U_j(\tau)$. From Eqs. (18) and (18a) we notice that the effects of a growing damping or a decrease in the nonlinear coupling coefficients are generally, as expected, to move the time value corresponding to a certain mode amplitude towards a higher value, and eventually to infinity for critical values of the time-dependent coefficients. The effect of, e.g., an increased damping in a limited time domain of the evolution of an explosive instability is therefore to flatten the curve in this region and to postpone the explosion. This is an interesting observation with respect to saturation effects originating, e.g., from higher-order nonlinear contributions.

It should be mentioned in conclusion, that it is a straightforward procedure to extend the methods that we have exploited here to a discussion of the evolution of explosively unstable systems consisting of more than three waves, i.e., to an n-wave system.

[1]P. A. Sturrock, Phys. Rev. Letters 16, 270 (1966).
[2]B. B. Kadomtsev, A. B. Mikhailovskii, and A. V. Timofeev, Zh. Eksperim. i Teor. Fiz. 47, 2266 (1964) [Sov. Phys. JETP 20, 1517 (1965)].
[3]V. M. Dikasov, L. I. Rudakov, and D. D. Ryutov, Zh. Eksperim. i Teor. Fiz. 48, 913 (1965) [Sov. Phys. JETP 21, 608 (1965)].
[4]R. Z. Sagdeev and A. A. Galeev, Lectures on the Non-Linear Theory of Plasma, Trieste, 1966 (unpublished).
[5]B. Coppi, M. N. Rosenbluth, and R. N. Sudan, Ann. Phys. (N.Y.) 55, 207 (1969).
[6]F. Engelmann and H. Wilhelmsson, Z. Naturforsch. 24a, 206 (1969).
[7]L. Stenflo and H. Wilhelmsson, Phys. Letters 29A, 217 (1969).
[8]A. Jarmén, L. Stenflo, H. Wilhelmsson, and F. Engelmann, Phys. Letters 28A, 748 (1969).
[9]H. Wilhelmsson, L. Stenflo, and F. Engelmann, J. Math. Phys. 11, 1738 (1970).
[10]H. Wilhelmsson and K. Östberg, Phys. Scripta (Sweden) 1, 267 (1970).
[11]H. Wilhelmsson, Phys. Scripta (Sweden) 2, 113 (1970).
[12]J. Weiland, J. Plasma Phys. (to be published).
[13]H. J. Hopman, Stanford University Internal Progress Report No. 402, 1970 (unpublished).
[14]H. J. Hopman, Oxford Conference on Ionized Gases, 1971 (unpublished).

PHYSICAL REVIEW A VOLUME 6, NUMBER 5 NOVEMBER 1972

Plane Waves of Constant Amplitude in Nonlinear Dielectrics

M. M. Carroll
Division of Applied Mechanics, University of California, Berkeley, California 94720
(Received 22 May 1972)

Two rather unusual effects which are associated with the propagation of plane waves of constant amplitude in nondispersive isotropic nonlinear electromagnetic media are presented. One is the distortion-free transmission of information through a nonlinear medium, by "polarization modulation" of a plane wave of constant amplitude. The other is the reflection and refraction of a monochromatic wave, obliquely incident on the plane interface between a linear medium and a nonlinear one, so that the transmitted wave, in the nonlinear medium, is also monochromatic. These effects are described by exact solutions of the nonlinear system of equations and they should be of interest, in particular, in laser physics.

I. INTRODUCTION

In a previous paper,[1] it was shown that a *circularly polarized* monochromatic plane electromagnetic wave can propagate in a nondispersive isotropic nonlinear medium just as it does in a linear medium. The problem of the reflection and transmission of such waves at normal incidence on the plane interface between a linear medium and a nonlinear one was solved by elementary means, requiring only the solution of an algebraic equation for the amplitude of the transmitted wave (in the nonlinear medium). This transmitted wave is also circularly polarized and it is monochromatic, which is rather surprising.

The existence of a more general wave of *constant amplitude*, of which the circularly polarized wave is a special case, was pointed out in a footnote in Ref. 1. The primary purpose of this present paper is to draw attention to the unusual properties of waves of constant amplitude and especially to point to their possible use for distortion-free transmission of information through a nonlinear medium. There is also the hope that it might lead someone to do the experiments which would check the validity of the theoretical result (which is based on a plane-wave analysis) for laser beams.

Our secondary purpose is to consider the possibility of transmitting a monochromatic wave to a nonlinear medium at *oblique* incidence. This can, in fact, be done, as shown by the exact solution obtained for arbitrary angle of incidence or refraction. The polarization of the incident wave must be suitably chosen (the choice depends on the angle of incidence, the incident wave amplitude and the properties of the media) and the reflected and transmitted waves are monochromatic, the latter wave being circularly polarized. The situation, although somewhat artificial, is experimentally realizable and possibly useful. As an exact solution of the reflection-refraction problem, involving a nonlinear medium and oblique incidence, it is a curiosity. Indeed, the only other exact solutions that I know of are the special case of normal incidence given in Ref. 1 and Broer's result[2] for a linearly polarized wave at normal incidence. A systematic approach to propagation and reflection-refraction problems in nonlinear media, which allows for dispersion and anisotropy and involves successive approximations to the nonlinear constitutive equations, may be found in Ref. 3.

The analysis in Ref. 1 was carried out for a general (nondispersive and isotropic) electromagnetic medium. The present paper deals with a nonmagnetic dielectric but corresponding results can be readily obtained for the more general medium.

II. PLANE WAVES OF CONSTANT AMPLITUDE

Let $\vec{i}$, $\vec{j}$ and $\vec{k}$ be the orthonormal base vectors associated with a rectangular Cartesian coordinate system (x, y, z). Consider an electromagnetic field for which the electric vector $\vec{E}$ and the magnetic vector $\vec{H}$ are of the form

$$\vec{E} = e\cos\varphi\,\vec{i} + e\sin\varphi\,\vec{j}, \quad \vec{H} = -h\sin\varphi\,\vec{i} + h\cos\varphi\,\vec{j}, \tag{1}$$

where φ is some function of the variable $t - z/c$,

$$\varphi = \hat{\varphi}(t - z/c)\ , \tag{2}$$

and e, h, and c are constants. Equations (1) and (2) describe a plane wave of constant amplitude propagating in the positive z direction with speed c. If $\hat{\varphi}$ is a linear function then the wave is circularly polarized.

It is well known that such a wave can propagate in every nondispersive homogeneous isotropic linear dielectric medium. It was shown in Ref. 1 that the same is true for nonlinear media described by general constitutive relations

$$\vec{D} = \hat{\vec{D}}(\vec{E}, \vec{B})\ , \quad \vec{H} = \hat{\vec{H}}(\vec{E}, \vec{B})\ , \tag{3}$$

the only difference being that the wave speed c depends on the constant amplitudes e and h. Indeed the result still holds in the presence of uniform static electric or magnetic fields applied in the z direction. In the present paper, for simplicity, we limit our attention to the case of a nonmagnetic dielectric with constitutive relation

$$\vec{D} = \hat{\epsilon}(E)\,\vec{E}\ , \quad \vec{H} = (1/\mu)\,\vec{B}\ , \tag{4}$$

where the magnetic permeability μ is constant and the effective "dielectric constant" is an even function of the electric field strength $E = (\vec{E}\cdot\vec{E})^{1/2}$. Since waves of constant amplitude (2) have constant field strength e, the response of the nonlinear medium (4) to such waves is precisely that of a linear medium with dielectric constant $\epsilon = \hat{\epsilon}(e)$. Thus, the wave (2) satisfies Maxwell's equations

$$\frac{\partial\vec{B}}{\partial t} + \mathrm{curl}\vec{E} = 0\ , \quad \mathrm{div}\vec{B} = 0\ ,$$
$$\frac{\partial\vec{D}}{\partial t} - \mathrm{curl}\vec{H} = 0\ , \quad \mathrm{div}\vec{D} = 0\ , \tag{5}$$

with $\vec{D}$ and $\vec{B}$ given by (4), provided

$$h = e\,[\hat{\epsilon}(e)/\mu]^{1/2}\ , \quad c = [\mu\hat{\epsilon}(e)]^{-1/2}\ . \tag{6}$$

The analysis in Ref. 1 dealt exclusively with circularly polarized waves, although the more general waves of constant amplitude were mentioned. These latter waves, however, are much more interesting, since they provide (in theory, at any rate) a means of sending information, without any distortion, through a nonlinear medium. This can be done by modulating the polarization of a wave of constant amplitude. Thus, if the directions of the vectors $\vec{E}$ and $\vec{H}$ in the xy plane, which are defined by the angle φ in (2), are subjected to time variation $\hat{\varphi}(t)$ at the station $z = 0$, while their magnitudes e and h [satisfying Eq. (6a)] are held constant, then the solution in the region $z > 0$ is given by (2) and (3). Consequently, the information $\hat{\varphi}(t)$, which might be called the polarization modulation, is transmitted without distortion through the nonlinear medium.

This should be a useful result. However its practical applicability may be limited, in particular, in view of the plane-wave assumption (2), which implies infinite extent in the xy plane and no transverse variation of the fields. The assumption of infinite extent in $t - z/c$ can be relaxed, for example, by considering square-wave forms. The assumption of zero dispersion in (4) should be a good one, in many situations. It would be of interest to check experimentally the extent to which beam width and the intensity profile across the beam might invalidate the result for high-intensity laser beams. Thus, one might check to see whether or not a third harmonic can be detected for circularly polarized beams. (It may well be that at the present state of technology there is still a greater interest in seeing a third harmonic than in not seeing one!) Even if a harmonic can be detected, it might also be of interest to investigate the effect of the polarization of a high-intensity beam on its harmonic generation. The theoretical result for circularly polarized plane waves suggests that the harmonic generation would be minimal, among all possible polarizations, at circular polarization, and possibly maximal at linear polarization, since these represent the extremes of minimal and maximal amplitude variations.

There is also the question of implementing the boundary condition at $z=0$. The solution given in Ref. 1 for transmission of a circularly polarized wave, at normal incidence, from a linear medium to a nonlinear one, applies equally for waves of constant amplitude. This may be relevant to the question of implementing the boundary condition at $z=0$, since it implies that if the polarization modulation can be effected in a linear medium it can then be transmitted to the nonlinear one.

III. REFLECTION AND REFRACTION AT PLANE INTERFACE

We now turn our attention to the problem of reflection and refraction at the plane interface $z=0$ between a linear medium, with dielectric constant ϵ_0 and magnetic permeability μ_0, say, occupying the region $z<0$, and a nonlinear medium, with the constitutive relations (4), occupying the region $z>0$. The incident wave is a monochromatic plane wave incident, at some angle θ, from the linear medium. We seek an elementary solution of the problem, using the simple propagation property of circularly polarized waves discussed in Sec. II.

For normal incidence ($\theta=0$), a simple solution obtains, involving circularly polarized incident, reflected, and transmitted waves. For oblique incidence, however, a circularly polarized incident wave will not give rise to a circularly polarized transmitted wave. Our approach is to find, for any given angle of incidence θ, those polarizations of the incident wave which would give rise to a circularly polarized transmitted wave at the interface between two linear media. These linear solutions can then be used to obtain elementary solutions of the nonlinear problem.

So, we consider first the situation in which the region $z>0$ is occupied by a linear medium with dielectric constant ϵ and magnetic permeability μ. The reflection-refraction problem is easily solved (see, for example, Stratton[4]). The electric vectors $\vec{E}$, $\vec{E}'$, and $\vec{E}''$ for the incident, reflected, and transmitted waves, respectively, are given by

$$\vec{E}=[P(\vec{n}\times\vec{j})+Q\vec{j}]\exp[-i\omega(t-\vec{n}\cdot\vec{r}/c_0)]\,, \tag{7a}$$

$$\vec{E}'=[\alpha' P(\vec{n}'\times\vec{j}\,')+\beta' Q\vec{j}]\exp[-i\omega(t-\vec{n}'\cdot\vec{r}/c_0)]\,, \tag{7b}$$

$$\vec{E}''=[\alpha'' P(\vec{n}''\times\vec{j})+\beta'' Q\vec{j}]\exp[-i\omega(t-\vec{n}''\cdot\vec{r}/c)] \tag{7c}$$

[adopting the usual convention in which the vectors $\vec{E}$, $\vec{E}'$, and $\vec{E}''$ are the real parts of the quantities on the right-hand side of Eq. (7)], where the propagation direction vectors $\vec{n}$, $\vec{n}'$, and $\vec{n}''$ for the incident, reflected, and transmitted waves are given by

$$\begin{aligned}\vec{n}&=\sin\theta\,\vec{i}+\cos\theta\,\vec{k}\,,\\ \vec{n}'&=\sin\theta\,\vec{i}-\cos\theta\,\vec{k}\,,\\ \vec{n}''&=\sin\theta''\,\vec{i}+\cos\theta''\,\vec{k}\,.\end{aligned} \tag{8}$$

All three waves have angular frequency ω, $c_0[=1/(\epsilon_0\mu_0)^{1/2}]$ and $c[=1/(\epsilon\mu)^{1/2}]$ are the velocities of light in the two media, and the angles of incidence and reflection are related through Snell's law

$$c\sin\theta=c_0\sin\theta''\,, \tag{9}$$

which serves to determine θ'' for assigned value of θ or *vice versa*. The reflection and transmission coefficients α' and α'' for waves polarized in the plane of incidence, and the corresponding coefficients β' and β'' for waves polarized normal to the plane of incidence, are given by

$$\alpha'=\frac{\mu_0 c_0\cos\theta-\mu c\cos\theta''}{\mu_0 c_0\cos\theta+\mu c\cos\theta''}\,, \tag{10a}$$

$$\alpha''=\frac{2\mu c\cos\theta}{\mu_0 c_0\cos\theta+\mu c\cos\theta''} \tag{10b}$$

and

$$\beta'=\frac{\mu c\cos\theta-\mu_0 c_0\cos\theta''}{\mu c\cos\theta+\mu_0 c_0\cos\theta''}\,, \tag{11a}$$

$$\beta''=\frac{2\mu c\cos\theta}{\mu c\cos\theta+\mu_0 c_0\cos\theta''}\,. \tag{11b}$$

The transmitted wave (7c) is circularly polarized if

$$Q=\pm i\,\alpha'' P/\beta''\,. \tag{12}$$

This equation, together with (7a) and the relation

$$\frac{\alpha''}{\beta''}=\frac{\mu c\cos\theta+\mu_0 c_0\cos\theta''}{\mu_0 c_0\cos\theta+\mu c\cos\theta''}\,, \tag{13}$$

describes the incident waves which give rise to circularly polarized transmitted waves. For each assigned value of θ or θ'', such that all quantities in (9) are real, they are a right and a left elliptically polarized wave, whose eccentricity (α''/β'' or its inverse) depends on the properties of the medium and on the assigned angle of incidence or refraction. We may assume, without loss in generality, that P is real, and the amplitude of the circularly polarized transmitted wave is then given by

$$(\vec{E}''\cdot\vec{E}'')^{1/2} = \alpha'' P \,. \tag{14}$$

We are now ready for the nonlinear problem. The solution described by Eqs. (7)–(12) remains valid, with

$$\epsilon = \hat{\epsilon}(e)\,, \quad e = \alpha'' P \,. \tag{15}$$

Equations (10b) and (15) give an equation

$$e = \frac{2\mu P c(e)\cos\theta}{\mu_0 c_0 \cos\theta + \mu c(e)\cos\theta''} \tag{16}$$

for the amplitude e of the transmitted wave, where P and θ are assigned and $c(e)$ and θ'' are defined by Eqs. (6b) and (9), respectively. This equation may also be written as

$$\left(\frac{\mu_0\hat{\epsilon}(e)}{\mu\epsilon_0}\right)^{1/2} + \left(\sec^2\theta - \frac{\mu_0\epsilon_0}{\mu\hat{\epsilon}(e)}\tan^2\theta\right)^{1/2} = \frac{2P}{e} \,. \tag{17}$$

When (17) is solved for e the effective dielectric constant $\hat{\epsilon}(e)$ of the nonlinear medium is determined. It should now be verified that the angles of incidence and refraction are such that

$$-1 \leq [c(e)/c_0]\sin\theta \leq 1 \quad \text{and} \quad -1 \leq [c_0/c(e)]\sin\theta'' \leq 1 \,. \tag{18}$$

The eccentricity of the elliptically polarized incident wave for which the elementary solution applies is given by (13). It depends on the properties of the media and on the angle of incidence and the amplitude of the incident wave, except in the case of normal incidence ($\theta = \theta'' = 0$) for which the incident wave must be a right or left circularly polarized wave. The electric fields for the incident, reflected and transmitted waves are given by (7) and the associated $\vec{B}$, $\vec{D}$, and $\vec{H}$ fields are the same as for the linear problem.

We have thus shown that a monochromatic circularly polarized wave can be transmitted, at oblique incidence, to a nonlinear medium. We remark that the result just obtained applies equally for waves of constant amplitude. Furthermore, similar results hold for general electromagnetic media described by the constitutive relations (8).

ACKNOWLEDGMENT

It is a pleasure to acknowledge my gratitude to the School of Theoretical Physics, Dublin Institute for Advanced Studies.

[1]M. M. Carroll, Quart. Appl. Math. 25, 319 (1967).

[2]L. J. F. Broer, Phys. Letters 4, 65 (1963).

[3]N. Bloembergen, *Nonlinear Optics* (Benjamin, New York, 1965).

[4]J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).