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Calculation of the Casimir force between imperfectly conducting plates

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The Lifshitz formalism for determining the attractive force between material bodies with generalized electromagnetic susceptibility is applied numerically to gold, copper, and aluminum. The deviation from the perfect conductivity Casimir force approximately agrees with a first-order plasma model calculation at large separation, but at separations corresponding to the plasma frequency, deviations of over 50% from the first-order model are found, while deviations from corrections up to second order are about 10%. These results are discussed in the context of recent measurements of the Casimir force. [S1050-2947(99)50605-7]

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I. INTRODUCTION

One of the surprising predictions of quantum electrodynamics, obtained by Casimir in 1948, is that two parallel, closely spaced mirrors (metallic plates) will be mutually attracted [1]. This attractive force is due to the modification of the electromagnetic mode structure between the plates as compared to free space; when a zero-point energy $\hbar\omega/2$ is assigned to each mode, there is a difference in the total energy between the plates as compared to free space. This leads to an attractive force between the plates of magnitude (per unit surface area when separated by a distance a)

$$F(a)=\frac{\pi^2}{240}\frac{\hbar c}{a^4}=0.013\ \ \mathrm{dyn}(\mu\mathrm{m})^4/\mathrm{cm}^2. \tag{1}$$

This prediction, of course, must break down when the mirror separation is sufficiently small, that is, when the mode frequencies being modified by the mirrors are near or above the plasma frequency of the metal used to make the mirrors; Casimir made use of this point as a physical reason for truncating an otherwise divergent integral in the force calculation.

In order to interpret precision experimental results, the attractive force as a function of separation for real metals must be theoretically determined. This can be done by use of the Lifshitz formalism for the determination of molecular attractive forces between bodies [2]. This formalism can be applied to any material for which the complex index of refraction as a function of frequency is known. In the limit of infinite conductivity, the Lifshitz and Casimir results for the attractive force are equal; the Lifshitz formalism can be thought of as a generalization of the Casimir calculation, although the source terms for each appear to be quite different. In the case of the Casimir calculation, the electromagnetic field is assumed to be quantized with a zero-point energy $\hbar\omega/2$ that serves as the source of the attractive force. In the Lifshitz calculation, the electromagnetic field is treated as classical; the source terms are the current and electrical polarization fluctuations in the material bodies. These fluctuations can be calculated from the quantum fluctuation-dissipation theorem and they persist at zero temperature [3,4]. The Casimir and Lifshitz approaches are different ways of looking at the same phenomenon: Since the material fluctuations persist at zero temperature, in order for the bodies to be in equilibrium with a radiation field at zero temperature, the classical field modes must also have a zero-point energy, but we need not assume that the field is quantized. In this regard, the Casimir force is analogous to the Planck calculation of the blackbody spectrum; one cannot decide between quantization of the source or the field in either situation. For the Casimir force, this point is addressed in [5].

Unfortunately, there is no simple form for the (complex) permittivity as a function of frequency for any particular metal, although a plasma (free-electron) model of a metal can be used for estimating the deviations from the perfectly conducting case. The first-order and second-order deviations of the force, assuming a plasma model of a metal with dielectric constant

$$\epsilon(\omega)\approx 1-\omega_p^2/\omega^2, \tag{2}$$

where $\omega_p^2=4\pi Ne^2/m_e$, with N the number density of the free electrons, is

$$\begin{aligned} F'(a)&=F(a)\left[1-\frac{16}{3}\frac{c}{\omega_p a}+24\left(\frac{c}{\omega_p a}\right)^2\right] \\ &=F(a)\left[1-\frac{8}{3\pi}\frac{\lambda_p}{a}+\frac{6}{\pi^2}\left(\frac{\lambda_p}{a}\right)^2\right], \end{aligned} \tag{3}$$

with $\lambda_p=2\pi c/\omega_p$. The first-order term was derived by Hargreaves [6] and by Schwinger *et al.* [7], and the second-order term by Bezzera *et al.* [8], in the limit $c/\omega_p a \ll 1$ (large plate separation). These corrections are nonphysical for $a<\lambda_p$; the second-order term in particular dominates as a approaches zero, and gives a larger force than the uncorrected Casimir force. A term-by-term expansion in a/λ_p is required in this region. For the case of dielectrics with an absorption resonance, when the plate separation is sufficiently small (compared to the resonance wavelength), the net force falls as $1/a^3$ (see [9], p. 230); we might expect a similar distance dependence for metals when the separation is small compared to λ_p. In any case, the simple plasma model is not a perfect description of a real metal.

It is possible to accurately determine the attractive force as a function of plate separation for any metal by a numerical calculation, provided we know its complex index of refraction as a function of frequency. The numerical calculation described here is based on using the tabulated complex index of refraction for the metal of interest as a function of frequency to determine its complex permittivity, the imaginary part of which is then used with the Kramers-Kronig relationships to find the permittivity along the positive imaginary frequency axis. This is then inserted into the Lifshitz expression for the Casimir force (see Sec. 90 of [4]) (at zero temperature, and assuming the plates are made of the same material)

$$\begin{aligned} F'(a)=\frac{\hbar}{2\pi^2c^3}\int_0^\infty\int_1^\infty p^2\xi^3\Bigg\{&\left[\frac{(s+p)^2}{(s-p)^2}e^{2p\xi a/c}-1\right]^{-1} \\ &+\left[\frac{[s+\epsilon(i\xi)p]^2}{[s-\epsilon(i\xi)p]^2}e^{2p\xi a/c}-1\right]^{-1}\Bigg\}dp\,d\xi, \end{aligned} \tag{4}$$

where $\omega=i\xi$ for imaginary ω, p is a real integration variable, and $\epsilon(i\xi)$ is a real function (the complex susceptibility has only a real component along the imaginary axis; this point will be discussed in the next section), and

$$s=\sqrt{\epsilon(i\xi)-1+p^2}. \tag{5}$$

Equation (4) is valid for a generalized complex permittivity [9], and makes possible, in principle, a calculation of the force of attraction for any distance if $\epsilon(i\xi)$ is known.

In some situations, such as when one of the surfaces is curved, the energy per unit area $E(a)$ is required to evaluate the force of attraction as a function of plate separation; the corrected energy $E'(a)$ can be easily determined by numerically integrating Eq. (4) (or, alternatively, the formalism developed in [9] can be used to determine the energy directly).

In the case where one plate is spherical, the force of attraction is given by

$$F_s(a)=2\pi RE(a), \tag{6}$$

where R is the radius of curvature. This result was first derived by Derjaguin and Abrikosova in 1957 [10], and has been applied more generally as the proximity force theorem [11]. The plasma correction for the Casimir energy, by integrating Eq. (3), is

$$E'(a)=E(a)\left[1-\frac{2}{\pi}\frac{\lambda_p}{a}+\frac{18}{5\pi^2}\left(\frac{\lambda_p}{a}\right)^2\right]. \tag{7}$$

II. NUMERICAL PROCEDURE

A. Determination of $\epsilon(i\xi)$

In Eq. (4), the permittivity of a material along the imaginary frequency axis $\epsilon(i\xi)$ is related to the force of attraction. The first step to determine $\epsilon(i\xi)$ is to find the complex permittivity as a function of frequency for the material of interest,

$$\epsilon(\omega)=\epsilon'(\omega)+i\epsilon''(\omega). \tag{8}$$

The complex index of refraction for a number of metals is tabulated as a function of frequency in several references [12,13], with the real part of the index listed as n and the imaginary part listed as k. The complex permittivity is given by

$$\epsilon'+i\epsilon''=n^2-k^2+2ink. \tag{9}$$

For the calculation described here, the tabulated data are interpolated in steps of 10^{12} rad/s (the distances we are interested in are of order 1 μm, corresponding to $\omega \approx 10^{15}$ rad/s). Values below those tabulated are obtained by extrapolating the lowest table value by $1/\omega$, the expected behavior for a metal; the upper limit is set to about 50×10^{15} rad/s, which the published tables extend to or beyond.

Next, the Kramers-Kronig relationships are used to determine ϵ along the imaginary axis (see [4], Sec. 82), noting that $\epsilon(i\xi)$ is real:

$$\epsilon(i\xi)=\frac{2}{\pi}\int_0^\infty\frac{x\epsilon''(x)}{x^2+\xi^2}dx+1. \tag{10}$$

For the metal of interest, $\epsilon(i\xi)$ is determined by a numerical integrating of this equation, using the values of $\epsilon''(\omega)$ as interpolated from the tabulated data. A simple rectangle rule numerical is used.

The calculational errors near the ends of the tabulated and extrapolated range do not significantly affect the integration of Eq. (4). The range was chosen to be large enough compared to the frequency range of interest so that the contributions from the end regions would be negligible.

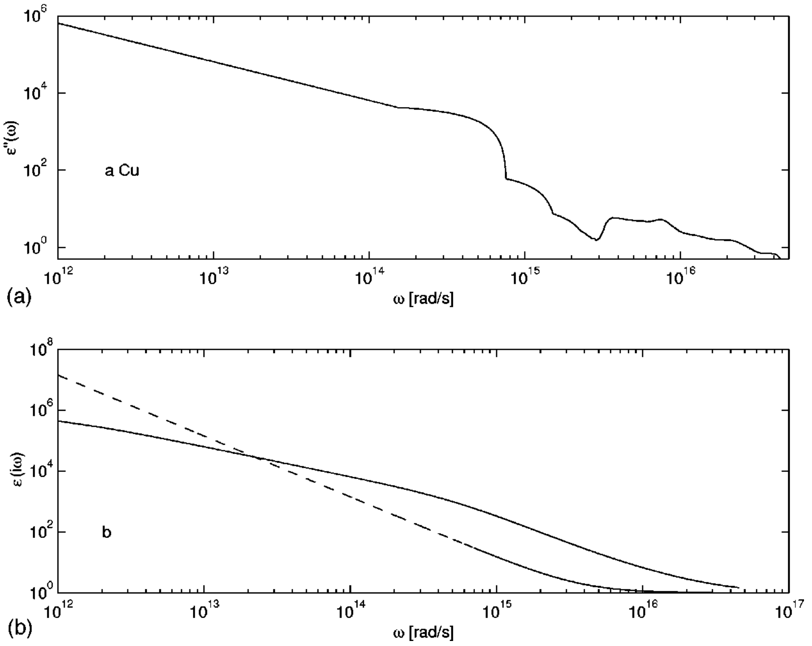

FIG. 1. (a) The imaginary component of the complex permittivity of Cu as a function of frequency from tabulated data. (b) The calculated permittivity along the imaginary axis (solid curve), compared to Eq. (12), with $\omega_p = 3.8 \times 10^{15}$ rad/s (dashed curve).

B. Determination of the force

Now that $\epsilon(i\xi)$ has been determined, the force of attraction can be found by numerically integrating Eq. (4). Following [4], Sec. 90, the substitution

$$x = 2p\xi a/c \tag{11}$$

is made in Eq. (4). The integration is done by first fixing a and p, then integrating over x (using a rectangle rule), yielding a function of p at fixed a; the direct dependence on a appears only as an overall factor of $1/a^4$, simplifying the comparison with the perfect conductivity case. The range and step size for the x integration are determined by the range of data for $\epsilon(i\xi)$, with step size set to $2pa/(c/1000)$. For the integration over p, the p step size is periodically adjusted so that $dp \approx p/100$, and the range of p is 1 to 900; this large upper limit ensures convergence of the integration for all a of interest. This is repeated as a function of a, in steps of 0.1 μm for $a = 0.05$ μm to 2.5 μm, and divided by the perfect conductivity Casimir force at that separation, yielding a table of correction factors as a function of a.

C. Determination of the energy

Simpson's rule is used to integrate the force found in the previous step, yielding an energy for each value of a in the table. The difficulty here is in the determination of the integration constant; this is found by extrapolating the numerical force calculation as $1/a$ to distances beyond 2.5 μm, and integrating the extrapolated force from ∞ to 2.5 μm. A slight error in this constant does not affect the integration to short distances. The integrated energy is divided by the perfect conductivity Casimir energy, again yielding a table of correction factors, as a function of a.

D. Accuracy of the calculations

The step sizes of the integrations were made a factor of 2 larger, and the changes in the final results were always less than 5%. We therefore claim that the numerical accuracy of the calculation is better than 5%. However, the limitation to the accuracy is certainly dominated by uncertainties in the tabulated complex index of refraction. In addition, the properties of a metallic surface are easily affected by surface contamination or oxidation, crystalline structure and preparation technique, and purity of the metal. In general, to make a comparison with theory, the complex surface index of the plates used in a Casimir force measurement would have to be measured as a function of frequency; there are simply too many variables to assume that the tabulated index is accurately representative of any specific sample. Nonetheless, the above calculation is useful for determining the general form of the force for imperfect conductors, and would be directly applicable to an experimental result, provided there is an accurate measurement of the complex index for the specific mirrors used in an experiment. Also, at frequencies above 5×10^{16} rad/s, the permittivity along the imaginary axis is very close to unity, so contributions to the force for higher frequencies are insignificant and we are justified in the choice of this as the upper integration limit [see Eq. (90.4) of [4]].

III. APPLICATIONS

The numerical determination of the Casimir force has been applied to three materials of recent experimental interest [14,15]: Au, Cu, and Al. The complex index of refraction as a function of frequency for these materials is tabulated in [12] pp. 12–131, 12–130, and 12–127, respectively). Much of these data also appear in [13].

For Au and Cu, the plasma wavelength $\lambda_p \approx 0.5$ μm. When $a = \lambda_p$, the plasma model gives first- and second-order force corrections terms of 0.848 and 0.607, resulting in an overall correction factor of 0.687. For Au, the numerically determined force correction factor at 0.5 μm is 0.657, while for Cu, it is 0.837. Similarly, Eq. (7) gives a plasma model energy correction factor 0.728 compared with numerical results of 0.719 (Au) and 0.874 (Cu). The plasma correction to second order is accurate to about 1% for Au, but is nearly

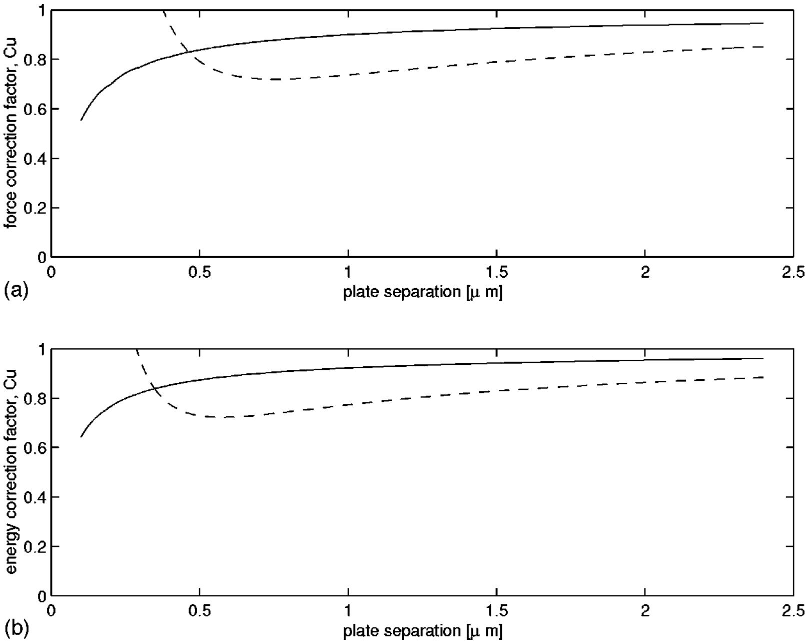

FIG. 2. The solid curve in each plot corresponds to the Lifshitz formalism calculation, while the dashed corresponds to the plasma correction to second order. (a) Force correction factor as a function of plate separation for Cu. (b) Energy correction factor for Cu.

20% below the numerical result for Cu (see Figs. 1 and 2).

The origin of the discrepancy in the case of Cu is evident in Fig. 1(b). For the plasma model, the permittivity along the imaginary axis should be

$$\epsilon(i\omega)=1+\omega_p^2/\omega, \tag{12}$$

but a significant deviation from the numerically calculated permittivity is evident in Fig. 1(b). This deviation is not simply due to the choice of λ_p, but due to the complicated structure of the Cu absorption spectrum. The deviation of the permittivity along the imaginary axis for Au as compared to Eq. (12) is not nearly as large as it is in the case of Cu, hence the closer agreement between the plasma and numerically determined corrections. The agreement for Au is likely in part coincidental because the convergence of the expansion for the plasma correction for $\lambda_p \approx 1$ is very slow; the first- and second-order corrections are nearly equal.

For Al, with $\lambda_p = 0.1\ \mu\text{m}$, the plasma force and energy correction factors at $a=\lambda_p$ are the same as above, 0.687 and 0.728, respectively, compared to 0.557 and 0.651, as determined numerically. The discrepancy in this case is likely due to a small resonance in Al absorption near $\omega=3\times10^{15}$ rad/s. The deviation between the plasma model and numerical calculation is about 11% for Al.

IV. CONCLUSION

In this paper, a general numerical technique for calculating the attractive force between plates with a known frequency-dependent complex index of refraction is developed and applied to three metals, Au, Cu, and Al. Except in the case of Au, the results from the numerical calculation are significantly different from the corrections (to second order) based on a plasma model description of the metals at separations approaching the plasma wavelength. In the case of Au, the agreement between the numerical and plasma corrections is probably coincidental because the expansions in Eqs. (3) and (7) are not necessarily valid when $\lambda_p/a \approx 1$. The results of this numerical analysis have already been applied to a previous measurement of the Casimir force [14], and an error relating to the calibration in the original measurement was found [16].

A more recent measurement of the Casimir force [15] made use of atomic force microscopy between Al coated surfaces and has a claimed precision of 1%. This precision is based on a comparison of experimental data with the plasma correction (to second order) for the Al films, and also takes into account surface roughness. Based on the above analysis, the simple plasma correction is about 11% different from a more accurate calculation (using the tabulated properties of Al) at separations around 100 nm (the minimum distance in [15]).

Furthermore, the roughness correction in [15] is on the order of 75% at 100 nm. The roughness correction in [8] was derived by simply performing a geometrical average of the distance variations between the two surfaces for a $1/a^4$ attractive force (or $1/a^3$ for the energy). Since the terms in the plasma force correction scale as $1/a$ and $1/a^2$, and the corrections are of order unity, the geometrical average should be done for the $1/a^5$ and $1/a^6$ functional dependencies separately. The multiplicative technique used in [15] for combining corrections is only valid when the corrections are small.

In conclusion, testing the theory of attractive force between metallic surfaces to high precision requires a number of auxiliary measurements of the physical properties of the surfaces and an extensive theoretical analysis for the effects of nonidealities. For example, in the numerical analysis outlined above, the tabulated complex indices of refraction of the materials of interest were used. In reality, the frequency-dependent index depends on a number of factors, including film purity, surface contamination, and preparation technique. A fundamental test of the Lifshitz theory would require a direct measurement of the frequency dependent-index for the particular metal (coated) plate under study, and a very careful analysis of possible roughness corrections. Under the circumstances, reliably testing the theory to better than 5% would seem a daunting task.

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