($\ln E/E$) for optically allowed transitions $1s$-$2p$ and $1s$-$3p$, whereas for optically forbidden transitions $1s$-$2s$, $1s$-$3s$, amd $1s$-$3d$, the cross section has an asymptotic $1/E$ dependence. We also observe that for a given principal quantum number the excitation cross sections are in general very high when the azimuthal quantum numbers are such that the transitions are optically allowed. It is also noted that the curves for each spectral series (such as $1s$-ns) are almost identical in shape, and any one curve approximating the preceding curve is displaced downward. Furthermore, the magnitude of the cross sections for different excitations continue to increase as the atomic number of the target increases.

In conclusion, we can say that the form-factor description of the target provides a simple approach to the calculation of atom-atom inelastic-collision cross section in the Born approximation. At low energies the Born approximation can, however, be improved by the inclusion of distortion and coupling. The present results are expected to be accurate at high energies. The need for experimental investigation of the above excitations is obvious.

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Multiphoton Transition in Hydrogen between 1S and 2P Levels

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$1S$-$2P$ multiphoton transition amplitudes are calculated for hydrogen atoms in the presence of an intense electromagnetic field. Nonlinear behavior of the transition amplitudes appears as in the case of $1S$-$2S$ transitions considered earlier.

I. INTRODUCTION

The purpose of this paper is to present the results and the interpretations of calculations for $1S$-$2P$ multiphoton transition in a hydrogen atom under the influence of an intense plane-wave electromagnetic field. The general theory has been presented by Reiss,[1] and we refer to the papers in Ref. 1 for notations, for general background of the theory, and for the results of calculation for $1S$-$2S$ transitions. Our results are of interest, we believe, for a variety of subjects ranging from experiments to observe these specific transitions and their dependence on intensity to astrophysical effects in regions of space where very intense fields may be present.

The terminology of this paper is for excitation from the $1S$ to the $2P$ level. This is an arbitrary choice. The results can be applied directly to induced emission from $2P$ to $1S$.

II. THEORY AND CALCULATION

We shall not go into the details of the general theory which has been presented in Ref. 1. The solution of the Schrödinger equation is approximated by means of a unitary transformation [momentum translation approximation]:

$$\psi(\vec{x}, t) = e^{ie\vec{A}\cdot\vec{x}}\phi(\vec{x}, t), \tag{1}$$

where $\phi(\vec{x}, t)$ is the solution of the Schrödinger equation without the electromagnetic field. The T matrix for transition from state $|i\rangle$ to state

$|f\rangle$ is dependent upon $\langle f|H' e^{ie\vec{A}\cdot\vec{x}}|i\rangle$, where H' is that part of the perturbation Hamiltonian attributed to the external electromagnetic field. One can extract the N photon contribution in a closed form as

$$T_{fi}^{(N)} = i^N(E_i - E_f)\langle f|J_N(ea\vec{x}\cdot\vec{\epsilon})|i\rangle \; , \qquad (2)$$

where $\vec{\epsilon}$ is the polarization vector and a is the amplitude of the plane-wave electromagnetic field. The matrix element of more interest is the one in which N photons of frequency ω and one photon of frequency $\hat{\omega}$ are absorbed or emitted. The T matrix for this case becomes

$$T_{fi}^{(N)} = i^{N+1}(E_i - E_f)\, e^{\pm i\alpha} \times \langle i\,|\,J_N(ea\vec{x}\cdot\vec{\epsilon})J_1(e\hat{a}\vec{x}\cdot\hat{\vec{\epsilon}})\,|f\rangle \, . \qquad (3)$$

The notations are identical to Ref. 1. The calculation now proceeds with the proper hydrogenic wave functions and the integral representations of the Bessel functions. One obtains

$$T_{fi}^{(N,1)} = i^{N+1}(E_i - E_f)[\,e^{\pm i\alpha}/(2\pi)^2]\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta'\, r^2\, dr \times R_f^*(r)\, R_i(r)\, (4\pi)(2l_f+1)^{1/2} \times \sum_{l,m;\bar{l},\bar{m}} [(2l+1)(2\bar{l}+1)]^{1/2}\, i^{l+\bar{l}}\; j_l(ear\sin\theta) \times j_{\bar{l}}(e\hat{a}r\sin\theta')\, Y_l^{m*}(\theta,\phi)\, Y_{\bar{l}}^{\bar{m}}(\theta',\phi') \times \begin{pmatrix} l & \bar{l} & l_f \\ m & \bar{m} & -m_f \end{pmatrix} \begin{pmatrix} l & \bar{l} & l_f \\ 0 & 0 & 0 \end{pmatrix} , \qquad (4)$$

where we have already assumed an S state for the initial state.

For specific $1S$-$2P$ transition, we set $l_f = 1$ and consider three matrix elements for $m_f = (1, 0, -1)$. One get the following expression for the transitions:

$$T_{fi}^{(N,1)}(m_f=0) = (\vec{\epsilon}\cdot\hat{\vec{\epsilon}})[y/(B^{1/2}+1)]^N[(N^2-1)B^2 + N(N^2-1)B^{3/2} + 6(N^2-1)B + 15NB^{1/2} + 15]F, \quad (5)$$

$$T_{fi}^{(N,1)}(m_f=\pm 1) = |\vec{\epsilon}\times\hat{\vec{\epsilon}}|[y/(B^{1/2}+1)]^N[3B^3 + 3NB^{5/2} + (N^2+2)B^2 + 3NB^{3/2} + 3B]F, \quad (6)$$

where

$$F \equiv (\tfrac{1}{3}e\hat{a}a_0)\, e^{\pm i\alpha}(-)^{N/2}\,(3/8ma_0^2),$$

$$B \equiv 1 + y^2, \quad y \equiv \tfrac{2}{3}\, eaa_0 \, .$$

In order to discuss the intensity dependence of these results, we shall find it convenient to introduce the reduced amplitudes $\mathcal{T}_N(m_f=0)$ and $\mathcal{T}_N(m_f=\pm 1)$, which will be defined as

$$T_{fi}^{(N,1)}(m_f=0) = (\vec{\epsilon}\cdot\hat{\vec{\epsilon}})\, F\mathcal{T}_N(m_f=0)\, , \qquad (7)$$

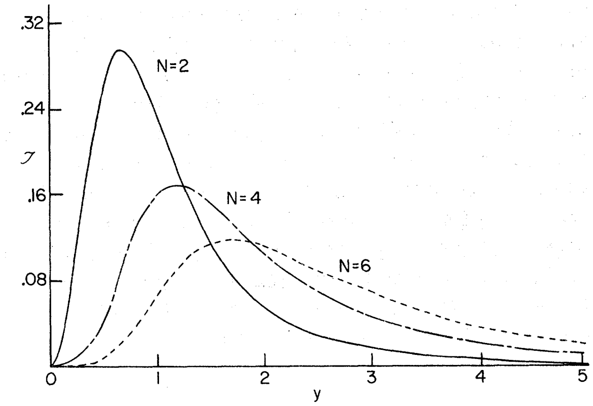

FIG. 1. "Reduced" transition amplitude $\mathcal{T}(m_f=0)$ as a function of the intensity parameter y for several values of the photon multiplicity N.

$$T_{fi}^{(N,1)}(m_f=\pm 1) = |\vec{\epsilon}\times\hat{\vec{\epsilon}}|\, F\mathcal{T}_N(m_f=\pm 1)\, . \qquad (8)$$

III. DISCUSSION

In order to discuss the results of the above calculation, we have plotted the results for various intensities as well as photon number. In Figs. 1 and 2, we plot the transition amplitudes for $m_f = 0$ and $m_f = \pm 1$, respectively, against intensity. In Figs. 3 and 4, we have the amplitudes plotted against N. Even though the last two figures look like continuous curves, it is obvious that only even values of N are the physical points.

A. Low-Intensity Limit

The low-intensity limit is described by $y \to 0$ or $B \to 1$. The reduced amplitudes, then, take the forms

$$\mathcal{T}_N(m_f=0) \to \tfrac{1}{8}(\tfrac{1}{2}y)^N(N+1)(N+2)(N+4)\, , \qquad (9)$$

$$\mathcal{T}_N(m_f=\pm 1) \to \tfrac{1}{16}\sqrt{2}\;(\tfrac{1}{2}y)^N(N+2)(N+4)\, . \qquad (10)$$

Both the amplitudes have the perturbation theory type of behavior, each being proportional to y^N with the additional fact of a polynomial dependence on N. These results also show that

$$\mathcal{T}_N(m_f=0)/\,\mathcal{T}_N(m_f=\pm 1) \to \sqrt{2}\,(N+1), \qquad (11)$$

which means that at low intensity, the transition probability to the $m_f = 0$ state is always greater than to the $m_f = \pm 1$ states. This can also be seen by comparing Figs. 1 and 2 in the low-intensity area. Gontier and Trahin[2] have recently published perturbation-theory calculations which correspond to those given here. A detailed comparison of momentum-translation results and the perturbative work of Gontier and Trahin has been done by Rogerson, DeWitt, and Reiss.[3]

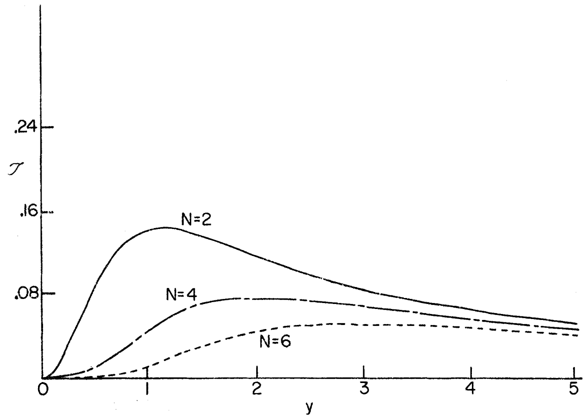

FIG. 2. "Reduced" transition amplitude $\mathcal{T}(m_f=\pm 1)$ as a function of the intensity parameter y for several values of the photon multiplicity N.

B. High-Order High-Intensity Limit

The high-order high-intensity limit is the region where perturbation theory is invalid. Closely paralleling the last paper in Ref. 1, we set $N = \beta B^{1/2}$. The reduced amplitudes take the form

$$\mathcal{T}_N(m_f=0) \to (e^{-\beta}/8B^{1/2})(\beta^2+\beta^3)\,, \qquad (12)$$

$$\mathcal{T}_N(m_f=\pm 1) \to (\sqrt{2}\,e^{-\beta}/16B^{1/2})(3+3\beta+\beta^2)\,. \qquad (13)$$

The extrema of (12) are located by solving $(\partial/\partial\beta) \times [\mathcal{T}_N(m_f=0)]=0$. The only physical solution other than $\beta=0$ is at $\beta=1+\sqrt{3}=2.732$. For $y=10$, one gets $N=27.4$. One can look at Fig. 3 to see that this is the case. Analyzing (13), there are no physical solutions of

$$\frac{\partial}{\partial\beta}[\mathcal{T}_N(m_f=\pm 1)]=\tfrac{1}{16}\sqrt{2}\,(-\beta-\beta^2)=0\,.$$

As seen from Fig. 4, we also note the absence of peaks at any value of the intensity.

C. General Behavior

Figures 1 and 2 tell us the intensity dependence of the transition amplitude for $N=2$, 4, 6 cases, i.e., 3-, 5-, and 7-photon cases, respectively. In Figs. 3 and 4, the amplitudes are plotted as functions of N, the even integer values of N being the only physical points. Figures 1 and 2 illustrate the nonlinear behavior at large intensities of the external electromagnetic field. All the curves deviate from the y^N behavior characteristic of perturbation theory. Each of them demonstrate a maximum at a certain intensity. As intensity increases for $\mathcal{T}_N(m_f=0)$, we see (Fig. 3) that the cross section increases at large photon numbers when the intensity is high and dominates the small photon cross section. In the case of $\mathcal{T}_N(m_f=\pm 1)$, the large photon processes show increasing probability when the intensity becomes high, but never dominate over the small photon processes even though they become comparable. Regarding the relative importance of $\mathcal{T}_N(m_f=0)$ and $\mathcal{T}_N(m_f=\pm 1)$, we have already stated the dominance of $\mathcal{T}_N(m_f=0)$ over $\mathcal{T}_N(m_f=\pm 1)$ at small intensities. At large intensities, one sees a reversal of the role, and $\mathcal{T}_N\,(m_f\pm 1)$ becomes larger than $\mathcal{T}_N(m_f=0)$.

IV. CONCLUSION

We have calculated the miltiphoton $1S\to 2P$ transitions induced by external electromagnetic plane waves. The results, as anticipated, show marked nonperturbative behavior at large intensities.

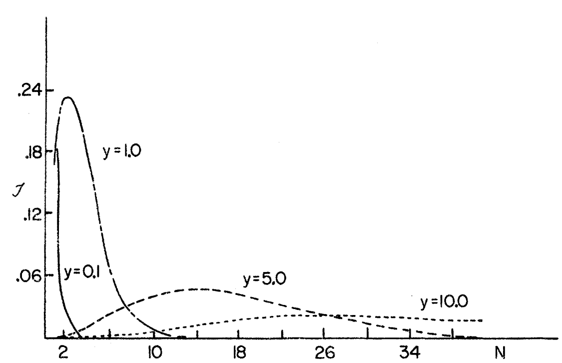

FIG. 3. "Reduced" transition amplitude $\mathcal{T}(m_f=0)$ as a function of photon multiplicity N for several values of the intensity parameter y. The multiplicity N is regarded as a continuous parameter for convenience. Only even-integer values of N are physical.

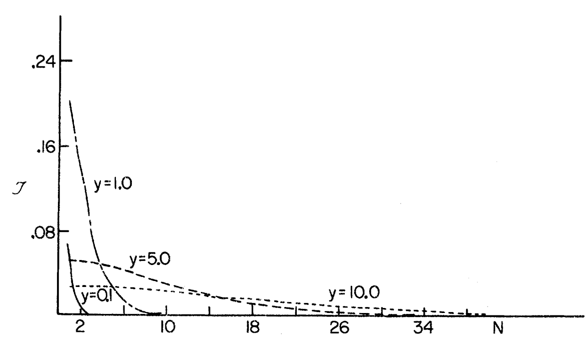

FIG. 4. "Reduced" transition amplitude $\mathcal{T}(m_f=\pm 1)$ as a function of photon multiplicity N for several values of the intensity parameter y. The multiplicity N is regarded as a continuous parameter for convenience. Only even-integer values of N are physical.

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ERRATA

Eikonal Theory of Intermediate-Energy Electron-Atom Scattering, Charles J. Joachain and Marvin H. Mittleman [Phys. Rev. A 4, 1492 (1971)]. The following typographical errors should be corrected: In Eq. (2.7) the quantity in brackets should read $[\delta(x, x') - \Phi_0(x)\Phi_0^*(x')]$. Four lines below Eq. (3.2), read "eikonal" instead of "eikon." The denominator in the integrand of Eq. (3.18) should read $[1+(\xi/\alpha)^2\sinh^2 u]^2$. Finally, the quantity displayed in Fig. 1 is $\bar{W}=4W$, not W.

Transport and Relaxation Phenomena in the Hydrogen Isotopes, F. R. McCourt and H. Moraal [Phys. Rev. A 5, 2000 (1972)]. A number of printing errors in this paper should be corrected: (i) In Eq. (2.8), the quantity $(2j+1)$ should read $(2j+1)^{-1}$. (ii) In Eq. (4.2′), the third term in the brackets multiplying $d^2\tau''/150$ should read $2/(1+4\omega^2\tau''^2)$. (iii) $\sigma_{\rm DPR}$ in Eq. (5.1) should be multiplied by $n\langle v_{\rm rel}\rangle_0$. (iv) Replace Eq. (5.2) by

$$\sigma_{\rm DPR} = 4\left\langle\frac{j(j+1)(4j^2+4j-7)}{(2j-1)^3(2j+3)^3}\right\rangle_0 \times\left\langle\frac{j(j+1)}{(2j-1)(2j+3)}\right\rangle_0^{-1}\sigma_{\rm H_2}^{(0)}.$$

Self-Induced Transparency in Atomic Rubidium, R. E. Slusher and H. M. Gibbs [Phys. Rev. A 5, 1634 (1972)]. A computer program error has been found which changes the theoretical results shown in Figs. 2–4 and 13. The computer program used to calculate pulse shapes, energies, and areas included relaxation terms. These relaxation rates were a factor of 2 too small in the computer program making $T_2'=112$ nsec and $T_1=67.2$ nsec instead of the correct values $T_2'=56$ nsec and $T_1=33.6$ nsec. The computer programs have been rerun and the corrected figures are shown here.

In Fig. 2 the dashed curve is lower than before. T_1 and T_2' losses are predominately responsible for the energy ratio not reaching unity at an input area of 2π; thus, the corrected computer results with increased relaxation rates give lower values for the energy rates than the previous erroneous rates.

Theoretical output pulse shapes in Figs. 3(a′)–3(b′) are also slightly modified. The lowest area pulse input has been changed from $A=5.7$ to $A=6.28$ for better agreement with experimental results. Experimental and theoretical pulse shapes are still in good agreement because the relaxation times are so much longer than the 7-nsec pulse length.

Figure 4 shows the theoretical effect of relaxa-

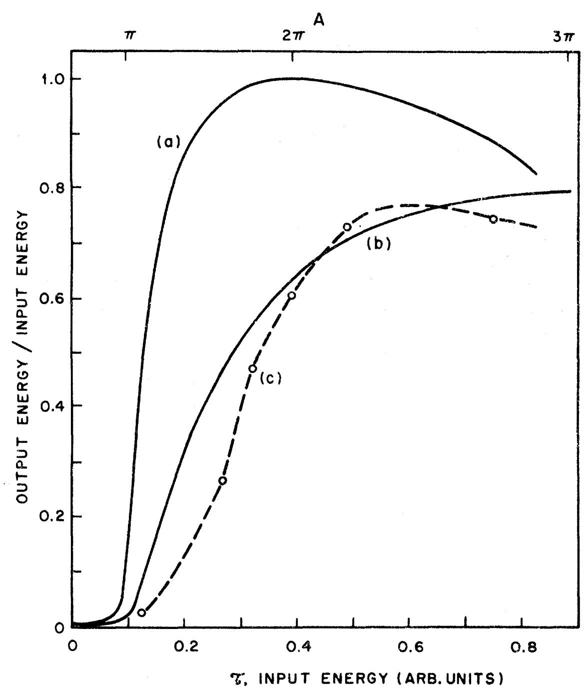

FIG. 2. Computer solutions for the output-input energy ratio of SIT pulses as a function of input energy and area; (a) assumes uniform plane wave and no losses, (b) Gaussian profile plane wave with no losses, (c) experimental pulse shapes, T_1 and T_2' losses for Rb, and uniform plane wave. All curves assume $\alpha L=5$. Points on (c) are the only computed points; (a) and (b) used closely spaced input areas because of simplicity of program.