straight lines drawn through the origin by less than 5%." This statement does not appear to be strictly true. Figure 7 shows the range in units of $(Z^2/A)R$ as a function of the velocity of the projectile where Z is the atomic number of the projectile and A is its atomic weight. The results for N ions show a break from the straight line through the origin at about 3.5×10^8 cm/sec and for Na ions at about 2.7×10^8 cm/sec. For Ar ions the present results do not go high enough in velocity to ascertain the exact situation, but they nevertheless suggest that a break might occur at 2.7×10^8 cm/sec. Since the range is zero at zero velocity, the slope of each curve must decrease as $V=0$ is approached, as is demonstrated by observing the low-energy curve of Evans *et al.* for Ar ions in N_2. It appears that the straight-line-through-the-origin approximation of Teplova *et al.* is not correct below a velocity of about 3×10^8 cm/sec; at higher velocities it appears to be valid, but it is not expected to be correct at velocities higher than the maximum of the stopping-power curve.[1]

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Multiphoton Ionization of Hydrogen Atoms in the Semiclassical Treatment of an Intense Radiation Field

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The transition probability for the multiphoton ionization of the ground-state hydrogen atom by an intense electromagnetic field has been calculated following a nonperturbative method developed by Reiss. One distinctive feature of the calculation is a minor reduction in the power-law dependence of the interaction parameter in the transition probability.

I. INTRODUCTION

Recently Bebb and Gold[1] calculated the ionization probability of the hydrogen atom by simultaneous absorption of several photons using the perturbation technique. Although the perturbation technique is useful in ordinary quantum electrodynamics, its application to the interaction of an intense radiation field with matter suffers from difficulties. For instance, at high intensities of the electromagnetic field, higher-order corrections cannot be tackled in the perturbative way, because the expansion parameter here is eA/E, which could be as large as ten, e.g.,

$$\frac{eA}{E} \approx O\left[\frac{1}{\alpha^2}\left(\frac{\Delta m^2}{m^2}\right)^{1/2}\right],$$

where α is the fine-structure constant ($\alpha \approx 10^{-2}$), and $\Delta m^2/m^2$ is the free-electron intense-field parameter, which can be as high as 10^{-6} for presently available intense radiation field. This as compared to $\frac{1}{137}$ in ordinary quantum electrodynamics is large. To avoid this difficulty Reiss[2] has developed an approximate method which is particularly suited for the study of the interaction of the intense radiation field with matter. It is based, essentially, upon a unitary transformation, which takes account of the major contribution due to the electromagnetic field, leaving out terms that can be neglected in the long-wavelength approximation. The accuracy of the method increases when the number of photons involved in the transition is large. In this paper the multiphonon ionization of the hydrogen atom using Reiss's method is discussed.

Section II includes a brief discussion of Reiss's method and calculations of the amplitude for the ionization of the general bound-state hydrogen atom by absorption of n photons of the intense radiation field. The result is displayed in the form of a sum over allowed combinations of the three angular momenta, viz., the initial angular momentum l_i of the bound state, the intermediate angular momentum l_A occurring in the partial-wave expansion of $e^{ie\vec{A}\cdot\vec{x}}$, and the final angular momentum l_f of the outgoing free plane-wave electron. Each term of the sum is an infinite series in the interaction parameter $e^2a^2a_0^2$, where a is the amplitude of electromagnetic field and a_0 is the first Bohr radius of the hydrogen atom. The coefficients of this series are functions of parameter $p^2a_0^2$, where p is the momentum of the outgoing electron. The series represents the sum of all orders of the interaction parameter $e^2a^2a_0^2$ exhibiting the presence of the lowest-order as well as the higher-order correction terms.

In Sec. III, the ionization of the ground-state hydrogen atom is discussed. It is indicated that the presence of the higher orders tends to reduce the power-law dependence of the probability of ionization. Numerical calculations, performed for the estimation of the reduction in power-law dependence in the multiphonon ionization probability of the ground-state hydrogen atom by the absorption of eight photons of the Ruby-laser beam at various intensities, are presented. The results for different angular momentum combinations [running from $(0, l_A, l_f) = (0, 0, 0)$ to $(0, 14, 14)$] shown in Fig. 1 indicate that the pronounced effect of the presence of higher orders appears in the neighborhood, and above a certain value of the interaction parameter [different for different $(0, l_A, l_f)$ values], at which the amplitude vanishes. Just before this value of the interaction parameter, a reduction in the power law is observed, while above this value a sudden enhancement is noted. The total probability of ionization shows a minor reduction in the power-law dependence.

In Sec. IV the results are summarized and a few improvements over the present calculations suggested.

II. MULTIPHONON IONIZATION

A. Reiss's Method

In the presence of an external electromagnetic field $\vec{A}$, the Schrödinger equation for an electron bound by a Coulomb potential $V(r)$ is given by[3]

$$i\frac{\partial}{\partial t}\psi = \left(\frac{p^2}{2m} - \frac{e}{m}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2m} + V(r)\right)\psi . \tag{1}$$

With the help of the unitary transformation $\psi \to \overline{\psi} = e^{-ie\vec{A}\cdot\vec{x}}\psi$, Eq. (1) can be rewritten as

$$i\frac{\partial}{\partial t}\overline{\psi} = \left(\frac{p^2}{2m} + V(r)\right)\overline{\psi} . \tag{2}$$

In writing down Eq. (2), terms like $\partial_t(\vec{A}\cdot\vec{x})$ and $\vec{\nabla}(\vec{A}\cdot\vec{x})$ have been neglected. In the long-wavelength approximation, i.e., when $\omega/E \ll 1$, where ω is single-photon energy of the radiation field and E is the transition energy in question, such approximation is well justified. Solution of Eq. (2) will then give a bound-state wave function in presence of the field $\vec{A}$. The S-matrix element of relevant transitions involving the states satisfying the dynamical equation (1) can now be developed using the time-development-operator formalism. Following the calculations of Reiss, the bound-to-bound transition-matrix element is as follows:

$$\tau_{fi} = -i\int_{-\infty}^{+\infty} dt\, e^{i(E_f-E_i)t}(\phi_f, H'\, e^{ie\vec{A}\cdot\vec{x}}\phi_i) , \tag{3a}$$

$$\tau_{fi} \simeq -i\,(E_i - E_f)\int_{-\infty}^{+\infty} dt\, e^{i(E_f-E_i)t}\,(\phi_f, e^{ie\vec{A}\cdot\vec{x}}\phi_i) , \tag{3b}$$

where in Eq. (3a)

$$H' = -\frac{e}{m}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2m} .$$

In going from Eq. (3a) to (3b) use has been made of a commutator theorem, which is valid in the long-wavelength approximation of an intense radiation field. ϕ_i and ϕ_f are time-independent initial and final bound-state solutions of Eq. (2). The presence of $e^{ie\vec{A}\cdot\vec{x}}$ in the amplitude indicates the presence of multiple powers of $\vec{A}$, showing the nonlinear character of an intense radiation field in the transition.

The matrix element for the process of multiphonon ionization is similarly obtained and is given by ex-

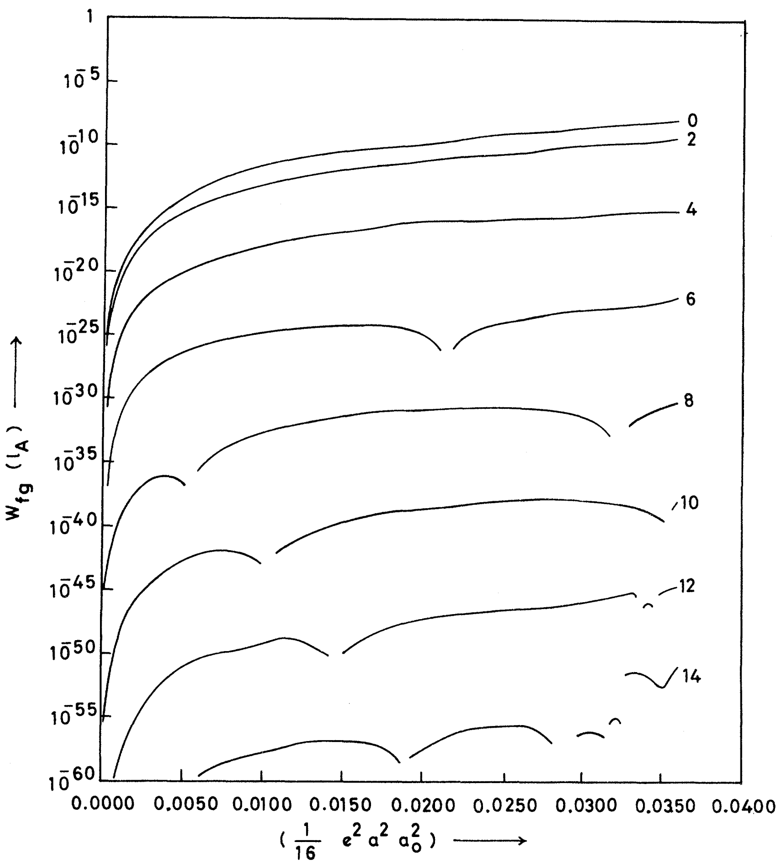

FIG. 1. Results of Eq. (16) for $l_A = 0, 2, 4, \ldots, 14$ are shown. The discontinuities in curves for a particular l_A value show the points where contribution to the probability of transition vanishes.

pression (3b) with final bound-state wave function ϕ_f replaced by free-electron wave function ϕ_f^{free}.

B. Ionization Amplitude

In this section the multiphonon ionization amplitude for the hydrogen atom is calculated. The bound-state wave function ϕ_i which is required for the calculations is given by[4]

$$\phi_i = U_{n_i l_i m_i}(r, \theta, \varphi)$$

$$= Q(n_i, l_i)(2\beta_i r)^{l_i} e^{-\beta_i r} L_{n_i+l_i}^{2l_i+1}(2\beta_i r) Y_{l_i m_i}(\theta, \varphi) , \tag{4}$$

where

$$Q(n_i, l_i) = -\left((2\beta_i)^3 \frac{(n_i - l_i - 1)!}{2n_i[(n_i + l_i)!]^3}\right)^{1/2} ,$$

$$\beta_i = Z/n_i a_0 ,$$

$$L_{n_i+l_i}^{2l_i+1}(2\beta_i r) = \sum_{k_i=0}^{n_i-l_i-1} P(n_i, l_i, k_i)(2\beta_i r)^{k_i} ,$$

and

$$P(n_i, l_i, k_i) = (-1)^{k_i+2l_i+1} \times \frac{[(n_i + l_i)!]^2}{(n_i - l_i - k_i - 1)!\,(2l_i + k_i + 1)!\,k_i!} ,$$

when n_i, l_i, m_i are quantum numbers of the initial bound-state hydrogen atom. The final free-electron state is taken as the usual plane wave

$$\phi_f^{\text{free}^*} = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{p}\cdot\vec{x}} ,$$

the partial-wave expansion of which is

$$\phi_f^{\text{free}^*} = \frac{4\pi}{(2\pi)^{3/2}} \sum_{l_f=0}^{\infty} (-i)^{l_f} j_{l_f}(pr) \times \sum_{m_f=-l_f}^{+l_f} Y_{l_f m_f}(\theta_p \varphi_p) Y^*_{l_f m_f}(\theta, \varphi) , \tag{5}$$

where

$$\vec{p} = (p, \theta_p, \varphi_p), \quad \vec{r} = (r, \theta, \varphi) .$$

Further assuming $\vec{A}$ to be in the Z direction and $\vec{A} = \hat{Z}a \cos\omega t$ when a is the amplitude, $e^{ie\vec{A}\cdot\vec{x}}$ can be expanded as

$$e^{ie\vec{A}\cdot\vec{x}} = \sum_{l_A=0}^{\infty} [4\pi(2l_A + 1)]^{1/2} i^{l_A} j_{l_A}(eAr) Y_{l_A 0}(\theta, \varphi) . \tag{6}$$

Using Eqs. (4)–(6), the amplitude for the ionization of the hydrogen atom in the n_i, l_i, m_i state by absorbing photons of intense radiation field can be written as[5,6]

$$(\phi_f^{\text{free}}, e^{ie\vec{A}\cdot\vec{x}}\phi_i) = \frac{4\pi}{(2\pi)^{3/2}} \sum_{l_f=0}^{\infty} (-i)^{l_f} Y_{l_f m_i}(\theta_p \varphi_p) \sum_{l_A=0}^{\infty} [4\pi(2l_A+1)]^{1/2} i^{l_A}$$
$$\times \left(\frac{(2l_i+1)(2l_A+1)}{4\pi(2l_f+1)}\right)^{1/2} c(l_i l_A l_f; m_i 0) c(l_i, l_A l_f; 00) Q(n_i l_i)(2\beta_i)^{l_i}$$
$$\times \sum_{k_i=0}^{n_i-l_i-1} P(n_i, l_i, k_i)(2\beta_i)^{k_i} \frac{\pi p^{l_A+l_f}}{2^{l_A+l_f+2}(\beta_i^{l_i+k_i+l_f+l_A+3})\Gamma(l_A+\frac{3}{2})}$$
$$\times \left(\frac{eA}{p}\right)^{l_A} \sum_{m_J=0}^{\infty} \frac{\Gamma(l_i+k_i+l_f+l_A+3+2m_J)}{\Gamma(m_J+1)\Gamma(l_f+m_J+\frac{3}{2})} (-\tfrac{1}{4}p^2\beta_i^{-2})^{m_J} {}_2F_1(-m_J, -m_J-l_f-\tfrac{1}{2}; l_A+\tfrac{3}{2}; e^2A^2/p^2), \quad (7)$$

where ${}_2F_1$ is a finite polynomial of degree m_J given by

$$\sum_{r=0}^{m_J} \frac{(-m_J)_r(-l_f-m_J-\frac{1}{2})_r}{r!\,(l_A+\frac{3}{2})_r} \frac{e^2A^2}{p^2} .$$

It is to be noted that because of the presence of the Clebsch-Gordan coefficients, the matrix elements [Eq. (7)] vanish unless the following conditions hold:

$$|l_i - l_f| \leq l_A \leq |l_i + l_f| ,$$
$$l_i + l_A + l_f = \text{even integer}, \quad (7')$$
$$m_i = m_f \text{ [already used in Eq. (7)]} .$$

The time integration in (3a) is performed by collecting terms proportional to $e^{\pm in\omega t}$ in Eq. (7) with the result

$$\tau_{fi} = -2\pi i(E_i - E_f)\textstyle\sum_n \delta(E_f - E_i \pm n\omega)\, T_{fi}^{(\pm n)} . \quad (8)$$

$T_{fi}^{(\pm n)}$ can be easily obtained by noting that for

$$A = a\cos\omega t = \tfrac{1}{2}a(e^{i\omega t} + e^{-i\omega t})$$

one can write

$$\left(\frac{eA}{p}\right)^{2r+l_A} = \left(\frac{ea}{2p}\right)^{2r+l_A} \sum_{S=0}^{2r+l_A} \binom{2r+l_A}{S} Z^{2S-2r-l_A}, \quad (9)$$

where $Z = e^{i\omega t}$.

For n-photon absorption, one needs only to know the coefficient of Z^{-n}. The relevant coefficient from above is

$$\left(\frac{ea}{2p}\right)^{2r+l_A} \frac{\Gamma(2r+l_A+1)}{\Gamma(r+\frac{1}{2}l_A-\frac{1}{2}n+1)\Gamma(r+\frac{1}{2}l_A+\frac{1}{2}n+1)} , \quad (10)$$

with $r+\frac{1}{2}l_A-\frac{1}{2}n \geq 0$, since index S in Eq. (9) can have only positive integral values. This implies that if the absorbed photons are odd (even), the intermediate angular momentum must also be odd (even). Furthermore, if $l_A > n$, r can go from 0 to m_J; but if $l_A < n$, r can start from $r = \frac{1}{2}|n - l_A|$ only.

Using this condition the expression for the amplitude can be written in new indices M and K defined by

$$m_J = M + \tfrac{1}{2}n - \tfrac{1}{2}l_A ,$$
$$r = K + \tfrac{1}{2}n - \tfrac{1}{2}l_A . \quad (11)$$

The new indices M and K can start from 0 or $\frac{1}{2}|l_A - n|$ provided that $l_A < n$ or $l_A > n$, whichever the case may be. Using less than (<) and greater than (>) signs as subscript for $l_A < n$ and $l_A > n$ cases, respectively, the matrix element for the photoionization by absorption of n photons reads

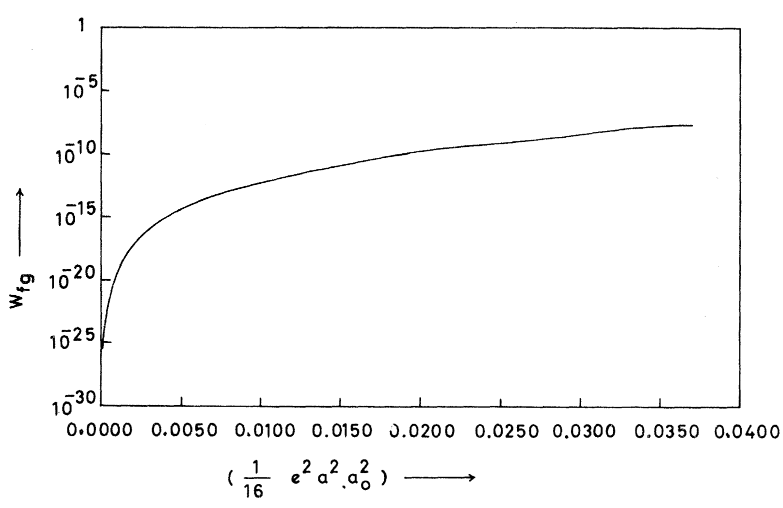

FIG. 2. Total transition probability [Eq. (16)] is plotted. No discontinuities occur.

$$\tau_{fi}^{(-n)} = -2\pi i(E_i - E_f)\delta(E_f - E_i - n\omega)\sum_{l_f=0}^{\infty}(-i)^{l_f} Y_{l_f m_i}(\theta_p\varphi_p)$$
$$\times \sum_{l_A=0}^{\infty}[4\pi(2l_A+1)]^{1/2} i^{l_A}\left(\frac{(2l_i+1)(2l_A+1)}{4\pi(2l_f+1)}\right)^{1/2} c(l_i l_A l_f; m_i 0)c(l_i l_A l_f;\ 00)$$
$$\times Q(n_i,\ l_i)(2\beta_i)^{l_i}\sum_{k_i=0}^{n_i-l_i-1} P(n_i,\ l_i,\ k_i)(2\beta_i)^{k_i}\frac{\pi}{4}\frac{(\frac{1}{4}p^2\beta_i^{-2})^{l_f/2}(-1)^{(n-l_A)/2}}{\beta_i^{l_i+k_i+3}\Gamma(l_A+\frac{3}{2})}$$
$$\times\Bigg[(\tfrac{1}{4}p^2\beta_i^{-2})^{n/2}(\tfrac{1}{4}e^2a^2/p^2)^{n/2}\sum_{\substack{M_<=0\\ M_>=(1/2)|l_A-n|}}^{\infty}\Gamma(l_i+k_i+l_f+n+3+2M)(-\tfrac{1}{4}p^2\beta_i^{-2})^M$$
$$\times\sum_{\substack{K_<=0\\ K_>=(1/2)|l_A-n|}}^{M}\frac{\Gamma(2K+n+1)}{\Gamma(M-K+1)\Gamma(M-K+l_f+\frac{3}{2})\Gamma(K+\frac{1}{2}n-\frac{1}{2}l_A+1)\Gamma(K+\frac{1}{2}n+\frac{1}{2}l_A+\frac{3}{2})}\frac{(e^2a^2/4p^2)^K}{\Gamma(K+n+1)\Gamma(K+1)}\Bigg]. \quad (12)$$

The series in square brackets in Eq. (12) can be rearranged to give a double series in terms of two independent parameters, viz., the interaction parameter $\frac{1}{16}e^2a^2\beta_i^{-2}$ and the parameter $\frac{1}{4}p^2\beta_i^{-2}$ with the result

$$(\tfrac{1}{16}e^2a^2\beta_i^{-2})^{n/2}\sum_{\substack{K_<=0\\ K_>=(1/2)|l_A-n|}}^{\infty}\frac{\Gamma(2K+n+1)(-\frac{1}{16}e^2a^2\beta_i^{-2})^k}{\Gamma(K+\frac{1}{2}n-\frac{1}{2}l_A+1)\Gamma(K+\frac{1}{2}n+\frac{1}{2}l_A+\frac{3}{2})\Gamma(K+n+1)\Gamma(K+1)}$$
$$\times\sum_{v=0}^{\infty}\frac{\Gamma(2K+l_i+k_i+l_f+n+3+2v)}{\Gamma(v+1)\Gamma(v+l_f+\frac{3}{2})}(-\tfrac{1}{4}p^2\beta_i^{-2})^v. \quad (13)$$

Equation (13) tells us that the least power of the interaction parameter $\frac{1}{16}e^2a^2\beta_i^{-2}$ involved in amplitude of the n-photon absorption process is $\frac{1}{2}n$, which is consistent with the ordinary perturbation theory. All other higher powers are the correction terms to the lowest-order term.

The series in index v of Eq. (13) can be summed up. This can be done by first writing each series in its hypergeometric form and then utilizing a well-known transformation,

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ x) = (1-x)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\ \gamma-\beta;\ \gamma;\ x).$$

Equation (13) transforms to the result

$$(\tfrac{1}{16}e^2a^2\beta_i^{-2})^{n/2}\sum_{\substack{K_<=0\\ K_>=(1/2)|l_A-n|}}^{\infty}\frac{2^{n+l_i+k_i+2+2K}}{\pi^{1/2}\Gamma(l_A+\frac{3}{2})}\frac{\Gamma(2K+n+1)}{\Gamma(K+1)\Gamma(K+n+1)}\frac{\Gamma[K+\frac{1}{2}(l_i+k_i+l_f+n+3)]\Gamma[K+\frac{1}{2}(l_i+k_i+l_f+n+4)]}{\Gamma(K+\frac{1}{2}n-\frac{1}{2}l_A+1)\Gamma(K+\frac{1}{2}n+\frac{1}{2}l_A+\frac{3}{2})}$$
$$\times\frac{(-\frac{1}{16}e^2a^2\beta_i^{-2})^K}{(1+p^2\beta_i^{-2})^{2K+l_i+k_i+l_f-l_A+n+2}}\,{}_2F_1[-K-\tfrac{1}{2}(l_i+k_i+l_f+n)+l_A,\ -K-\tfrac{1}{2}(l_i+l_f+k_i+n+1)+l_A;\ l_A+\tfrac{3}{2};\ -p^2\beta_i^{-2}]. \quad (14)$$

The ${}_2F_1$ involved in (14) are all polynomials. The convergence domain of the series in index K is given by

$$\lim_{K\to\infty}\left(\frac{16}{K^2}\frac{f_{K+1}}{f_K}\right)(\tfrac{1}{16}e^2a^2\beta_i^{-2}) < 1. \quad (14')$$

Here f_{K+1} and f_K are values of expressions involving $p^2\beta_i^{-2}$ for the $(K+1)$th- and Kth-order terms, respectively. Keeping only the first term of the polynomials, the domain of convergence is

$$[e^2a^2\beta_i^{-2}/(1+p^2\beta_i^{-2})^2] < 1.$$

Since this includes the presently available maximum value of $e^2a^2a_0^2 \simeq 0.01$, it is worthwhile to do some numerical computation. In Sec. III are presented numerical results for the ionization of ground-state hydrogen atoms.

III. GROUND-STATE IONIZATION

The ground-state ionization potential of the hydrogen atom is 13.6 eV. The minimum number of photons (each of energy 1.78 eV) required for ionization is eight. The energy of the outgoing electron is then 0.064 eV and the parameter $p^2a_0^2$ has the value 0.047. Since the number of photons is even, the intermediate angular momentum l_A must also be even.

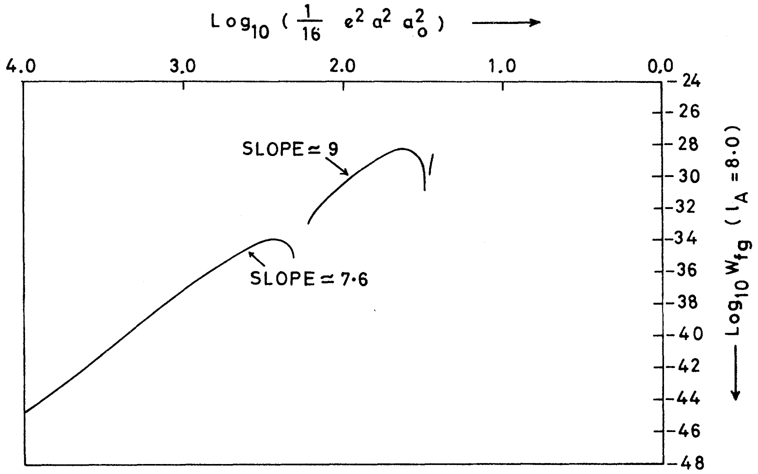

FIG. 3. Curve shows the effects on the power-law dependence owing to the presence of higher-order terms for l_A =8 term of Eq. (16). The slope of the curve gives the power-law dependence.

The amplitude for the ground-state ionization by absorption of n photons can be easily obtained by using the ground-state quantum numbers $n_i = 1$, $l_i = m_i = 0$ in the general expression [Eq. (12)]. Equation (7′) allows only $l_A = l_f$. Consequently only one of the l_A and l_f summations would contribute. The Clebsch–Gordan coefficients involved are all unity. So the ground-state ionization amplitude is

$$T_{fi}^{(-n)} = -2\pi i\delta(E_f - E_i - n\omega)(E_i - E_f)\frac{2\pi^2}{(2\pi)^{3/2}}a_0^{3/2}\sum_{l_A=0}^{\infty}\frac{(-1)^{(1/2)(n-l_A)}(2l_A+1)^{1/2}}{(\Gamma(l_A+\frac{3}{2}))^2}(\tfrac{1}{4}p^2a_0^2)^{l_A/2}S(l_A, n)Y_{l_A0}(\theta_p\varphi_p)\,, \tag{15}$$

where

$$S(l_A, n) = \frac{(\frac{1}{16}e^2a_2a_0^2)^{n/2}}{(1+p^2a_0^2)^{n+2}}\sum_{K_>=(1/2)|l_A-n|}^{\infty}\!\!\!\!\!\!{}_{K_<=0}\;\frac{\Gamma(2K+n+1)\Gamma(2K+l_A+n+3)}{\Gamma(K+\frac{1}{2}n-\frac{1}{2}l_A+1)\Gamma(K+\frac{1}{2}n+\frac{1}{2}l_A+\frac{3}{2})\Gamma(K+n+1)\Gamma(K+1)}$$

$$\times\left(\frac{-\frac{1}{16}e^2a^2a_0^2}{(1+p^2a_0^2)^2}\right)^K {}_2F_1(-K-\tfrac{1}{2}n+\tfrac{1}{2}l_A, -K-\tfrac{1}{2}n-\tfrac{1}{2}+\tfrac{1}{2}l_A;\; l_A+\tfrac{3}{2};\; -p^2a_0^2)\,.$$

The probability of ionization integrated over the full solid angle $\Omega_p(\theta_p\varphi_p)$ is

$$\omega_{fg} = 2\pi\left((E_i - E_f)^2\int|T_{fg}^{(-n)}|^2\,d\Omega_p\right)\delta(E_f - E_i - n\omega)\,,$$

$$= (2\pi)^2(E_i - E_f)^2a_0^3\sum_{l_A=0}^{\infty}\frac{(2l_A+1)}{[\Gamma(l_A+\frac{3}{2})]^4}$$

$$\times(\tfrac{1}{4}p^2a_0^2)^{l_A}|S(l_A, n)|^2\,. \tag{16}$$

An interesting experimental result in the multiphoton ionization of some noble gases is the reduction in the power dependence of the interaction parameter in the transition probability.[7] The power-law dependence of the transition probability given by Eq. (16) can be determined by using the formula

$$\text{power of } x = \frac{d\ln\omega_{fg}}{d\ln x}\,,\quad x = \tfrac{1}{16}e^2a^2a_0^2\,. \tag{17}$$

Taking, for example, only the $l_A = 0$ term Eq. (17) reduces to

$$\text{power of } x = n\frac{a_n a_{n+1} f_0 f_1 x}{a_n^2 f_0^2 - x a_n a_{n+1} f_0 f_1 + x^2 a_{n+1}^2 f_1^2}\cdots,$$

where a_n and a_{n+1} are coefficients of x^n and x^{n+1} in (16). Equation (18) predicts a reduction in the value of n for small values of x. Numerical results of Eq. (16) at various values of the interaction parameter are given in Figs. 1 and 2. Figure 1 shows the contribution to the probability of transition due to terms with $l_A = 0, 2, 4, \ldots, 14$. The results show that for a particular l_A, the contribution to the probability of ionization vanishes at certain values of the interaction parameter, e.g., for $l_A = 8$, the vanishing of the amplitude occurs at $x \approx 0.0050$ and $x \approx 0.0310$. Around such a point the power-law dependence for the term of a particular l_A first decreases and then shows an immediate increase as the value of the interaction parameter increases. To demonstrate this effect, $\ln\omega_{fg}(l_A = 8)$ vs $\ln x$ is plotted in Fig. 3. The slope of this curve gives the power-law dependence. The slope of the curve in Fig. 3 for $x < 0.0050$ is ≈ 7.6, while for $x > 0.0060$ is ≈ 9. This shows that in the neighborhood where the amplitude vanishes, the contributions from higher orders become significant.

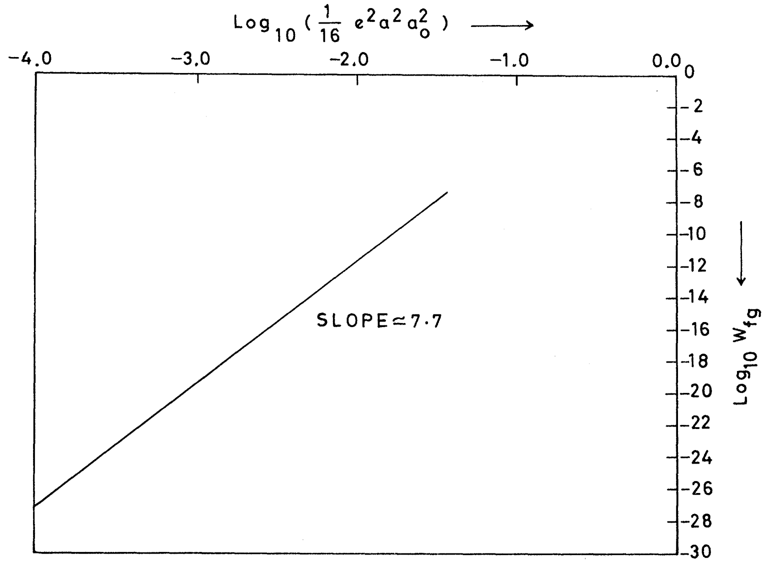

FIG. 4. Power-law dependence of the total probability of transition is exhibited.

Figure 2 gives the variation of the total transition probability at various intensities. No vanishing point occurs within the range of values of the interaction parameter considered; however, a reduction in power-law dependence occurs (see Fig. 4).

It is to be noted that there exists no restriction on the maximum value of the intermediate angular momentum l_A, and it is difficult from Eq. (16) to draw conclusions about the convergence of the sum over the angular momentum l_A. However, the results up to $l_A = 14$ in Fig. 1 tend to show a reasonable convergence of the l_A summation.

IV. SUMMARY

Numerical results obtained indicate a minor reduction in the power-law dependence of the interaction parameter in the total transition probability for ionization of the ground-state hydrogen atom by Ruby-laser light of photon energy 1.78 eV. In the present calculations, the effects that would arise due to the intermediate bound-to-bound transitions have not been taken account of. Also the final wave function of the outgoing electron that was used was the plane-wave function. The results would certainly be modified if one used the modified wave function of the outgoing electron in the presence of the intense electromagnetic field.[8]

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