duces the probability of multiple capture but enhances multiple loss. Little is known about the times which are necessary to complete the rearrangement processes of ions initially highly excited. When these times are longer than the times between two successive collisions, multiple-loss cross sections would to some extent change into excitation cross sections; this would particularly influence the balance of electron capture and loss by heavy ions penetrating through large molecules or solids. Given the major qualitative understanding of multiple-electron loss of heavy ions, one may hope that qualitatively satisfactory theories can be worked out in the future.

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[1]N. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 18, No. 8 (1948).

[2]G. I. Bell, Phys. Rev. 90, 548 (1953).

[3]N. Bohr and J. Lindhard, Kgl. Danske Vindenskab. Selskab, Mat.-Fys. Medd. 28, No. 7 (1954).

[4]See review by V. S. Nikolaev, Usp. Fiz. Nauk 85, 679 (1965) [Sov. Phys. Usp. 8, 269 (1965)].

[5]C. D. Moak, H. O. Lutz, L. B. Bridwell, L. C. Northcliffe, and S. Datz, Phys. Rev. 170, 427 (1968).

[6]N. Angert, B. Franzke, A. Möller, and Ch. Schmelzer, Phys. Letters 27A, 28 (1968).

[7]A. Möller, N. Angert, B. Franzke, and Ch. Schmelzer, Phys. Letters 27A, 621 (1968).

[8]S. Datz, H. O. Lutz, L. B. Bridwell, C. D. Moak, H. D. Betz, and L. D. Ellsworth, Phys. Rev. A 2, 430 (1970).

[9]H. D. Betz, G. Ryding, and A. B. Wittkower, Phys. Rev. A 3, 197 (1971).

[10]H. D. Betz, Rev. Mod. Phys. 44, 465 (1972).

[11]G. Ryding, A. B. Wittkower, and P. H. Rose, Phys. Rev. 185, 129 (1969).

[12]G. Ryding, H. D. Betz, and A. B. Wittkower, Phys. Rev. Letters 24, 123 (1970).

[13]H. D. Betz, Phys. Rev. Letters 25, 903 (1970).

[14]G. Ryding, A. Wittkower, and P. H. Rose, Phys. Rev. A 3, 1658 (1971).

[15]G. Ryding, A. B. Wittkower, and P. H. Rose, Phys. Rev. 184, 93 (1969).

[16]A. B. Wittkower and H. D. Betz, J. Phys. B 4, 1173 (1971).

[17]Q. C. Kessel, Phys. Rev. A 2, 1881 (1970).

[18]I. S. Dmitriev, V. S. Nikolaev, L. N. Fateeva, and Ya. A. Teplova, Zh. Eksperim. i Teor. Fiz. 43, 361 (1962) [Sov. Phys. JETP 16, 259 (1963)].

[19]U. Fano and W. Lichten, Phys. Rev. Letters 14, 627 (1965).

[20]W. Lichten, Phys. Rev. 164, 131 (1967).

[21]W. Brandt and R. Laubert, Phys. Rev. Letters 24, 1037 (1970).

[22]H. D. Betz, J. P. Delvaille, K. Kalata, H. W. Schnopper, A. R. Sohval, K. W. Jones, and H. E. Wegner, in *Proceedings of the International Conference on Inner Shell Ionization Phenomena, Atlanta*, 1972 (North-Holland, Amsterdam, to be published).

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Theory of Low-Energy Scattering by a Long-Range r^{-8} Potential

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The low-energy scattering by a potential consisting of a long- and a short-range part is discussed. A general expression for the phase shift δ_L is derived starting from the radial Schrödinger equation. Effective-range expansions are presented for partial waves of all orders for the case of a long-range r^{-8} potential. The quantity $\tan\delta_L$ is calculated up to and including the term $k^{2L+7}\ln k$. It is found that in the low-energy limit, $\tan\delta_L$ is proportional to k^{2L+1} for s, p, and d waves, and $\tan\delta_L$ is proportional to k^6 for all the higher partial waves.

I. INTRODUCTION

The purpose of this paper is to present a derivation of an effective-range theory for r^{-8} potential scattering. The significance of the r^{-8} potential lies in the fact that it appears as a correction of the van der Waals potential in the description of the long-range interaction of two atoms, and represents dipole-quadrupole effects.[1,2] It also appears as a correction to the polarization potential in the description of long-range electron-atom interaction.[3-5] It is generally a repulsive potential, while the van der Waals and polarization potentials are attractive.

In Sec. II we derive a general formula for the phase shift due to scattering by a potential consisting of a short- and a long-range part. In Sec. III we present effective-range expansions for r^{-8} potential scattering for all partial waves, based on the formula derived in Sec. II.

II. FORMULA FOR PHASE SHIFTS

We consider a scattering potential consisting of a short-range part $V(r)$ and a long-range part $U(r)$ such that

$$V(r)=0 \quad \text{for } r>d\,, \qquad U(r)=0 \quad \text{for } r<d\,. \tag{1}$$

Let $u_L(r)$ be the eigenfunction of the radial Schrödinger equation with both U and V acting, and $u_{Ls}(r)$ the eigenfunction with only V acting. Then

$$\frac{d^2u_L}{dr^2}+\left(k^2-\frac{L(L+1)}{r^2}-(\bar{U}+\bar{V})\right)u_L=0\,, \tag{2}$$

$$\frac{d^2u_{Ls}}{dr^2}+\left(k^2-\frac{L(L+1)}{r^2}-\bar{V}\right)u_{Ls}=0\,. \tag{3}$$

Here $\bar{U}=(2m/\hbar^2)U$, $\bar{V}=(2m/\hbar^2)V$, L is the orbital-angular-momentum quantum number, and k is the wave number of the scattered particle. Multiplying Eq. (2) by u_{Ls} and Eq. (3) by u_L and subtracting, we obtain

$$u_{Ls}\frac{d^2u_L}{dr^2}-u_L\frac{d^2u_{Ls}}{dr^2}=u_{Ls}\bar{U}u_L\,, \tag{4}$$

which may be expressed as

$$\frac{d}{dr}\left(u_{Ls}\frac{du_L}{dr}-u_L\frac{du_{Ls}}{dr}\right)=u_{Ls}\bar{U}u_L\,. \tag{5}$$

Integrating (5) from 0 to r, we obtain

$$\left(u_{Ls}\frac{du_L}{dr}-u_L\frac{du_{Ls}}{dr}\right)_0^r=\int_0^r u_{Ls}\bar{U}u_L\,dr\,. \tag{6}$$

Now u_L and u_{Ls} satisfy the boundary conditions

$$\begin{gathered} u_L(0)=u_{Ls}(0)=0\,,\\ u_L\to\sin(kr-\tfrac{1}{2}L\pi)+\tan\delta_L\cos(kr-\tfrac{1}{2}L\pi) \quad \text{as } r\to\infty\,,\\ u_{Ls}\to\sin(kr-\tfrac{1}{2}L\pi)+\tan\delta_{Ls}\cos(kr-\tfrac{1}{2}L\pi) \quad \text{as } r\to\infty. \end{gathered} \tag{7}$$

Here δ_L is the phase shift when the scattering potential is $U+V$, while δ_{Ls} is the phase shift when the scattering potential is V alone. Letting $r\to\infty$ in Eq. (6), using the boundary conditions (7), and noting that $\bar{U}=0$ for $r<d$, we arrive at the formula

$$\tan\delta_L=\tan\delta_{Ls}-k^{-1}\int_d^\infty u_{Ls}\bar{U}u_L\,dr\,. \tag{8}$$

We will only consider first-order effects due to U. Since u_L differs from u_{Ls} by a term of order U, Eq. (8) becomes

$$\tan\delta_L=\tan\delta_{Ls}-k^{-1}\int_d^\infty u_{Ls}^2\bar{U}dr+O(U^2)\,. \tag{9}$$

For $r>d$, u_{Ls} is the radial wave function for free-particle scattering:

$$u_{Ls}=krj_L(kr)-\tan\delta_{Ls}krn_L(kr) \quad \text{for } r>d\,, \tag{10}$$

where j_L and n_L are the usual spherical Bessel and Neumann functions, respectively. Substituting Eq. (10) into Eq. (9) and adding and subtracting a term $[\tan\delta_{Ls}krj_L(kr)]^2$ in the integrand, we find

$$\tan\delta_L=\tan\delta_{Ls}-k^{-1}(1-\tan^2\delta_{Ls})I(L)+2k^{-1}\tan\delta_{Ls}J(L)-k^{-1}\tan^2\delta_{Ls}K(L)+O(U^2)\,, \tag{11}$$

where $I(L)$, $J(L)$, $K(L)$ are integrals defined by

$$I(L)=\int_d^\infty (kr)^2j_L^2(kr)\bar{U}(r)\,dr\,, \tag{12}$$

$$J(L)=\int_d^\infty (kr)^2j_L(kr)n_L(kr)\bar{U}(r)\,dr\,, \tag{13}$$

$$K(L)=\int_d^\infty (kr)^2\left[j_L^2(kr)+n_L^2(kr)\right]\bar{U}(r)\,dr\,. \tag{14}$$

Since $V(r)$ is short ranged, the phase shift from $V(r)$ is given by conventional effective-range theory[6]:

$$k^{2L+1}\cot\delta_{Ls}=-A_{Ls}^{-1}+\tfrac{1}{2}r_{Ls}k^2-Q_{Ls}r_{Ls}^3k^4+\cdots\,, \tag{15}$$

where A_{Ls}, r_{Ls}, and Q_{Ls} are the scattering length, effective range, and shape-dependent parameter, respectively, for the potential V. When the potential is $U+V$, the corresponding parameters will be denoted by A_L, r_L, and Q_L. For our purposes it is convenient to rewrite Eq. (15) in the form

$$\tan\delta_{Ls}=-A_{Ls}k^{2L+1}-\tfrac{1}{2}r_{Ls}A_{Ls}^2k^{2L+3}+(Q_{Ls}r_{Ls}^3A_{Ls}^2-\tfrac{1}{4}r_{Ls}^2A_{Ls}^3)k^{2L+5}+\cdots\,. \tag{16}$$

We see from Eq. (11) that the determination of an effective-range expansion for each partial wave involves the evaluation of the integrals $I(L)$, $J(L)$, and $K(L)$.

III. EFFECTIVE-RANGE EXPANSIONS

In this section we present effective-range expansions for a r^{-8} long-range potential, which are based on formula (11). We consider a r^{-8} repulsion:

$$\bar{U}(r)=\beta r^{-8}\,, \qquad r>d \tag{17}$$

where β is a parameter which characterizes the strength of the potential. The short-range potential $\bar{V}(r)$ is arbitrary. On inserting Eq. (16) and the results of Appendices A–C into Eq. (11), we obtain effective-range expansions for all partial waves. We find for the s wave:

$$\tan\delta_0=-A_0k-\tfrac{1}{2}r_0A_0^2k^3+(Q_0r_0^3A_0^2-\tfrac{1}{4}r_0^2A_0^3)k^5+\tfrac{2}{315}\pi\beta k^6+\tfrac{8}{315}\beta A_0k^7\ln|2kd|+O(k^7)\,, \tag{18}$$

where

$$A_0=A_{0s}+\beta\theta_0\,, \tag{19a}$$

$$\theta_0=(\tfrac{1}{5}d^2-\tfrac{1}{3}A_{0s}d+\tfrac{1}{7}A_{0s}^2)/d^7\,, \tag{19b}$$

$$r_0A_0^2=r_{0s}A_{0s}^2+\beta\theta_1\,, \tag{19c}$$

$$\theta_1=(-\tfrac{2}{9}d^4+\tfrac{2}{3}A_{0s}d^3-\tfrac{2}{5}A_{0s}^2d^2-\tfrac{1}{3}r_{0s}A_{0s}^2d+\tfrac{2}{7}r_{0s}A_{0s}^3)/d^7\,, \tag{19d}$$

$$Q_0r_0^3A_0^2-\tfrac{1}{4}r_0^2A_0^3=Q_{0s}r_{0s}^3A_{0s}^2-\tfrac{1}{4}r_{0s}^2A_{0s}^3+\beta\theta_2\,, \tag{19e}$$

$$\theta_2=[-\tfrac{2}{45}d^6+\tfrac{2}{15}A_{0s}d^5-\tfrac{1}{9}A_{0s}^2d^4-\tfrac{1}{6}r_{0s}A_{0s}^2d^3$$

$$+\tfrac{1}{5}r_{0s}A_{0s}^3d^2+\tfrac{1}{3}(\tfrac{1}{4}r_{0s}^2A_{0s}^3-Q_{0s}r_{0s}^3A_{0s}^2)d -\tfrac{1}{7}(\tfrac{3}{4}r_{0s}^2A_{0s}^4-2Q_{0s}r_{0s}^3A_{0s}^3)]/d^7\,. \quad (19f)$$

We find for the p wave:

$$\tan\delta_1=-A_1k^3-\tfrac{1}{2}r_1A_1^2k^5-\tfrac{2}{567}\pi\beta k^6+(Q_1r_1^3A_1^2-\tfrac{1}{4}r_1^2A_1^3)k^7 -\tfrac{8}{567}\beta A_1k^9\ln|2kd|+O(k^9)\,, \quad (20)$$

where

$$A_1=A_{1s}+\beta\theta_3\,, \quad (21a)$$

$$\theta_3=(\tfrac{1}{27}d^6-\tfrac{1}{9}A_{1s}d^3+\tfrac{1}{9}A_{1s}^2)/d^9\,, \quad (21b)$$

$$r_1A_1^2=r_{1s}A_{1s}^2+\beta\theta_4\,, \quad (21c)$$

$$\theta_4=(-\tfrac{2}{45}d^8-\tfrac{2}{15}A_{1s}d^5-\tfrac{1}{9}r_{1s}A_{1s}^2d^3 +\tfrac{2}{7}A_{1s}^2d^2+\tfrac{2}{9}r_{1s}A_{1s}^3)/d^9\,, \quad (21d)$$

$$Q_1r_1^3A_1^2-\tfrac{1}{4}r_1^2A_1^3=Q_{1s}r_{1s}^3A_{1s}^2-\tfrac{1}{4}r_{1s}^2A_{1s}^3+\beta\theta_5\,, \quad (21e)$$

$$\theta_5=[\tfrac{1}{525}d^{10}-\tfrac{2}{35}A_{1s}d^7+\tfrac{1}{30}r_{1s}A_{1s}^2d^5 +\tfrac{1}{9}(\tfrac{1}{4}r_{1s}^2A_{1s}^3-Q_{1s}r_{1s}^3A_{1s}^2)d^3-\tfrac{1}{7}r_{1s}A_{1s}^3d^2-\tfrac{1}{9}(\tfrac{3}{4}r_{1s}^2A_{1s}^4 -2Q_{1s}r_{1s}^3A_{1s}^3)]/d^9\,. \quad (21f)$$

We find for the d wave:

$$\tan\delta_2=-A_2k^5+\tfrac{2}{2079}\pi\beta k^6-\tfrac{1}{2}r_2A_2^2k^7+O(k^9)\,, \quad (22)$$

where

$$A_2=A_{2s}+\beta\theta_6\,, \quad (23a)$$

$$\theta_6=(\tfrac{1}{225}d^{10}-\tfrac{1}{15}A_{2s}d^5+\tfrac{9}{11}A_{2s}^2)/d^{11}\,, \quad (23b)$$

$$r_2A_2^2=r_{2s}A_{2s}^2+\beta\theta_7\,, \quad (23c)$$

$$\theta_7=(\tfrac{2}{1575}d^{12}-\tfrac{2}{105}A_{2s}d^7-\tfrac{1}{15}r_{2s}A_{2s}^2d^5 +\tfrac{2}{3}A_{2s}^2d^2+\tfrac{18}{11}r_{2s}A_{2s}^3)/d^{11}\,. \quad (23d)$$

The expansion (22) also contains a higher-order logarithmic term, $\frac{8}{2079}\beta A_2k^{11}\ln|2kd|$. We find for the higher partial waves:

$$\tan\delta_L=-10\pi\beta w(L)k^6-A_Lk^{2L+1}-\tfrac{1}{2}r_LA_L^2k^{2L+3}+(Q_Lr_L^3A_L^2-\tfrac{1}{4}r_L^2A_L^3)k^{2L+5}+O(\beta^2)\,,\quad L>2 \quad (24)$$

where

$$A_L=A_{Ls}+\beta\theta_8\,, \quad (25a)$$

$$\theta_8=-\frac{\pi d^{2L-5}}{2^{2L+2}(2L-5)[\Gamma(L+\frac{3}{2})]^2}-\frac{A_{Ls}}{3(2L+1)d^6}+\frac{[(2L)!]^2A_{Ls}^2}{2^{2L}(L!)^2(2L+7)d^{2L+7}}\,, \quad (25b)$$

$$r_LA_L^2=r_{Ls}A_{Ls}^2+\beta\theta_9\,, \quad (25c)$$

$$\theta_9=\frac{\pi(2L+3)d^{2L-3}}{2^{2L+3}(2L-3)[\Gamma(L+\frac{5}{2})]^2}-\frac{A_{Ls}g(L)}{4d^4}-\frac{r_{Ls}A_{Ls}^2}{3(2L+1)d^6}+\frac{[(2L)!]^2r_{Ls}A_{Ls}^3}{2^{2L-1}(L!)^2(2L+7)d^{2L+7}}+\frac{(2L-1)!(2L-2)!A_{Ls}^2}{2^{2L-3}[(L-1)!]^2(2L+5)d^{2L+5}}\,, \quad (25d)$$

$$Q_Lr_L^3A_L^2-\tfrac{1}{4}r_L^2A_L^3=Q_{Ls}r_{Ls}^3A_{Ls}^2-\tfrac{1}{4}r_{Ls}^2A_{Ls}^2+\beta\theta_{10}\,, \quad (25e)$$

$$\theta_{10}=\frac{\pi(2L+4)(2L+5)d^{2L-1}}{2^{2L+7}(2L-1)[\Gamma(L+\frac{7}{2})]^2}+\frac{6A_{Ls}h(L)}{d^2}+\frac{r_{Ls}A_{Ls}^2g(L)}{16d^4}+\frac{\frac{1}{4}r_{Ls}^2A_{Ls}^3-Q_{Ls}r_{Ls}^3A_{Ls}^2}{3(2L+1)d^6}-\frac{[(2L)!]^2(\frac{3}{4}r_{Ls}^2A_{Ls}^4-2Q_{Ls}r_{Ls}^3A_{Ls}^3)}{2^{2L}(L!)^2(2L+7)d^{2L+7}} -\frac{(2L-1)!(2L-2)!r_{Ls}A_{Ls}^3}{2^{2L-2}[(L-1)!]^2(2L+5)d^{2L+5}}-\frac{(2L-2)!(2L-4)!A_{Ls}^2}{2^{2L-3}[(L-2)!]^2(2L+3)d^{2L+3}}\,. \quad (25f)$$

The functions $g(L)$, $h(L)$, and $w(L)$ in Eqs. (24)–(25f) are defined by Eqs. (B6), (B7), and (A3) in the Appendices. The following points should be noted concerning the general validity of the expansion (24). Only those terms k^{2L+1}, ..., are to be retained for which the exponent is less than 12. The reason why the terms with exponents greater than 12 are invalid is the approximation made in going from Eq. (8) to Eq. (9), i.e., the neglect of any terms of second order in the long-range potential. This rules out all second-order terms in β, the leading one being a β^2k^{12} term for all L. This restriction eliminates all terms except the k^6 term for all $L>5$.

It may be noted that all the terms through k^6 are anticipated in the general results of Levy and Keller[7] who discussed the general r^{-n} long-range potential using a different approach from the present one. The higher-order k^7, k^9, and $k^{2L+7}\ln k$ terms for $L=0$, 1, 2 were not explicitly considered by Levy and Keller, and are therefore new. Thus the present work is an extension of the results of Levy and Keller, filling in some higher-order terms.

In Eqs. (18) and (20), the coefficient of $k^{2L+7}\ln k$ actually contains A_{Ls} rather than A_L, but the replacement of A_{Ls} by A_L is legitimate since the error introduced is of second order in the long-range potential. The expansion (24) for $L>2$ does not contain any logarithmic term in first order. Equations (18), (20), (22), and (24) show that in the extreme low-energy region, $\tan\delta_L$ varies as k^{2L+1} for s, p, and d waves, and $\tan\delta_L$ varies as k^6 for all partial waves higher than the d wave. By combining these results with the corresponding results for a polarization r^{-4} potential[8,9] and a van der Waals r^{-6} potential,[9,10] we can infer a general rule for any potential with a r^{-2n} tail (where n is an integer >1). At sufficiently low energies, $\tan\delta_L$ varies at k^{2L+1} for all partial waves up to and including the partial wave of order $L=n-2$. Further, $\tan\delta_L$ varies as k^{2n-2} for all partial waves higher than the partial wave of order $L=n-2$. The k^{2L+1} dependence is characteristic of short-range forces (e.g., nuclear forces), while the k^{2n-2} dependence is characteristic of long-range forces.

We have examined the r^{-8} potential in the Born approximation, and we find that only the k^6 term for $L>2$ is given correctly. More generally, for any potential with a r^{-2n} tail, the Born approximation gives the correct leading term in the low-energy limit for all partial waves of order higher than $L=n-2$. The fact that the Born approximation gives substantial agreement with Schrödinger theory for the long-range contribution has been pointed out previously.[7,8,11]

APPENDIX A: EVALUATION OF $I(L)$

On inserting Eq. (17) into Eq. (12) and making the substitution $z=kr$, we obtain

$$I(L)=\tfrac{1}{2}\pi\beta k^7\int_{kd}^{\infty} z^{-7}J^2_{L+1/2}(z)\,dz\ . \tag{A1}$$

The integral in (A1) may be evaluated for $L>2$ by dividing the range of integration into two parts, one part extending from zero to infinity, and the other from zero to kd. The integral with range zero to infinity may be evaluated using Eq. 6.574-2 of Ref. 12. To evaluate the integral from zero to kd, we may use the power series expansion for the square of a Bessel function as given by Eq. 5.4-6 of Ref. 13.

We find that the value of $I(L)$ for $L>2$ is

$$I(L>2)=\tfrac{1}{2}\pi\beta k^7\left(20w(L)-\sum_{m=0}^{\infty}\frac{a_{Lm}}{2L+2m-5}(kd)^{2L+2m-5}\right), \tag{A2}$$

where

$$w(L)=[(2L+7)(2L+5)(2L+3)(2L+1)\times(2L-1)(2L-3)(2L-5)]^{-1}\ , \tag{A3}$$

$$a_{Lm}=\frac{(-1)^m(2L+2m+1)!}{2^{2L+2m+1}(2L+m+1)![\Gamma(L+m+\frac{3}{2})]^2m!}\ . \tag{A4}$$

Considering the case $L=0$, we make the substitution $z=kr$ in Eq. (12), and use the trigonometric expression for $j_0(z)$,[14] thus obtaining

$$I(L=0)=\beta k^7\int_{kd}^{\infty} z^{-8}\sin^2 z\,dz\ . \tag{A5}$$

The integral in (A5) may be evaluated by repeated integration by parts. The final result is

$$I(L=0)=\beta\left(\frac{k^2}{5d^5}-\frac{k^4}{9d^3}+\frac{2k^6}{45d}-\frac{2\pi k^7}{315}+O(k^8)\right). \tag{A6}$$

For the case $L=1$, making the substitution $z=kr$ in Eq. (12), and using the trigonometric expression for $j_1(z)$,[14] we obtain

$$I(L=1)=\beta k^7\int_{kd}^{\infty}(z^{-10}\sin^2 z-z^{-9}\sin 2z+z^{-8}\cos^2 z)\,dz\ . \tag{A7}$$

The three integrals in (A7) may be evaluated by repeated integration by parts. After somewhat tedious but straightforward algebra, the following expansion is obtained:

$$I(L=1)=\beta\left(\frac{k^4}{27d^3}-\frac{k^6}{45d}+\frac{2\pi k^7}{567}-\frac{dk^8}{525}+O(k^{10})\right)\ . \tag{A8}$$

Considering the case $L=2$, and making the substitution $z=kr$ in Eq. (12), and using the trigonometric expression for $j_2(z)$,[14] we obtain the expression

$$I(L=2)=\beta k^7\int_{kd}^{\infty}[(9z^{-12}-6z^{-10}+z^{-8})\sin^2 z-(9z^{-11}-3z^{-9})\sin 2z+9z^{-10}\cos^2 z]\,dz\ . \tag{A9}$$

The integrals in (A9) may be evaluated straightforwardly by repeated integration by parts. Since the algebra becomes rather tedious, we only give the first three terms in the final expansion:

$$I(L=2)=\beta\left(\frac{k^6}{225d}-\frac{2\pi k^7}{2079}+\frac{dk^8}{1575}+O(k^{10})\right). \tag{A10}$$

APPENDIX B: EVALUATION OF $J(L)$

On inserting (17) in (13), and making the substitution $z=kr$, we obtain the expression

$$J(L)=(-1)^{L+1}\tfrac{1}{2}\pi\beta k^7\int_{kd}^{\infty} z^{-7}J_{L+1/2}(z)J_{-L-1/2}(z)\,dz\ . \tag{B1}$$

The integral in (B1) may be evaluated by dividing the range of integration into two parts, one part extending from kd to 1, and the other from 1 to ∞. To evaluate the integral with range kd to 1, we use the power series given by Eq. 5.4-7 of Ref. 13, which leads to the result

$$J(L)=(-1)^{L+1}\tfrac{1}{2}\pi\beta k^7\left(\frac{b_{L0}}{6(kd)^6}+\frac{b_{L1}}{4(kd)^4}+\frac{b_{L2}}{2(kd)^2}-b_{L3}\ln|2kd|+B-\sum_{m=4}^{\infty}\frac{b_{Lm}}{2m-6}(kd)^{2m-6}\right), \tag{B2}$$

where

$$B=\int_1^{\infty} z^{-7}J_{L+1/2}(z)J_{-L-1/2}(z)\,dz-\tfrac{1}{6}b_{L0}-\tfrac{1}{4}b_{L1}-\tfrac{1}{2}b_{L2}+b_{L3}\ln 2+\sum_{m=4}^{\infty}\frac{b_{Lm}}{2m-6}\ , \tag{B3}$$

$$b_{Lm}=\frac{(-1)^m(2m)!}{2^{2m}(m!)^2\Gamma(L+m+\frac{3}{2})\,\Gamma(-L+m+\frac{1}{2})}\ . \tag{B4}$$

We note that B is independent of k, and that Eq. (B2) holds for all values of L. Inserting the values of b_{L0}, b_{L1}, and b_{L2} explicitly, we obtain the result

$$J(L) = -\beta\left(\frac{k}{6(2L+1)d^6} + \frac{g(L)k^3}{16d^4} + \frac{3h(L)k^5}{d^2} - 20w(L)k^7 \ln|2kd| + O(k^7)\right), \quad \text{(B5)}$$

where

$$g(L) = [(L+\tfrac{3}{2})(L+\tfrac{1}{2})(L-\tfrac{1}{2})]^{-1}, \quad \text{(B6)}$$

$$h(L) = [(2L+5)(2L+3)(2L+1)(2L-1)(2L-3)]^{-1}, \quad \text{(B7)}$$

and $w(L)$ is defined by (A3).

APPENDIX C: EVALUATION OF $K(L)$

In order to evaluate the integral $K(L)$, we insert (17) in (14) and make the substitution $z = kr$, which gives

$$K(L) = \tfrac{1}{2}\pi\beta k^7 \int_{kd}^{\infty} z^{-7}\left[J^2_{L+1/2}(z) + J^2_{-L-1/2}(z)\right] dz . \quad \text{(C1)}$$

The integral in (C1) may be evaluated using Eq. 9.62-5 of Ref. 13. We find

$$K(L) = -\frac{\beta}{d^7} \sum_{m=0}^{L} \frac{c_{Lm}}{2m-2L-7}(kd)^{2m-2L}, \quad \text{(C2)}$$

where the coefficients c_{Lm} are given by

$$c_{Lm} = \frac{2^{2m-2L}(2L-m)!\,(2L-2m)!}{[(L-m)!]^2 m!} . \quad \text{(C3)}$$

We note that Eq. (C2) holds for all values of L.

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[1]C. H. Page, Phys. Rev. 53, 426 (1938).
[2]G. Das and A. C. Wahl, Phys. Rev. A 4, 825 (1971).
[3]R. W. La Bahn and J. Callaway, Phys. Rev. 180, 91 (1969).
[4]M. H. Mittleman, Ann. Phys. (N.Y.) 14, 94 (1961).
[5]C. J. Kleinman, Y. Hahn, and L. Spruch, Phys. Rev. 165, 53 (1968).
[6]H. A. Bethe, Phys. Rev. 76, 38 (1949).
[7]B. R. Levy and J. B. Keller, J. Math. Phys. 4, 54 (1963).
[8]T. F. O'Malley, L. Spruch, and L. Rosenberg, J. Math. Phys. 2, 491 (1961).
[9]O. Hinckelmann and L. Spruch, Phys. Rev. A 3, 642 (1971).
[10]P. S. Ganas, Phys. Rev. A 5, 1684 (1972).
[11]E. Gerjuoy and S. Stein, Phys. Rev. 97, 1671 (1955).
[12]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).
[13]G. N. Watson, *Theory of Bessel Functions* (Cambridge U. P., London, 1966).
[14]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

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Absolute Cross Sections for Excitation of Neon by Impact of 20–180-keV H^+, H_2^+, and He^+†

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The technique of heavy-ion energy-loss spectrometry has been used to measure excitation cross sections for the $(2p^5)3s$ and $(2p^5)3p$ electronic configurations of neon by impact of heavy ions upon ground-state neon. The incident particles used were H^+, H_2^+, and He^+ at impact energies from 20 to 180 keV. The results are compared with previous optical measurements of the emission cross sections of lines from these levels as excited by H^+ and He^+ impact. Agreement is not good, neither in shape nor in absolute magnitude, for excitation of the $(2p^5)3s$ configuration. However, agreement is surprisingly good for excitation of the $(2p^5)3p$ configuration. A curve-fitting technique has been applied to extract relative singlet-triplet cross sections for levels within the $(2p^5)3s$ configuration. Almost no triplet excitation is observed for H^+ and H_2^+ impact. Significant triplet excitation is observed only for He^+ impact.

I. INTRODUCTION

There has been considerable recent interest in the properties of neon as embodied in collision cross sections. Investigations have been conducted by bombarding neon with low-energy ions[1-3] and with electrons at energies ranging from threshold to several hundreds of eV.[4,5]

The extensive work of Coffey *et al.*[3] on inelastic and elastic scattering of He^+ by Ne at energies below 500 eV has indicated the wealth of information obtainable by collision spectroscopy. In this low-energy range, the observed patterns in the data can be explained quite reasonably in terms of molecular curve crossings which, in turn, yield valuable information concerning the nature of inter-