PHYSICAL REVIEW A VOLUME 59, NUMBER 2 FEBRUARY 1999

Order-α radiative correction to the rate for parapositronium decay to four photons

Gregory S. Adkins and Eric D. Pfahl
Franklin and Marshall College, Lancaster, Pennsylvania 17604

(Received 13 July 1998)

We have calculated the $O(\alpha)$ radiative correction affecting the decay rate of parapositronium to four photons. Our result for the rate is $\Gamma(4\gamma)=\Gamma_{\rm LO}(4\gamma)[1-14.5(6)\alpha/\pi+O(\alpha^2)]$, where $\Gamma_{\rm LO}(4\gamma)$ is the lowest order four-photon decay rate. Our prediction for the four-photon to two-photon branching ratio, including all $O(\alpha)$ corrections, is $R=1.4388(21)\times10^{-6}$. We also have a higher precision result for the lowest-order rate: $\Gamma_{\rm LO}(4\gamma)=0.013\,895\,7(4)m\alpha^7$. [S1050-2947(99)50202-3]

PACS number(s): 36.10.Dr, 12.20.Ds, 31.15.Md, 31.30.Jv

It has recently become possible to measure the branching ratio R for the four-photon vs two-photon decay of parapositronium. A measurement of the four-photon decay, reported in 1990, yielded a result [1]

$$R=[1.30\pm0.26(\text{statistical})\pm0.16(\text{systematic})]\times10^{-6}. \tag{1}$$

Two measurements reported in 1994 gave more precise results [2,3]:

$$R=[1.50\pm0.07(\text{statistical})\pm0.09(\text{systematic})]\times10^{-6}, \tag{2}$$

$$R=[1.48\pm0.13(\text{statistical})\pm0.12(\text{systematic})]\times10^{-6}. \tag{3}$$

Another measurement was obtained in 1996 as a by-product of a search for C-violating decays [4]:

$$R=[1.19\pm0.14(\text{statistical})\pm0.22(\text{systematic})]\times10^{-6}. \tag{4}$$

Theoretically, the branching ratio $R=\Gamma(4\gamma)/\Gamma(2\gamma)$ is obtained as the ratio of four-photon to two-photon decay rates, which are calculated individually. The $O(\alpha)$ corrected two-photon decay rate was found by Harris and Brown to be [5,6]

$$\Gamma(2\gamma)=\Gamma_{\rm LO}(2\gamma)\left\{1-\left(5-\frac{\pi^2}{4}\right)\frac{\alpha}{\pi}+O(\alpha^2)\right\}, \tag{5}$$

where the lowest-order two-photon rate

$$\Gamma_{\rm LO}(2\gamma)=\frac{1}{2}m\alpha^5 \tag{6}$$

was calculated long ago by Wheeler [7] and by Pirenne [8]. The lowest-order four-photon rate has been obtained by several groups [9–13]. The most precise of these is due to Adachi [13], who found a lowest-order QED prediction of $R_{\rm LO}=1.4796(6)\times10^{-6}$, which translates into

$$\Gamma_{\rm LO}(4\gamma)=0.013\,893(6)m\alpha^7. \tag{7}$$

The $O(\alpha)$ corrections to multiphoton decay rates tend to enter at the few percent level [0.6% for the two-photon rate of Eq. (5) and 2.4% for the three-photon decay of orthopositronium [14–16]]. Since the combined uncertainty of the results in Eqs. (2) and (3) is 6.4%, it seems safe to compare these results to the lowest-order value of R. However, the $O(\alpha)$ correction to R will have to be known in order to analyze future, more precise experimental results. Our main result in this Rapid Communication is a calculation of the $O(\alpha)$ correction to $\Gamma(4\gamma)$, and thus to R.

The general expression for the parapositronium to four-photon decay rate is

$$\Gamma(4\gamma)=\int d(PS)\,\frac{1}{4!}\overline{|M|^2}, \tag{8}$$

where M is the amplitude for parapositronium decay at rest into four photons having momenta k_i and polarization ϵ_i, and $\overline{|M|^2}$ is the spin sum

$$\overline{|M|^2}=\sum_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}|M|^2. \tag{9}$$

The four-photon phase-space integral is [17]

$$\int d(PS)=\int\frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3}\frac{d^3k_4}{(2\pi)^3 2\omega_4}\times(2\pi)^4\delta(P-k_1-k_2-k_3-k_4), \tag{10}$$

where $\omega_i=|\vec{k}_i|$ and $P\approx(2m,\vec{0})$. (It is adequate at this order of approximation to ignore the binding energy and take the parapositronium rest mass to be simply $2m$.) After using the four-dimensional δ function and performing a trivial integration over the three Euler angles describing the overall orientation in space of the decay configuration, the phase space is described by five variables. We made use of two explicit parametrizations of the phase space. The first,

$$\int d(PS)=\frac{m^4}{2^{10}\pi^6}\int_0^1 dx_1\int_0^{x_2^{\rm max}}dx_2\int_0^{\pi}d\theta_2\sin\theta_2\times\int_0^{\pi}d\theta_3\sin\theta_3\int_0^{2\pi}d\phi_3 x_1x_2H, \tag{11}$$

where $x_i=\omega_i/m$, is in terms of the (normalized) energies of photons 1 and 2, the polar angles of photons 2 and 3, and the

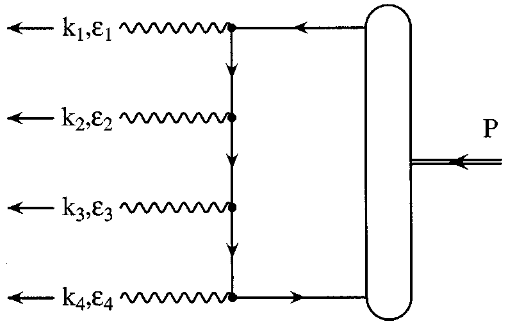

FIG. 1. Lowest-order parapositronium to four-photon decay diagram.

azimuthal angle of photon 3. (The direction of photon 1 is taken to define the z axis, and photon 2 is taken to lie in the xz plane.) Here one has

$$x_2^{\max}=\frac{2(1-x_1)}{2-x_1(1-\cos\theta_2)}, \tag{12}$$

$$x_3=\frac{Q^2}{2m(Q_0-\vec{Q}\cdot\hat{\vec{k}}_3)}, \tag{13}$$

where $Q=P-k_1-k_2$. The function H has the form $H=4m^2x_3^2/Q^2$. The second phase-space parametrization is based on invariant variables, and has the form [18,19]

$$\int d(PS)=\frac{m^4}{2^7\pi^6}\int_0^1 ds_1\int_0^{s_1}ds_2\int_{s_2/s_1}^{1-s_1+s_2}du_1 \times\int_{u_2^-}^{u_2^+}du_2\int_{-1}^{1}d\zeta\frac{1}{\sqrt{\lambda(1,s_2,s_2')(1-\zeta^2)}}, \tag{14}$$

where $u_2^{\pm}$ and λ are defined in [19], as are expressions for the inner products $k_i\cdot k_j$ in terms of the integration variables. We found that the decay rate contributions were generally smoother when using the second phase-space parametrization. Unfortunately, the second parametrization led more rapidly to floating-point errors, often forcing us to stick with the first parametrization or to use quadruple precision variables in our numerical routines.

The lowest-order decay rate comes from Eq. (8), with M replaced by the lowest-order decay amplitude $M_{\rm LO}$ (depicted in Fig. 1). The lowest-order amplitude can be written as

$$M_{\rm LO}=\frac{1}{m^3}\sum_{S_4}\ {\rm tr}\left[(-ie\gamma\epsilon_4^*)\frac{i}{-\gamma r_4-1}(-ie\gamma\epsilon_3^*)\frac{i}{\gamma r_{12}-1} \times(-ie\gamma\epsilon_2^*)\frac{i}{\gamma r_1-1}(-ie\gamma\epsilon_1^*)\Psi\right], \tag{15}$$

where the sum is over the 4! elements of the permutation group S_4, $r_i=n-k_i/m$, $r_{ij}=n-(k_i+k_j)/m$, $n=(1,\vec{0})$, and the wave-function factor Ψ is the product of the nonrelativistic wave function at contact $\phi_0=[m^3\alpha^3/(8\pi)]^{1/2}$ times a normalized spin-zero matrix factor [20]:

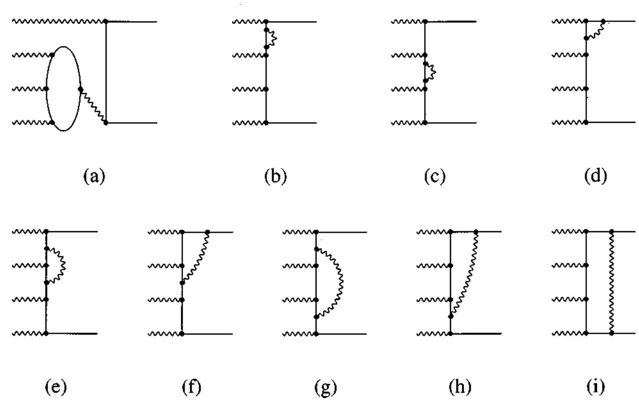

FIG. 2. Diagrams contributing $O(\alpha)$ corrections to the four-photon decay of parapositronium. They represent (a) light-by-light scattering, (b) outer self-energy, (c) inner self-energy, (d) outer single vertex, (e) inner single vertex, (f) outer double vertex, (g) inner double vertex, (h) triple vertex, and (i) ladder contributions to the decay rate. The bound-state wave function is implicit in each diagram. Note that diagrams (a), (b), (d)–(f), and (h) must be doubled.

$$\Psi=\phi_0\begin{pmatrix}0 & 1/\sqrt{2}\\ 0 & 0\end{pmatrix}. \tag{16}$$

One finds that

$$M_{\rm LO}=\frac{i\pi^{3/2}\alpha^{7/2}}{m^{3/2}}\sum_{S_4}\frac{1}{x_1x_{12}x_4}\frac{1}{4} \times{\rm tr}[\gamma\epsilon_4^*(-\gamma r_4+1)\gamma\epsilon_3^*(\gamma r_{12}+1)\gamma\epsilon_2^* \times(\gamma r_1+1)\gamma\epsilon_1^*(\gamma n+1)\gamma_5], \tag{17}$$

where $x_i=n\cdot k_i/m$ and $x_{ij}=x_i+x_j-k_i\cdot k_j/m^2$. The polarization sums were done using the replacement

TABLE I. Contributions to the parapositronium to four-photon decay rate in units of $(\alpha/\pi)\Gamma_{\rm LO}(4\gamma)$. Infrared divergences are regulated by a photon mass λ (measured in terms of the electron mass).

Graph	Contribution
Light-by-light	$-0.0271(3)$
Outer self-energy	$4\ln\lambda+3.178(12)$
Inner self-energy	$2\ln\lambda+2.690(5)$
Outer single vertex	$-4\ln\lambda+0.351(9)$
Inner single vertex	$-4\ln\lambda-0.987(3)$
Outer double vertex	$-1.430(6)$
Inner double vertex	$-3.280(8)$
Triple vertex	$-1.78(30)$
Ladder	$\frac{2\pi}{\lambda}+2\ln\lambda-13.21(52)$
Total	$\frac{2\pi}{\lambda}-14.5(6)$

TABLE II. Results from numerical integration for $I_L(3)$, the $T(0)$ part of $T(q)$, at various values of the photon mass parameter λ [in units of $(\alpha/\pi)\Gamma_{\rm LO}(4\gamma)$]. Removal of the threshold and logarithmic singularities gives $I_L'(3)=I_L(3)-(2\pi/\lambda+2\ln\lambda)$.

λ	$I_L(3)$	$I_L'(3)$
0.20	20.025(34)	−8.172(34)
0.15	29.250(57)	−8.844(57)
0.12	38.842(78)	−9.277(78)
0.10	48.527(109)	−9.699(109)
0.07	74.138(209)	−10.303(209)
0.05	108.537(396)	−11.135(396)

$$\sum_{\epsilon} \epsilon^*_\mu \epsilon_\nu \to -g_{\mu\nu}. \tag{18}$$

We performed the traces with the symbolic manipulation system REDUCE [21], and did the resulting five-dimensional integral using the adaptive Monte Carlo integration routine, VEGAS [22]. Our result for the lowest-order rate is

$$\Gamma_{\rm LO}(4\gamma)=0.013\,895\,7(4)m\alpha^7. \tag{19}$$

The $O(\alpha)$ contribution to the decay rate was computed following the procedure of Stroscio and Holt [23] and Caswell, Lepage, and Sapirstein [14] for the analogous problem of orthopositronium decay to three photons, except that the interpretation and removal of the ladder graph threshold singularity was handled by way of nonrelativistic QED (NRQED) [24,25]. The contributions for the various diagrams of Fig. 2 are shown in Table I. The unrenormalized self-energy and single vertex diagrams are divergent in the ultraviolet but finite in the infrared. The renormalized diagrams, however, contain infrared divergences that are logarithmic in the photon mass that is used for infrared regularization. We used known analytic forms for the self-energy and single-vertex functions, and used Feynman parameters to combine denominators for the double-vertex, triple-vertex, ladder, and light-by-light graphs. The integrals (of up to ten dimensions) were performed numerically using VEGAS. For the nine-dimensional triple-vertex integral we found it necessary to use the second phase-space parametrization, and break up the integral over the angular variable ζ into a number of parts that were evaluated individually.

The ladder graph contains a threshold singularity and requires special consideration. The integral for this graph has the form

$$I_L=\int d(\mathrm{PS})\sum_{S_4}\int\frac{d^4q}{i\pi^2}\frac{T(q)}{Z(q)} \times[(-q^2+\lambda^2)(-q^2+2qn)(-q^2-2qn)]^{-1}, \tag{20}$$

where $Z(q)=[-(q+r_1)^2+1][-(q+r_{12})^2+1][-(q-r_4)^2+1]$ and $T(q)$ is a trace factor. (The loop momentum q and photon mass λ are measured in terms of the electron mass m.) We found it useful to write $T(q)$ as the sum of three terms: $T(q)-T(0)-T_1$, T_1, and $T(0)$, where T_1 is the part of $T(q)$ that is linear in q. In the first two terms we set $\lambda\to 0$ and used the second phase-space parametrization to do the integrals. We found values of $-13.82(7)$ and $12.82(19)$, respectively [in units of $(\alpha/\pi)\Gamma_{\rm LO}(4\gamma)$]. For the third term $I_L(3)$ we used the first phase-space parametrization to perform the integrals at various values of λ as shown in Table II. The uncertainties grew rapidly for λ smaller than 0.10. The $1/\lambda$ threshold singularity is evident in the results of Table II. Analysis shows that the integral behaves like $2\pi/\lambda+2\ln\lambda+O(1)$ for small λ. [The analogous integral for the two-photon decay of parapositronium can be done exactly, and has an expansion $2\pi/\lambda+2\ln\lambda+A+B\lambda+C\lambda^2+D\lambda^3+(E\ln\lambda+F)\lambda^4+G\lambda^5+H\lambda^6+O(\lambda^7)$, where A-H are all constants, with $A=-2-2\ln 2$, $B=\pi/12$, $C=-1/3$, etc.] We subtracted $2\pi/\lambda+2\ln\lambda$ from the results shown in the second column of Table II to obtain the third column $I_L'(3)$ of that table. We fit the $I_l'(3)$ numbers to a quadratic form $A+B\lambda+C\lambda^2$ (see Fig. 3). Our best-fit value for A was -12.21 with a fitting error of 0.47. The $2\pi/\lambda$ in the final result represents the threshold singularity, and is removed in the process of NRQED matching [25].

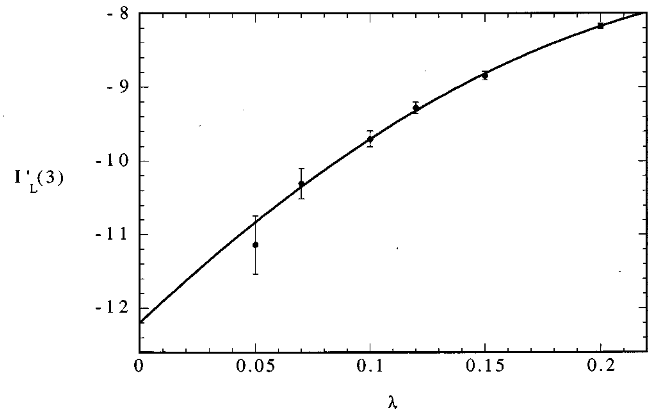

FIG. 3. Graph shows $I_L'(3)$ as a function of the photon mass λ. The data points with error bars represent results from numerical integration (the third column of Table II). The solid line is the quadratic best fit $A+B\lambda+C\lambda^2$, with $A=-12.21(47)$, $B=29.9(6.3)$, and $C=-49(21)$.

Our final result for the $O(\alpha)$ corrected decay rate is

$$\Gamma(4\gamma)=\Gamma_{\rm LO}(4\gamma)\left\{1-14.5(6)\frac{\alpha}{\pi}+O(\alpha^2)\right\}. \tag{21}$$

This corresponds to a 3.4% correction to the lowest-order rate. Our prediction for the branching ratio is

$$R=1.4388(21)\times 10^{-6}. \tag{22}$$

Corrections of order α^2 have not been computed, but should be negligible compared to the order-α uncertainties given.

G.S.A. acknowledges the support of the National Science Foundation (through Grant Nos. PHY-9722074 and PHY-9711991) and the Franklin and Marshall College Grants Committee. We would like to thank Mark Skalsey for useful discussions and encouragement, and Dave Crandles for advice on the fitting procedure.

[1] S. Adachi, M. Chiba, T. Hirose, S. Nagayama, Y. Nakamitsu, T. Sato, and T. Yamada, Phys. Rev. Lett. **65**, 2634 (1990).
[2] H. von Busch, P. Thirolf, C. Ender, D. Habs, F. Köck, T. Schulze, and D. Schwalm, Phys. Lett. B **325**, 300 (1994).
[3] S. Adachi, M. Chiba, T. Hirose, S. Nagayama, Y. Nakamitsu, T. Sato, and T. Yamada, Phys. Rev. A **49**, 3201 (1994).
[4] J. Yang, M. Chiba, R. Hamatsu, T. Hirose, T. Matsumoto, and J. Yu, Phys. Rev. A **54**, 1952 (1996).
[5] I. Harris and L. M. Brown, Phys. Rev. **105**, 1656 (1957).
[6] We use the conventions and natural units [$\hbar = c = 1$, $\alpha = e^2/4\pi \approx (137)^{-1}$] of J. D. Bjorken and S. D. Drell [*Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964)]. The symbol m represents the electron mass.
[7] J. A. Wheeler, Ann. (N.Y.) Acad. Sci. **48**, 219 (1946).
[8] J. Pirenne, Arch. Sci. Phys. Nat. **29**, 265 (1947).
[9] A. Billoire, R. Lacaze, A. Morel, and H. Navelet, Phys. Lett. B **78**, 140 (1978).
[10] T. Muta and T. Niuya, Prog. Theor. Phys. **68**, 1735 (1982).
[11] G. S. Adkins and F. R. Brown, Phys. Rev. A **28**, 1164 (1983).
[12] G. P. Lepage, P. B. Mackenzie, K. H. Streng, and P. M. Zerwas, Phys. Rev. A **28**, 3090 (1983).
[13] S. Adachi, Bachelor thesis, Tokyo Metropolitan University, 1990 (unpublished). (This reference was obtained from [1].)
[14] W. E. Caswell, G. P. Lepage, and J. Sapirstein, Phys. Rev. Lett. **38**, 488 (1977).
[15] G. S. Adkins, Ann. Phys. (N.Y.) **146**, 78 (1983).
[16] G. S. Adkins, Phys. Rev. Lett. **76**, 4903 (1996).
[17] We choose to absorb a factor of $1/\sqrt{2M}$ (where $M \approx 2m$ is the bound-state mass) into the wave function. This leads to a different normalization for the wave function compared to [15].
[18] R. Kumar, Phys. Rev. **185**, 1865 (1969).
[19] E. L. Bratkovskaya, E. A. Kuraev, and Z. K. Silagadze, Phys. Lett. B **359**, 217 (1995).
[20] The wave-function factor Ψ is the integral over relative momentum p of a bound-state wave function $\Psi(p)$. To lowest-order approximation, p can be neglected in the rest of the graph compared to the other momentum variables, for which the scale is set by the electron mass. See [15] and [17].
[21] REDUCE *User's Manual 3.6*, edited by A. C. Hearn and J. P. Fitch (Konrad-Zuse-Zentrum, Berlin, 1996).
[22] G. P. Lepage, J. Comput. Phys. **27**, 192 (1978).
[23] M. A. Stroscio and J. M. Holt, Phys. Rev. A **10**, 749 (1974).
[24] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1986).
[25] P. Labelle, Ph.D. thesis, Cornell University, 1994 (unpublished).