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Local sampling of phase-space distributions by cascaded optical homodyning

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We propose the determination of phase-space distributions of optical fields by cascaded optical homodyning, where phase-randomized balanced homodyning is used for measuring the output photon-number statistics of an unbalanced homodyne detection scheme. The phase-space point of interest is controlled by the complex amplitude of the local oscillator and a universal sampling function is sufficient for mapping the measured quadrature statistics onto the phase-space distribution. [S1050-2947(99)50201-1]

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The first successful reconstruction of the quantum state of a radiation mode was based on balanced optical homodyne tomography [1]. In this method the quadrature distributions are measured for a (sufficiently dense) set of phases and the results of all these measurements are combined to numerically calculate a quasiprobability distribution that contains the full information on the quantum state. The mathematical procedure needed for homodyne tomography is the inverse Radon transform, which essentially consists of a threefold integral transform [2]. The pioneering tomography experiment has stimulated a manifold of theoretical work [3,4]. More direct ways have been derived for reconstructing the density matrix from the data recorded in balanced optical homodyning. The density matrix in a quadrature representation can be obtained via a twofold Fourier transform of the data [5]. Moreover, it is possible to derive the density matrix in a photon-number representation by a twofold integration of the data with appropriate sampling functions for each density-matrix element [6–8]. In these methods the accuracy of the reconstruction crucially depends on the number of adjusted local-oscillator phases [9].

The latter problem can be avoided in a local method for reconstructing the phase-space distribution of a light field, which is based on unbalanced homodyning [10,11]. The signal field is superimposed by the local oscillator whose intensity and phase uniquely define the point in phase space where the quasiprobability of the signal field is reconstructed. That is, this method yields a local reconstruction of the quantum state for each point in phase space independently of other points. Therefore, the density of the chosen set of phase-space points has no impact on the quality of the reconstructed quasiprobabilities, contrary to the situation in homodyne tomography. The phase-space distributions are simply derived as weighted sums of the photon-number statistics of the signal field that has been displaced by the local oscillator. For applying this method one has to discriminate between n and $n+1$ photons, which is a nontrivial task with available photodetection devices. To avoid these technical limitations, a multichannel detection scheme has been proposed [10,12], which requires exceeding both experimental and numerical efforts.

In the present paper we solve this problem by combining the unbalanced homodyne scheme for the local reconstruction of phase-space distributions [10,11] and the phase-randomized balanced homodyne detection of the photon statistics [13] to a unified detection scheme, which we call in the following cascaded optical homodyning. Here the phase-randomized balanced homodyne detection scheme serves as the photon counter in an unbalanced homodyne measurement. We derive a local sampling relation for the cascaded homodyne scheme that maps the measured phase-integrated quadrature statistics directly onto the quasiprobability of the signal field for the phase-space point defined by the setting of the local-oscillator amplitude in the unbalanced subdevice. An important result is that the complete phase space can be locally probed by a universal sampling function. This is in contrast to the, in principle, infinite set of sampling functions needed for the density-matrix reconstruction via balanced optical homodyning [6]. In our method rapidly oscillating sampling functions, which appear in balanced homodyning for large photon numbers, become superfluous. Moreover, we obtain an analytical expression for the sampling function for cascaded homodyning, which turns out to be an appropriately scaled zeroth-order pattern function known from quantum-state tomography. Consequently, the sampling function is well-behaved and the scheme is robust against noisy data.

Let us consider the cascaded optical homodyne scheme given in Fig. 1. The signal field is superimposed with the local-oscillator field LO1, which is in the coherent state $|\beta\rangle$, by the beam splitter BS1 whose amplitude-transmission coefficient T is close to unity. That is, $\epsilon=R/T\ll 1$ where R is the coefficient of amplitude reflection and $|T|^2+|R|^2=1$. The quantum state of the superimposed light field SL can be considered to be the signal state that has been displaced in phase space by the effective local-oscillator amplitude $\epsilon\beta$. The displaced signal field is then mixed by the second beam splitter BS2 with another local-oscillator field LO2 that is phase-randomized and strong compared with the displaced signal. The difference-count statistics of the output fields of BS2 is measured by the two photodetectors $D1$ and $D2$ of quantum efficiency η_D.

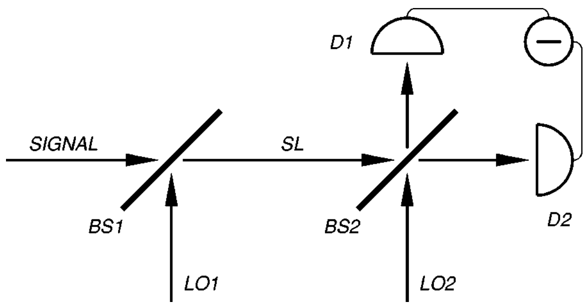

FIG. 1. Draft scheme of the cascaded optical homodyne setup. The signal field is measured with two detectors $D1$, $D2$, with the help of two local-oscillator fields LO1, LO2, and two beam splitters BS1, BS2.

The relation between the s-parametrized quasiprobability distribution $W(\alpha;s)$ and the photocount statistics $P_n(\alpha,\eta)$ of the displaced signal field (displacement amplitude $\alpha=-\epsilon\beta$) measured with overall quantum efficiency $\eta=\eta_D|T|^2$ reads as [10,11]

$$W(\alpha;s)=\frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}(-\xi)^n P_n(\alpha,\eta), \tag{1}$$

where the parameter $\xi=\xi(s,\eta)$ accounts for the detection losses and the ordering parameter of the desired quasiprobability distribution; it is given by

$$\xi(s,\eta)=\frac{2-\eta(1-s)}{\eta(1-s)}. \tag{2}$$

The photocount statistics $P_n(\alpha,\eta)$ of the displaced field SL can be obtained by balanced homodyne detection with a phase-randomized, strong local oscillator [13]. In our cascaded scheme this is realized by the second (50:50) beam splitter BS2, the strong (phase-randomized) local oscillator LO2, and the detectors $D1$ and $D2$.

Taking into account the losses due to imperfect detection of the displaced field, we easily obtain from the results of [13] the photocount statistics $P_n(\alpha,\eta)$ in the form

$$P_n(\alpha,\eta)=\int_{-\infty}^{\infty}dx f_{nn}(x)p(x;\alpha,\eta), \tag{3}$$

where $p(x;\alpha,\eta)$ is the phase-integrated quadrature distribution of the displaced signal field measured with the nonperfect detectors $D1$ and $D2$. The pattern functions $f_{nn}(x)$ can be expressed in terms of the regular and irregular eigenfunctions of the Schrödinger equation of the harmonic oscillator [7], $\psi_n(x)$ and $\varphi_n(x)$, respectively,

$$f_{nn}(x)=\frac{\partial}{\partial x}[\psi_n(x)\varphi_n(x)]. \tag{4}$$

Inserting Eq. (3) into Eq. (1) we obtain a formal expression for the direct sampling of the quasiprobability distribution $W(\alpha;s)$ from the measured data $p(x;\alpha,\eta)$

$$W(\alpha;s)=\int_{-\infty}^{\infty}dx S(x;s,\eta)p(x;\alpha,\eta). \tag{5}$$

The overall sampling function $S(x;s,\eta)$ is given by the infinite series of pattern functions

$$S(x;s,\eta)=\frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}[-\xi(s,\eta)]^n f_{nn}(x). \tag{6}$$

So far the sampling function is determined by an infinite number of pattern functions $f_{nn}(x)$, which become strongly oscillating functions of x for large values of the photon number n. Note that the sampling function $S(x;s,\eta)$ does not depend on the phase-space amplitude α determined by the local-oscillator field LO1. That is, the quasiprobability distribution can be obtained on the complete phase space from the measured data by integration with only one unique sampling function.

The calculation of the sampling function $S(x;s,\eta)$ as given in Eq. (6) is numerically not very stable due to the alternating sum over highly oscillating pattern functions. In the following we derive a closed analytic form of the sampling function. The result turns out to be surprisingly simple: it contains only the appropriately rescaled zeroth-order pattern function $f_{00}(x)$. No higher-order sampling functions are needed to determine the quantum state by the cascaded homodyne method. Note that in balanced optical homodyning one even needs the pattern functions $f_{nm}(x)$ $(n\neq m)$ in order to obtain off-diagonal density-matrix elements; cf. Refs. [6,7].

The task is now to give an analytic expression for the alternating sum $\Sigma_n(-\xi)^n f_{nn}(x)$ that determines our unique sampling function (6). Since the pattern functions $f_{nn}(x)$ are bound, for $\xi<1$ the sum in Eq. (6) converges and the sampling function exists. Therefore, in the following we will restrict ourselves to the case where $\xi<1$, giving us an upper bound for the s parameter of the reconstructable quasiprobability distributions of $s<(\eta-1)/\eta$.

Using the definition of the pattern functions (4) and exploiting the properties of the regular and irregular eigenfunctions of the harmonic oscillator [14], we obtain the relation

$$f_{nn}+f_{n+1,n+1}=2x(\psi_n\varphi_n-\psi_{n+1}\varphi_{n+1}). \tag{7}$$

For notational simplicity we ignore the argument x of the eigenfunctions. Summing up the left-hand side of Eq. (7) with alternating signs, $\Sigma_{k=0}^{n-1}(-1)^k(f_{kk}+f_{k+1,k+1})=f_{00}-(-1)^n f_{nn}$, we get

$$f_{00}-(-1)^n f_{nn} =2x\left[\psi_0\varphi_0+2\sum_{k=1}^{n}(-1)^k\psi_k\varphi_k-(-1)^n\psi_n\varphi_n\right]. \tag{8}$$

Multiplying both sides of this equation with ξ^n and summing up from $n=1$ to N, we obtain after rearranging some terms

$$\sum_{n=0}^{N}(-\xi)^n f_{nn}=\frac{1-\xi^{N+1}}{1-\xi}(f_{00}+2x\psi_0\varphi_0)-2x\,\frac{1+\xi}{1-\xi}\times\sum_{n=0}^{N}(-\xi)^n\psi_n\varphi_n+4x\,\frac{\xi^{N+1}}{1-\xi}\times\sum_{n=0}^{N}(-1)^n\psi_n\varphi_n. \quad (9)$$

With the help of the asymptotic forms of the regular and irregular wave functions [3,7], it can be shown that the growth of the last sum in Eq. (9) is linear in N, i.e., $\xi^{N+1}\Sigma_{n=0}^{N}(-1)^n\psi_n\varphi_n<cN\xi^{N+1}$, where c is a positive constant. Thus, for $\xi<1$ in the limit $N\to\infty$ the last term in Eq. (9) vanishes and we obtain

$$\sum_{n=0}^{\infty}(-\xi)^n f_{nn}=\frac{1}{1-\xi}(f_{00}+2x\psi_0\varphi_0)-2x\,\frac{1+\xi}{1-\xi}\times\sum_{n=0}^{\infty}(-\xi)^n\psi_n\varphi_n. \quad (10)$$

For obtaining a solution for the desired sum from Eq. (10), we now introduce the function

$$M(x;\xi)=\sum_{n=0}^{\infty}(-\xi)^n\psi_n(x)\varphi_n(x), \quad (11)$$

whose derivative can be identified via Eq. (4) as the left-hand side of Eq. (10). The relation given in Eq. (10) can then be rewritten as a first-order differential equation for $M(x;\xi)$,

$$\frac{\partial}{\partial x}M(x;\xi)=-2x\,\frac{1+\xi}{1-\xi}\,M(x;\xi)+\frac{1}{1-\xi}\times[f_{00}(x)+2x\psi_0(x)\varphi_0(x)], \quad (12)$$

with the initial condition $M(0;\xi)=0$ [15]. The solution of Eq. (12) can be written, after some algebra, in the compact form

$$M(x;\xi)=\frac{1}{\sqrt{1-\xi^2}}\,\psi_0\left(\sqrt{\frac{1+\xi}{1-\xi}}\,x\right)\varphi_0\left(\sqrt{\frac{1+\xi}{1-\xi}}\,x\right), \quad (13)$$

where we have used the structure of the zeroth-order eigenfunctions $\psi_0(x)$ and $\varphi_0(x)$ [16,17]. Combining Eqs. (13) and (4) we get the solution for the alternating sum over the pattern functions, which can be used to obtain the explicit form of the sampling function (6) of the cascaded optical homodyne scheme,

$$S(x;s,\eta)=\frac{\eta}{\pi[\eta(1-s)-1]}\,f_{00}\left(\frac{x}{\sqrt{\eta(1-s)-1}}\right), \quad (14)$$

where we used the expression for $\xi(s,\eta)$ given in Eq. (2). This result is strikingly simple in that we need only the lowest-order pattern function, which is rescaled to take into account the detection efficiency η and the s parameter. The zeroth-order pattern function $f_{00}(x)$ can be numerically calculated very efficiently by the use of Dawson's integral $F(x)$ [18], as $f_{00}(x)=2-4xF(x)$. Examples for the sampling function are shown in Fig. 2.

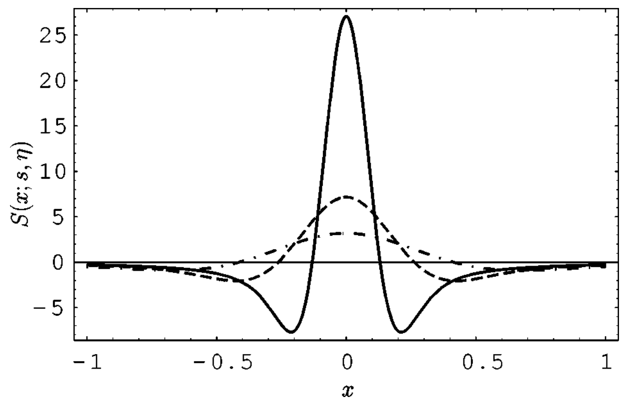

FIG. 2. Sampling function for $s=-0.2$ and efficiencies η=0.85 (solid), η=0.9 (dashed), and η=1.0 (dot-dashed).

To illustrate the applicability of our method we simulate a cascaded-homodyne experiment. In agreement with recent experiments [8,13] for obtaining the phase-integrated quadrature distribution $p(x;\alpha,\eta)$, we simulate for each phase-space amplitude 5000 measurements of the difference current of the detectors. The events are accumulated in 128 bins to derive the probabilities of obtaining a difference current in intervals of size Δx, the latter being chosen such that the relevant range of x values is contained in the set of bins. The quantum state of the signal field is supposed to be an odd coherent state. The overall efficiency and the s parameter of the desired quasiprobability distribution have been chosen as $\eta=0.9$ and $s=-0.2$, respectively. That is, we sample a phase-space distribution that is close to the Wigner function. The reconstructed phase-space distribution is shown in Fig. 3 along the real axis of phase space, together with the exact distribution, where the statistical variances of the reconstructed quasiprobabilities are given as error bars. Within the statistical uncertainties the reconstructed interference structure of the cat state is in good agreement with the exact one.

While the classical noise of the strong phase-randomized local oscillator LO2 is compensated by the balanced setup,

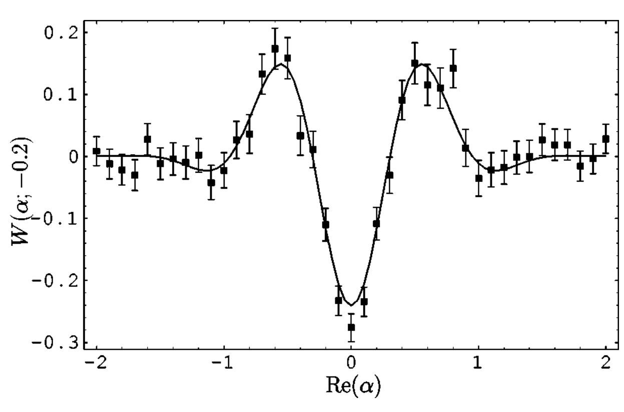

FIG. 3. Simulation of the reconstruction of $W(\alpha;-0.2)$ along the real axis for an odd cat state $|\alpha\rangle_-\propto(|\alpha\rangle-|-\alpha\rangle)$ with $\alpha=1.5i$ and $\eta=0.9$. For each phase-space point 5000 events have been sampled in 128 bins.

the noise of the first local oscillator LO1 leads to a smearing of the displacement of the signal field. It has been shown, that due to the fact that the reflected amplitude of LO1 and the amplitude of the transmitted signal field are of comparable magnitude, the effects of the noise of LO1 are of minor importance [10]. For a Gaussian smearing of the local-oscillator amplitude they effectively lead to a decrease of the overall detection efficiency η.

In summary we have shown that cascaded optical homodyning is suited for locally sampling phase-space distributions from data obtained by difference-count measurements. The sampling function needed for the reconstruction is a unique one; that is, regardless of the amplitude in phase space where the quasiprobability is probed for, the sampling function is always the same. An analytic expression for the sampling function has been derived that is simply given by a scaled zeroth-order pattern function known from homodyne tomography. So far, the cascaded homodyne method avoids the use of highly oscillating sampling functions and is therefore expected to be more stable with respect to noisy data. The method works for realistic experimental parameters and can be regarded as an alternative scheme to balanced optical homodyne tomography. The main advantage of the cascaded scheme consists in the local probing of phase space, which gives a more direct connection of the measured data to the desired quasiprobabilities via a single, universal sampling function.

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[14] We use the recursion relations $\chi_n' = x\chi_n - \sqrt{2(n+1)}\chi_{n+1}$ and $\sqrt{n+1}\chi_{n+1} = \sqrt{2}x\chi_n - \sqrt{n}\chi_{n-1}$ for the regular and irregular harmonic-oscillator eigenfunctions $\chi_n = \psi_n, \varphi_n$.

[15] $M(0;\xi)=0$, since in Eq. (11) $\psi_n(0)\varphi_n(0)=0$, because for all n either the regular or the irregular eigenfunctions are antisymmetric in x, i.e., $\psi_n(0)=0$ or $\varphi_n(0)=0$.

[16] The regular and irregular ground states read $\psi_0(x) = \pi^{-1/4}\exp(-x^2/2)$ and $\varphi_0(x) = 2\pi^{1/4}\exp(x^2/2)F(x)$, with $F(x)$ being Dawson's integral [18].

[17] Note that in this way we derived also the generating function $G(x;\xi) = \Sigma_n(-\xi)^n f_{nn}(x) = \partial M(x;\xi)/\partial x$ of the pattern functions, $f_{nn}(x) = 1/n!(-\partial/\partial\xi)^n G(x;\xi)|_{\xi=0}$, as $G(x;\xi) = f_{00}[\sqrt{(1+\xi)/(1-\xi)}x]/(1-\xi)$.

[18] For an efficient numerical calculation of Dawson's integral, $F(x) = \exp(-x^2)\int_0^x dt \exp(t^2)$, see Chap. 6.10 of W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C* (Cambridge University Press, Cambridge, 1995).