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# Simple scheme for state measurement

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We present a simple and fast scheme to measure the state of a trapped atom. Our method directly yields the characteristic function of the Wigner function avoiding a demanding data analysis. We show that our method can be readily applied to an actual experiment. [S1050-2947(99)50702-6]

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In recent years the experimental progress in the preparation and reconstruction of quantum states has been enormous in the case of high-$Q$ cavities and trapped atoms. It is now possible to create and measure a variety of nonclassical quantum states [1–3] and to perform simple quantum logic operations [4]. Particularly worth mentioning is the complete reconstruction of a nonclassical quantum state [5]. Another topic of interest is the study of the decoherence of states of the Schrödinger-cat type [6], where the loss of coherence increases the more classically distinguishable the two constituent states are [7]. This imposes a limit on the size of a Schrödinger cat that can be reconstructed. In order to push this limit, it is important to have methods to measure the quantum state in a time that is much shorter than the decoherence time. This can only be done by a method that consists of a few fast steps.

There has also been a large number of theoretical ideas for state reconstruction [8,9]. But only a few of the proposals have a strikingly simple data analysis. The scheme for directly measuring the characteristic function of the Wigner function was made by Wilkens and Meystre [10]. Recently, a similar suggestion was made by Kim *et al.* [11]. A scheme that directly yields the Wigner function was suggested by Lutterbach and Davidovich [12]. What these schemes all have in common is that the data analysis is very simple. The function representing the quantum state is directly given by measuring the population of an auxiliary two-level system.

We present a method to directly measure the characteristic function of the Wigner function of the motional state of a trapped particle. The method consists of only three simple steps. All the steps of our state measurement scheme closely follow the scheme to prepare a Schrödinger cat in the Paul trap as presented by Monroe *et al.* [2]. Before we discuss the relation of our findings to their work, we will describe our scheme in detail. Finally, we show its experimental feasibility.

Because the dynamics of a mode of a cavity field and a trapped atom are mathematically closely related, measurement schemes are, in general, suitable for both systems. Therefore, we have discussed the achievements in both cases together. However, in order to describe our method, we will focus on a trapped atom because of the strong relation between our scheme and the experimental realization of a Schrödinger-cat state mentioned above. The case of a cavity mode that is experimentally more complicated is discussed briefly at the end.

The system considered is the one-dimensional center-of-mass motion of a particle in an effective harmonic trap potential with frequency $\omega$ and two internal levels separated by a transition frequency $\Omega$. For all the steps, the coupling of the internal levels and the oscillator levels can be seen in Fig. 1. In the experiment [1,2,5], the internal states are long-living hyperfine spin states with a forbidden direct transition. Hence it is possible to select the various required couplings between the levels by using appropriate frequencies and polarizations of the Raman laser pulses, and by exploiting the selection rules for transitions. The successive transformation of the state can be followed in Table I.

The starting point for the state measurement is the state

$$\hat{\boldsymbol{\rho}}_0 = |g\rangle\langle g|\hat{\boldsymbol{\rho}}, \tag{1}$$

with the unknown quantum state $\hat{\rho}$ of one dimension of the center-of-mass motion and the lower internal state $|g\rangle$. First, we transform the initial internal state $|g\rangle$ into the superposition $(|g\rangle + e^{i\varphi_1}|e\rangle)/\sqrt{2}$ via a Raman laser-pulse resonant with $\Omega$, using the linearly polarized lasers $a$ and $b$ in Fig. 1. Then we have the total state

$$\hat{\boldsymbol{\rho}}_1 = \frac{1}{2}(|g\rangle + e^{i\varphi_1}|e\rangle)\hat{\boldsymbol{\rho}}(\langle g| + e^{-i\varphi_1}\langle e|). \tag{2}$$

The relative phase $\varphi_1$ between $|g\rangle$ and $|e\rangle$ is adjusted by the phase difference of the two lasers.

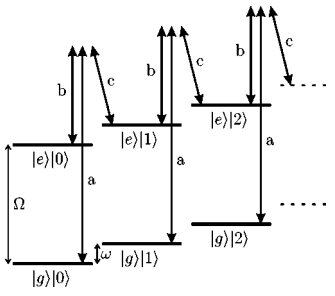


FIG. 1. Schematic level structure and couplings. The transition frequency between the vibrational levels $|n=0\rangle, |n=1\rangle, \ldots,$ is $\omega$ and between the two internal states, $|g\rangle$ and $|e\rangle$, is $\Omega$. The linearly polarized Raman laser beams $a$ and $b$ resonantly couple levels $|g\rangle|n\rangle$ and $|e\rangle|n\rangle$, whereas the circularly polarized Raman laser beams $b$ and $c$ couple levels $|e\rangle|n\rangle$ and $|e\rangle|n+1\rangle$.

 

TABLE I. We summarize the transformation of the total system for each step of the measurement scheme. The density operator to be measured is $\hat{\boldsymbol{\rho}}$, whereas $|g\rangle$ and $|e\rangle$ denote two auxiliary internal states. The parameters of the transformation—the phases $\varphi_1$, $\varphi_3$, $\varphi=\varphi_1-\varphi_3$, and the amplitude $\alpha$ of the shift—are determined by the applied laser pulses.

| Step | Function | State of system |
|---|---|---|
| 0 | initial | $\|g\rangle\langle g\|\hat{\boldsymbol{\rho}}$ |
| 1 | $\pi/2$-pulse | $\frac{1}{2}(\|g\rangle+e^{i\varphi_1}\|e\rangle)\hat{\boldsymbol{\rho}}(\langle g\|+e^{-i\varphi_1}\langle e\|)$ |
| 2 | shift by $\alpha$ | $\frac{1}{2}(\|g\rangle+e^{i\varphi_1}\|e\rangle\hat{\mathbf{D}}(\alpha))\hat{\boldsymbol{\rho}}(\langle g\|+e^{-i\varphi_1}\hat{\mathbf{D}}^\dagger(\alpha)\langle e\|)$ |
| 3 | $\pi/2$-pulse | $\frac{1}{4}[\|g\rangle(1-e^{i\varphi}\hat{\mathbf{D}}(\alpha))+e^{i\varphi_3}\|e\rangle(1+e^{i\varphi}\hat{\mathbf{D}}(\alpha))]\hat{\boldsymbol{\rho}}$ $\times[(1-e^{-i\varphi}\hat{\mathbf{D}}^\dagger(\alpha))\langle g\|+e^{-i\varphi_3}(1+e^{-i\varphi}\hat{\mathbf{D}}^\dagger(\alpha))\langle e\|]$ |
| 4 | projection on $\|g\rangle\langle g\|$ | $\frac{1}{4}\|g\rangle\langle g\|(1-e^{i\varphi}\hat{\mathbf{D}}(\alpha))\hat{\boldsymbol{\rho}}(1-e^{-i\varphi}\hat{\mathbf{D}}^\dagger(\alpha))/P_g$ |

The next step is a Raman laser pulse tuned resonant to the frequency $\omega$ of the trapping potential using the circularly polarized lasers $c$ and $b$. The density operator of the combined system is then given by

$$\hat{\boldsymbol{\rho}}_2=\frac{1}{2}(|g\rangle+e^{i\varphi_1}|e\rangle\hat{\mathbf{D}}(\alpha))\hat{\boldsymbol{\rho}}(\langle g|+e^{-i\varphi_1}\hat{\mathbf{D}}^\dagger(\alpha)\langle e|). \quad (3)$$

This is a shift $\hat{\mathbf{D}}(\alpha)=e^{\alpha\hat{\mathbf{a}}^\dagger-\alpha^*\hat{\mathbf{a}}}$ in phase space just for that part of the motional state that is correlated with the internal state $|e\rangle$. Here $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ are the usual annihilation and creation operators of the harmonic trap oscillator. The modulus of $\alpha$ is controlled by the duration and strength of the pulse, and the phase is again controlled by the phase difference of the lasers.

So far, the shift operator $\hat{\mathbf{D}}(\alpha)$ is still entangled with the internal state $|e\rangle$. We can now remove this entanglement by projecting on a superposition of the two states $|g\rangle$ and $|e\rangle$. This is achieved by applying another resonant Raman laser pulse, such as the first pulse, yielding the transformation $|g\rangle\rightarrow(|g\rangle+e^{i\varphi_3}|e\rangle)/\sqrt{2}$ and $|e\rangle\rightarrow(-e^{-i\varphi_3}|g\rangle+|e\rangle)/\sqrt{2}$, where $\varphi_3$ is the phase difference of the two lasers $a$ and $b$. This gives the new total state

$$\begin{aligned}\hat{\boldsymbol{\rho}}_3=&\frac{1}{4}[|g\rangle(1-e^{i(\varphi_1-\varphi_3)}\hat{\mathbf{D}}(\alpha))+|e\rangle(e^{i\varphi_3}+e^{i\varphi_1}\hat{\mathbf{D}}(\alpha))]\hat{\boldsymbol{\rho}}\\&\times[(1-e^{-i(\varphi_1-\varphi_3)}\hat{\mathbf{D}}^\dagger(\alpha))\langle g|\\&+(e^{-i\varphi_3}+e^{-i\varphi_1}\hat{\mathbf{D}}^\dagger(\alpha))\langle e|]. \quad (4)\end{aligned}$$

Finally measuring the population of the lower internal state $|g\rangle$, we find the probability

$$\begin{aligned}P_g(\alpha,\varphi)&=\frac{1}{4}\mathrm{Tr}\{(1-e^{i\varphi}\hat{\mathbf{D}}(\alpha))\hat{\boldsymbol{\rho}}(1-e^{-i\varphi}\hat{\mathbf{D}}^\dagger(\alpha))\}\\&=\frac{1}{2}-\frac{1}{2}\mathrm{Re}[e^{i\varphi}\mathrm{Tr}\{\hat{\mathbf{D}}(\alpha)\hat{\boldsymbol{\rho}}\}], \quad (5)\end{aligned}$$

where $\varphi=\varphi_1-\varphi_3$. We now see that $P_g(\alpha,\varphi)$ is directly related to the real part of the characteristic function of the Wigner function

$$\chi(\alpha)=\mathrm{Tr}\{\hat{\boldsymbol{\rho}}\hat{\mathbf{D}}(\alpha)\}, \quad (6)$$

through

$$P_g(\alpha,\varphi)=\frac{1}{2}-\frac{1}{2}\mathrm{Re}[e^{i\varphi}\chi(\alpha)]. \quad (7)$$

Therefore, we obtain the whole complex function at the point $\alpha$,

$$\chi(\alpha)=1-2P_g(\alpha,0)+i\left[2P_g\left(\alpha,\frac{\pi}{2}\right)-1\right], \quad (8)$$

by a measurement of $P_g(\alpha,\varphi)$ for two phases $\varphi=0,\pi/2$. Due to the property $\chi^*(\alpha)=\chi(-\alpha)$ the number of necessary measurements effectively reduces to the number of desired different values of $\alpha$.

Since the characteristic function contains the full information about the state, we can easily calculate any expectation value from it [13], e.g., the density operator is given by

$$\hat{\boldsymbol{\rho}}=\int\frac{d^2\alpha}{\pi}\chi(\alpha)\hat{\mathbf{D}}^\dagger(\alpha). \quad (9)$$

The characteristic function is the Fourier transform of the Wigner function

$$W(\beta)=\int\frac{d^2\alpha}{\pi^2}\chi(\alpha)e^{\beta\alpha^*-\beta^*\alpha} \quad (10)$$

and, therefore, behaves in a complementary way: Displacements in phase space lead to phase factors in the characteristic function, whereas oscillations of the Wigner function are represented by shifts in the characteristic function. All the information about the state is as easily accessible as in the Wigner function. A detailed discussion about a direct interpretation of the characteristic function, which is also known as the Shirley or ambiguity function, can be found in Ref. [14].

The characteristic function of the Wigner function has a feature of particular use for state reconstruction. In the vicinity around $\alpha=0$, $\chi(\alpha)$ already contains enough information to get the $Q$ function. The reason for this is that the characteristic function of the $Q$ function $\chi_Q(\alpha)=\chi(\alpha)\exp(-|\alpha|^2/2)$ is approximately given by the characteristic function of the Wigner function in the neighborhood of $\alpha=0$, and is negligible elsewhere. Therefore, it is very easy to get a rough picture of the state $\hat{\boldsymbol{\rho}}$. This helps to make the experiment efficient since the regions of phase space, where it is relevant to measure $\chi(\alpha)$, are known from the

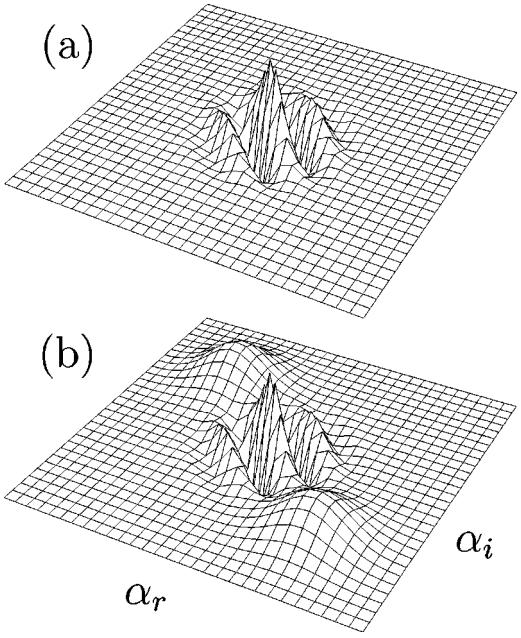


FIG. 2. Characteristic function $\chi(\alpha=\alpha_r+i\alpha_i)$ of an incoherent and a coherent Schrödinger-cat state. In (a) we show $\chi(\alpha)$ for an incoherent superposition $\hat{\boldsymbol{\rho}}=1/2|\beta\rangle\langle\beta|+1/2|-\beta\rangle\langle-\beta|$. The amplitude $\beta=1.5-1.5i$ is encoded in the oscillations. In (b) we show $\chi(\alpha)$ for an even Schrödinger-cat state with the same amplitude $\beta$ as above. The coherence is represented by the two additional bumps at $\pm 2\beta$ in comparison with (a). Note that in both cases $\chi(\alpha)$ is real.

knowledge of the $Q$ function. We show in Fig. 2 the characteristic functions of an incoherent and a coherent Schrödinger-cat state with the same amplitudes. Both characteristic functions show the same oscillations around the origin that encode the complex amplitudes. The coherence of the Schrödinger-cat state is reflected in the two additional bumps, away from the origin. The position of these bumps can already be predicted from the knowledge of $\chi(\alpha)$ around $\alpha=0$.

The main ingredient of our approach is a superposition of the quantum state and the quantum state shifted in phase space by an amplitude $\alpha$. From the interference term, we can then extract the characteristic function of the Wigner function. In principle, our approach has similarities with schemes to prepare a Schrödinger-cat state [2]. The main difference is that the two constituent coherent states of the Schrödinger cat are replaced by the shifted unknown state, which is reconstructed.

Actually all the experimentally required techniques for our scheme have already been successfully realized by Monroe *et al.* in Ref. [2]. But it has not been pointed out earlier that the data shown in their Figs. 4 and 5 can be interpreted as characteristic functions for various coherent states. The difference is that, in addition, they correlate the internal state $|g\rangle$ in Eq. (3) with another shift $\hat{\mathbf{D}}(-\tilde{\alpha})$, where $\tilde{\alpha}=\alpha e^{i\phi}$. With their starting state $\hat{\boldsymbol{\rho}}=|0\rangle\langle 0|$, this leads to

$$P_g=\frac{1}{2}-\frac{1}{2}\mathrm{Re}[e^{i\varphi}\mathrm{Tr}\{\hat{\mathbf{D}}(\alpha)\hat{\mathbf{D}}(\tilde{\alpha})|\alpha\rangle\langle\alpha|\}]. \qquad (11)$$

The two shift operators can be combined to a single one, except for a phase factor $e^{i\mathrm{Im}(\alpha\tilde{\alpha}^*)}$. By comparing Eqs. (11) and (5) we realize that their results are related to the characteristic function of a coherent state $|\alpha\rangle$ at the position $\alpha+\tilde{\alpha}$. Their theoretical curves show very good agreement with the experiment, directly demonstrating the feasibility of our approach.

An advantage of our scheme is that the time required to perform all the steps is very short, because we only use resonant interactions. This plays an essential role concerning the loss of coherence during the measurement procedure. The time scale of decoherence is about 84 $\mu$s [1]. The interactions of the first and third steps require 0.5 $\mu$s each, and the second step takes about 10 $\mu$s. The procedure for directly measuring the Wigner function proposed by Lutterbach and Davidovich [12] requires an additional 50 $\mu$s. That is because their scheme requires an accessory slow dispersive interaction to achieve a correlation of the type of Eq. (3), where the shift is replaced by a rotation. Likewise, the method by Leibfried *et al.* [5], which was used to measure the Wigner function, requires another step. It is based on an additional monitoring of the time evolution of the system according to a Jaynes-Cummings-type interaction. This complicates the data analysis further. The method by Kim *et al.* [11] requires an additional time evolution according to the Jaynes-Cummings model as well.

Despite the fact that we have only discussed the use of our scheme to measure the state of a trapped atom, it is, in principle, possible to apply it to measure the quantum state of a cavity field as well. The problem is to realize a displacement operator for the cavity field state correlated to the internal state of a passing atom. Such a displacement operation has been suggested by Davidovich *et al.* [15] in the context of quantum switches using a dispersive atom-field interaction. Unfortunately, it is necessary to compensate additional rotations of the field state in phase space, depending on the amplitude of the phase shift. Even though this can be done—for instance, by a more complicated data analysis—the experimental realization is more difficult. Also, the control of parameters, such as interaction strength and time, is less precise since the atoms can take different paths through the cavity due to imperfect collimation.

To summarize, we have introduced a method to directly measure the characteristic function of the Wigner function of the center-of-mass motion of a trapped atom. We have shown that our method can be readily used in an actual experiment. It is simple and fast with respect to both experiment and data analysis.

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[1] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **76**, 1796 (1996).

[2] C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, Science **272**, 1131 (1996).

[3] O. Benson, M. Weidinger, G. Gaithel, and H. Walther, J. Mod. Opt. **44**, 2011 (1997).

[4] X. Maître, E. Hagley, G. Nogues, C. Wunderlich, P. Goy, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 769 (1997); C. Monroe, D. Leibfried, B. E. King, D. M. Meekhof, W. M. Itano, and D. J. Wineland, Phys. Rev. A **55**, R2489 (1997).

[5] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).

[6] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **77**, 4887 (1996).

[7] W. H. Zurek, Phys. Today **44** (10), 36 (1991).

[8] For the measurement of the motional state of a trapped atom, see S. Wallentowitz and W. Vogel, Phys. Rev. Lett. **75**, 2932 (1995); J. F. Poyatos, R. Walser, J. I. Cirac, and P. Zoller, Phys. Rev. A **53**, R1966 (1996); C. D'Helon and G. J. Milburn, *ibid.* **54**, R25 (1996); P. J. Bardroff, C. Leichtle, G. Schrade, and W. P. Schleich, Phys. Rev. Lett. **77**, 2198 (1996); M. Freyberger, Phys. Rev. A **55**, 4120 (1997); C. T. Bodendorf, G. Antesberger, M. S. Kim, and H. Walther, *ibid.* **57**, 1371 (1998).

[9] For the measurement of the state in the cavity, see W. Vogel, D.-G. Welsch, and L. Leine, J. Opt. Soc. Am. B **4**, 1633 (1987); M. Freyberger and A. M. Herkommer, Phys. Rev. Lett. **72**, 1952 (1994); P. J. Bardroff, E. Mayr, and W. P. Schleich, Phys. Rev. A **51**, 4963 (1995); O. Steuernalgel, J. A. Vaccaro, Phys. Rev. Lett. **75**, 3201 (1995); R. Walser, J. I. Cirac, and P. Zoller, *ibid.* **77**, 2658 (1996); S. Schneider, A. M. Herkommer, U. Leonhardt, and W. P. Schleich, J. Mod. Opt. **44**, 2333 (1997); M. S. Zubairy, Phys. Rev. A **57**, 2066 (1998).

[10] M. Wilkens and P. Meystre, Phys. Rev. A **43**, 3832 (1991).

[11] M. S. Kim, G. Antesberger, C. T. Bodendorf, and H. Walther, Phys. Rev. A **58**, R65 (1998).

[12] L. G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).

[13] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).

[14] S. Stenholm and N. Vitanov, J. Mod. Opt. (to be published).

[15] L. Davidovich, A. Maali, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **71**, 2361 (1993).