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# Optical control and entanglement of atomic Schrödinger fields

M. G. Moore and P. Meystre
*Optical Sciences Center and Department of Physics, University of Arizona, Tucson, Arizona 85721*

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We develop a fully quantized model of a Bose-Einstein condensate driven by a far off-resonant pump laser and interacting with a single mode of an optical ring cavity. This geometry leads to the generation of two condensate side modes that grow exponentially and are strongly entangled with the cavity mode. By changing the initial state of the optical field one can vary the quantum-statistical properties of the atomic side modes between thermal and coherent limits, as well as vary the degree of quantum entanglement.
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The recent demonstration of Bose-Einstein condensation (BEC) in low-density alkali-metal vapors [1,2] opens up a new paradigm in atomic/optical physics. It is now possible to generate macroscopic atomic fields whose quantum-statistical properties can in principle be manipulated and controlled, very much like those of quantum-optical fields. One important consideration is to define to which extent the quantum state of a many-particle atomic field can be *optically* manipulated. In the single-particle case, the answer to this problem is known to a large extent. This is the domain of atom optics [3], where a number of optical elements for matter waves have now been developed, including gratings, mirrors, interferometers, resonators, etc. But these optical elements manipulate just the atomic field ''density,'' or at most first-order coherence properties. However, Schrödinger fields possess a wealth of further properties past their first-order coherence, including atom statistics, density correlation functions, etc. In analogy to the optical case, one can therefore think of ''quantum atom optics'' as that extension of atom optics where the quantum state of a many-particle matter-wave field is being controlled, characterized, and used in novel applications.

This Rapid Communication presents an analysis of a system where a quantized optical field is used to manipulate the quantum state of a matter-wave field, as well as to generate new forms of quantum entanglement. The principle of using a quantized electric field to manipulate the coherence of a matter-wave field was first proposed by Zeng and co-workers [4], where they treated a condensate subject to a Raman transition. That paper is flawed, however, in that it incorrectly concludes that all solutions are stable (sinusoidal), and that the system cannot be triggered by noise. Using a geometry identical to that used in the collective atomic recoil laser [5–7] and similar to that described in [4], as well as those used for atom interferometry [8] and recoil-induced resonances [9,10], we consider a Bose-Einstein condensate driven by a far off-resonant pump laser and coupled to a single mode of an optical ring cavity. This results in gain in the cavity mode, as well as the generation of momentum side modes of the BEC, which are assumed to be unbound by any magnetic or optical trap. In particular, we study the unstable (exponential) solutions, as these result in the largest generated fields. We show how the quantum-statistical properties of the side modes are strongly manipulated by varying the initial state of the optical cavity mode, and in addition, that a strong quantum-mechanical entanglement can develop between the optical and matter-wave fields, as well as between matter-wave side modes. The experimental realization of this system is currently feasible, in view of recent experiments on the diffraction of condensates by Deng *et al.* [11].

We consider an ultracold sample of bosonic atoms driven by a strong classical ''pump'' and a counterpropagating weak quantized ''probe'' optical field, both being far off-resonant from any electronic transition. Under these conditions the internal atomic degrees of freedom can be adiabatically eliminated and the matter-wave field is effectively scalar. The combined Hamiltonian for the atomic and probe fields is

$$\begin{aligned}\hat{H}=&\frac{\hbar^2}{2m}\sum_{\mathbf{q}} q^2\hat{c}^\dagger(\mathbf{q})\hat{c}(\mathbf{q})+\hbar ck\hat{A}^\dagger\hat{A}\\&+i\frac{\hbar}{2\Delta}\sum_{\mathbf{q}}[g\Omega_0e^{-i\omega_0t}\hat{A}^\dagger\hat{c}^\dagger(\mathbf{q}-\mathbf{K})\hat{c}(\mathbf{q})-\text{H.c.}]\\&+\frac{\hbar}{\Delta}\left(\frac{|\Omega_0|^2}{4}+|g|^2\hat{A}^\dagger\hat{A}\right)\sum_{\mathbf{q}}\hat{c}^\dagger(\mathbf{q})\hat{c}(\mathbf{q}).\end{aligned}\tag{1}$$

Here, $\Omega_0$ is the Rabi frequency of the pump laser of frequency $\omega_0$ and momentum $\mathbf{k}_0$, $\hat{A}$ is the annihilation operator of the probe field of frequency $\omega$ and momentum $\mathbf{k}$, satisfying $[A,A^\dagger]=1$, and $\hat{c}(\mathbf{q})$ is the annihilation operator for a ground state atom of momentum $\mathbf{q}$, satisfying $[\hat{c}(\mathbf{q}),\hat{c}^\dagger(\mathbf{q}')]=\delta_{\mathbf{q},\mathbf{q}'}$. In addition, $\Delta$ is the detuning between the pump frequency and the upper electronic level closest to resonance, and $g=d[ck/(2\hbar\epsilon_0LS)]^{1/2}$ is the atom-probe coupling constant. Here $d$ is the atomic dipole moment, $L$ the length of the ring cavity, and $S$ the cross section of the probe mode in the region of the atomic sample. Finally, $\mathbf{K}\equiv\mathbf{k}-\mathbf{k}_0$ is the atomic recoil momentum resulting from the absorption of a pump photon followed by the emission of a probe photon.

The first two terms in Eq. (1) are the free Hamiltonians of the atomic and probe fields, respectively. The remaining terms correspond to the various processes by which an atom undergoes a virtual transition under the influence of the optical fields. The first such term involves the exchange of a photon between the pump and probe fields, e.g., stimulated absorption of a pump photon followed by stimulated emission of a probe photon, or vice versa. In coordinate representation, this term would take the form of the familiar periodic optical potential generated by the counterpropagating pump and probe lasers fields. The last two terms in Eq. (1) correspond to processes where a photon is first absorbed and then reemitted into the same field. These transitions are recoilless, but contribute a cross-phase modulation between the atomic and optical fields.

Assuming that the initial momentum width of the condensate is small compared to the recoil momentum $K$, it is reasonable to treat it as a *single-mode* atomic field of momentum $\mathbf{q}=0$. We furthermore restrict our discussion to the case $T \ll T_c$, where $T_c$ is the critical temperature, and assume a large condensate for which the bare mode $q=0$ can then be described to a good approximation as a $c$ number, $\hat{c}(0) \rightarrow \sqrt{N}\exp(i|\Omega_0|^2 t/4\Delta)$, where $N$ is the mean number of atoms in the condensate. This approximation neglects both the depletion that occurs as atoms are transferred into the side modes $q \neq 0$ and the cross-phase modulation between the condensate and the probe field; thus it is valid for times short enough that $\Sigma_{\mathbf{q}\neq 0}\langle \hat{c}^\dagger(\mathbf{q})\hat{c}(\mathbf{q})\rangle \ll N$ and $\langle \hat{A}^\dagger \hat{A}\rangle \ll |\Omega_0|^2/4|g|^2$. This is the matter-wave optics analog of the familiar classical and undepleted pump approximation of nonlinear optics. Hence we describe the optical and matter-wave fields on equal footings, treating all strongly populated modes classically and all weakly populated modes quantum-mechanically.

Once we have replaced the condensate mode with its $c$-number counterpart, we then neglect all terms in the Hamiltonian (1) involving the product of three or more weakly populated field modes. This is a direct consequence of Bose enhancement, which strongly strengthens the interactions involving the central $q=0$ mode relative to those involving only the side modes, and leads us to the effective Hamiltonian

$$\begin{aligned}\hat{H}=\hbar\omega_r[&\hat{c}_+^\dagger\hat{c}_+ + \hat{c}_-^\dagger\hat{c}_- - \delta\hat{a}^\dagger\hat{a}\\ &+\chi(\hat{a}^\dagger\hat{c}_-^\dagger + \hat{a}^\dagger\hat{c}_+ + \hat{c}_+^\dagger\hat{a} + \hat{c}_-\hat{a})],\end{aligned} \tag{2}$$

where $\omega_r = \hbar K^2/2m$, and we have introduced the slowly varying operators $\hat{c}_\pm = \exp(i|\Omega_0|^2 t/4\Delta)\hat{c}(\pm\mathbf{K})$ and $\hat{a} = -i(g\Omega_0^*|\Delta|/|g|\Omega_0\Delta)\exp(i\omega_0 t)\hat{A}$. The system is fully characterized by the effective coupling constant $\chi = |g||\Omega_0|\sqrt{N}/2\omega_r|\Delta|$ and the dimensionless pump-probe detuning $\delta=(\omega_0-\omega)/\omega_r$.

The Hamiltonian (2) describes three coupled field modes: the optical probe and two atomic condensate side modes with wave numbers $\pm\mathbf{K}$. The term $\hat{a}^\dagger\hat{c}_-^\dagger$ in Eq. (2) describes the creation of correlated atom-photon pairs, and immediately brings to mind the optical parametric amplifier [12], a device known to generate highly nonclassical optical fields exhibiting two-mode intensity correlations and squeezing, and which has been extensively employed in the creation of entangled photon pairs for fundamental studies of quantum mechanics, quantum cryptography, and quantum computing. A novel aspect of the present system is that it offers a way to achieve quantum entanglement between atomic and optical fields.

The dynamics of the system can be determined by solving the three coupled-mode equations

$$\frac{d}{d\tau}\begin{pmatrix}\hat{d}_a\\ \hat{d}_-\\ \hat{d}_+\end{pmatrix} = i\begin{pmatrix}\delta & -\chi & -\chi\\ \chi & 1 & 0\\ -\chi & 0 & -1\end{pmatrix}\begin{pmatrix}\hat{d}_a\\ \hat{d}_-\\ \hat{d}_+\end{pmatrix}, \tag{3}$$

where $\tau=\omega_r t$, and we have introduced $\hat{d}_a(\tau)\equiv\hat{a}(\tau)$, $\hat{d}_-(\tau)\equiv\hat{c}_-^\dagger(\tau)$, and $\hat{d}_+(\tau)\equiv\hat{c}_+(\tau)$ for future notational compactness.

An analytic solution can be constructed explicitly from the eigenvalues $\{\lambda_j\}$ and eigenvectors $\{\mathbf{v}_j\}$ of the matrix on the right-hand side of Eq. (3). The eigenvalues $\{\lambda_j\}$ have been studied in detail in Ref. [7] in the context of the theory of the collective atom recoil laser (CARL). It was shown that, provided the system parameters $\chi$ and $\delta$ satisfy certain threshold conditions, they take the form $\lambda_1=\omega_1, \lambda_2=\lambda_3^* = \Omega + i\Gamma$, where $\omega_1$, $\Omega$, and $\Gamma$ are all real quantities. Hence we see that after an initial transient, the solution grows exponentially in time at the rate $\Gamma$. This regime of exponential growth is familiar from the physics of the free-electron laser and of the CARL, where it is usually studied at high temperatures. The explicit form of the eigenvalues and eigenvectors is not required for the current analysis and will be presented elsewhere. For our present purposes, it is sufficient to know that for a given set of parameters $\chi,\delta$ they are simply constants.

The solution of Eq. (3) is

$$\hat{d}_i(\tau) = \sum_j u_{ij}(\tau)\hat{d}_j(0), \tag{4}$$

where the coefficients $u_{ij}(\tau)$ are given by

$$u_{ij}(\tau) = \sum_k v_{ik}v_{kj}^{-1}e^{i\lambda_k\tau} \approx \zeta_{ij}e^{(\Gamma+i\Omega)\tau}. \tag{5}$$

Here $v_{ik}$ is the $i$th component of the eigenvector $\mathbf{v}_k$, $v_{kj}^{-1}$ satisfies $\Sigma_k v_{ik}v_{kj}^{-1} = \delta_{ij}$, and $\zeta_{ij}\equiv v_{i3}v_{3j}^{-1}$. The approximate equality in Eq. (5) is valid for times long enough that one can neglect all but the exponentially growing terms, henceforth referred to as the exponential growth regime.

This exponential growth of the system can be triggered either from vacuum fluctuations, as we discuss in more detail shortly, or by a weak injected probe signal. We investigate both situations by assuming that the probe field is initially in the coherent state $\alpha$, the vacuum state corresponding to $\alpha=0$. The condensate side modes, in contrast, are always taken to be in the vacuum state at $\tau=0$, so that the initial state of the system is $|\alpha,0,0\rangle$.

The expectation values $\langle\hat{d}_i\rangle$ of the three coupled modes are readily found to be

$$\langle\hat{d}_i(\tau)\rangle = \alpha u_{ia}(\tau) \approx \alpha\zeta_{ia}e^{(\Gamma+i\Omega)\tau}, \tag{6}$$

where the approximate result is for the exponential growth regime. As expected, in the absence of an injected signal the mean fields remain zero, but the injected probe breaks the symmetry of the system and leads to nonzero expectation values.

Decomposing the expectation values $\langle d_i\rangle$ in terms of an amplitude and phase as $\langle d_i\rangle = r_i(\tau)\exp[i\theta_i(\tau)]$, we find that in the exponential growth regime their uncertainties obey

$$\Delta r_i(\tau)/r_i(\tau) = \Delta\theta_i(\tau) \approx f(\chi,\delta)/(\sqrt{2}|\alpha|), \tag{7}$$

where the fluctuation function $f(\chi,\delta)=|v_{3-}^{-1}/v_{3a}^{-1}|$ has a relatively simple dependence on the control parameters $\chi$ and $\delta$. Specifically, for a given $\chi$, $f(\chi,\delta)$ is approximately unity at the $\delta$, which maximizes the growth rate $\Gamma$, and increases steadily away from this value. Clearly, Eq. (7) holds only in the case of an injected probe signal, $\alpha\neq 0$. In that case, the phase uncertainties of all three mean fields approach the same limiting value for large $\tau$, and this value approaches zero as $\alpha$ becomes very large; i.e., for large enough $\alpha$ all three modes are effectively in coherent states. We note that for large $\alpha$ the system is essentially equivalent to Kapitza-Dirac atomic diffraction of a condensate by a standing wave [11], which we now see produces atomic side modes in coherent states.

We now investigate the mean intensities $I_i(\tau) \equiv \langle d_i^\dagger(\tau)d_i(\tau)\rangle - \delta_{i,-}$. The $\delta$ function accounts for the fact that $\hat{d}_2(\tau)$ in Eq. (3) is a creation rather than an annihilation operator, thus guaranteeing that the initial intensity of the "$-$" side mode vanishes. These intensities are given explicitly by

$$I_i(\tau) = |\alpha|^2|u_{ia}(\tau)|^2 + |u_{i-}(\tau)|^2 - \delta_{i,-}. \tag{8}$$

In the exponential growth regime they reduce to

$$I_i(\tau) \approx (|\alpha|^2|\zeta_{ia}|^2 + |\zeta_{i-}|^2)e^{2\Gamma\tau}. \tag{9}$$

They have a stimulated component, proportional to $|\alpha|^2$, and a spontaneous component, which is present even when all three field modes begin in the vacuum state. The stimulated component is simply the squared amplitude of the mean field, while the spontaneous component has no mean field, as it originates from the amplification of vacuum fluctuations in the atomic bunching.

To help understand this in more detail, we introduce the atomic "bunching operator" $\hat{B}=(1/N)\Sigma_j \exp(iK\hat{z}_j)$, where $\hat{z}_j$ is the position operator of the $j$th atom. If the atoms in the sample are evenly distributed in space, then $\langle\hat{B}\rangle=0$. At the opposite extreme, if all the atoms are localized on a array of period $2\pi/K$, then $|\langle\hat{B}\rangle|=1$. Second-quantizing $\hat{B}$ and linearizing the result by treating the $q=0$ mode as a $c$ number and keeping as in the derivation of Eq. (2) only the lowest-order terms in the side mode operators, we can reexpress $\hat{B}$ in terms of the atomic field operators as $\hat{B} = (1/\sqrt{N})(\mathrm{c}-\dagger+\mathrm{c}+)$. It is immediately apparent from that definition that $\langle\hat{B}(0)\rangle=0$ for our initial state $|\alpha,0,0\rangle$. However, the fluctuations $\langle\hat{B}^2(0)\rangle$ are nonzero, due to the fact that $\langle\hat{c}_-(0)\hat{c}_-^\dagger(0)\rangle=1$. It is precisely this expectation value that leads to the spontaneous intensity component, which can therefore be attributed to vacuum fluctuations in the initial atomic bunching. These fluctuations play a role similar to that of vacuum fluctuations in spontaneous emission.

In addition to the side-mode intensity, it is instructive to also study their equal-time intensity correlation functions. For the probe mode, we have

$$g_a^{(2)}(\tau) = \frac{\langle\hat{a}^\dagger(\tau)\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}(\tau)\rangle}{\langle\hat{a}^\dagger(\tau)\hat{a}(\tau)\rangle^2}. \tag{10}$$

The side-mode correlation functions $g_-^{(2)}(\tau)$ and $g_+^{(2)}(\tau)$ are defined likewise but with $\hat{a}(\tau)$ replaced by $\hat{c}_-(\tau)$ and $\hat{c}_+(\tau)$, respectively. These correlation functions are given explicitly as

$$g_i^{(2)}(\tau) = 2 - \frac{|\alpha|^4|u_{ia}(\tau)|^4}{I_i^2(\tau)} \approx 2 - \frac{|\alpha|^4}{[|\alpha|^2+f^2(\chi,\delta)]^2}. \tag{11}$$

As before, the approximate result applies to the exponential growth regime, where the intensity correlation functions become constant in time and the same for each mode. In the case where the system builds up from noise ($|\alpha|^2=0$), we have then $g_i^{(2)}=2$, the signature of a thermal or chaotic field. As the injected signal strength is increased, however, $g_i^{(2)} \to 1$, which is characteristic of a Glauber coherent field with Poissonian excitation statistics. Note the important point that the state of the side modes can be continuously varied from thermal to coherent by varying the strength of the injected probe signal and/or the system parameters $\chi$ and $\delta$. Thus the coherence properties of the matter-wave fields are directly controlled by an optical field.

We have mentioned the analogy between the problem at hand and the parametric oscillator. It is the tool of choice for generating entangled quantum-optical states. We now investigate if similar entanglements can be obtained here. We proceed by investigating the equal-time two-mode intensity cross correlations, which are a measure of the degree of entanglement between the modes of the system. For example, the intensity cross-correlation function $g_{a-}^{(2)}(\tau)$ is defined as

$$g_{a-}^{(2)} = \frac{\langle\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{c}_-^\dagger(\tau)\hat{c}_-(\tau)\rangle}{\langle\hat{a}^\dagger(\tau)\hat{a}(\tau)\rangle\langle\hat{c}_-^\dagger(\tau)\hat{c}_-(\tau)\rangle}. \tag{12}$$

Other intensity cross-correlation functions, such as $g_{a+}^{[2]}(\tau)$ and $g_{-+}^{(2)}(\tau)$, are defined similarly.

For classical fields, there is an upper limit to the second-order equal-time correlation function. It is given by the Cauchy-Schwartz inequality [12]

$$g_{ij}^{(2)}(\tau) \leq [g_i^{(2)}(\tau)]^{1/2}[g_j^{(2)}(\tau)]^{1/2}. \tag{13}$$

Quantum-mechanical fields, however, can violate this inequality and are instead constrained by [12]

$$g_{ij}^{(2)}(\tau) \leq \left[g_i^{(2)}(\tau) + \frac{1}{I_i(\tau)}\right]^{1/2}\left[g_j^{(2)}(\tau) + \frac{1}{I_j(\tau)}\right]^{1/2}, \tag{14}$$

which reduces to the classical result in the limit of large intensities.

We focus our attention on the spontaneous case $\alpha=0$, where the single-mode intensity correlation functions are given by $g_i^{(2)}(\tau)=2$. In this case, the equal-time intensity cross-correlation functions are found to be

$$g_{a-}^{(2)}=g_{-+}^{(2)}=\left[2+\frac{1}{I_a(\tau)+I_+(\tau)}\right]^{1/2}\left[2+\frac{1}{I_-(\tau)}\right]^{1/2},$$

$$g_{a+}^{(2)}=2. \tag{15}$$

From Eq. (15) we see that both $g_{a-}^{(2)}(\tau)$ and $g_{-+}^{(2)}(\tau)$ violate the Cauchy-Schwartz inequality, while $g_{a+}^{(2)}(\tau)$ is consistent with classical cross correlations. Furthermore, the explicit evaluation of the $\zeta_{ij}$'s shows that $I_+(\tau)\ll I_a(\tau)$, which implies that $g_{a-}^{(2)}(\tau)$ is very close to the maximum violation of the classical inequality consistent with quantum mechanics, whereas for $g_{-+}^{(2)}(\tau)$ the violation is not close to the allowed maximum. In the two-mode parametric amplifier, the two-mode correlation function shows the maximum violation of the Cauchy-Schwartz inequality consistent with quantum mechanics. In the three-mode system, however, the two-mode cross-correlation functions involve a trace over the third mode; hence it is not surprising that the two-mode correlations are not maximized.

If we now allow for an injected coherent probe field $(\alpha\neq 0)$, we must first note that the intensities are increased by approximately $|\alpha|^2$, which means that the time scale on which the classical and quantum upper limits (13) and (14) converge is reduced by $1/|\alpha|^2$, making an experimental confirmation of quantum correlations more difficult. In addition, whereas for the spontaneous case $\alpha=0$, numerics show the cross correlation $g_{a-}^{(2)}$ follows almost exactly the quantum upper limit (14) for all $t>0$; for $\alpha\neq 0$, it lies somewhere in between the quantum (14) and classical (13) limits. As $\alpha$ is increased, it falls ever closer to the classical upper limit, so that in the limit of very large $\alpha$, the fields exhibit classical cross correlations only.

In summary, we have discussed how the quantum state of momentum side modes of a condensate can be varied continuously between two distinct limits by specifying the initial state of an optical cavity mode. When it begins in the vacuum state, the side-mode and the cavity-mode fields develop with zero mean fields, thermal intensity fluctuations, and strong quantum correlations between the modes. In contrast, when it is prepared in a strong coherent state, we approach a ''classical'' limit in which the fields develop with nonzero mean fields having well-defined phases, intensity fluctuations indicating a coherent state, and exhibiting classical correlations only. Condensate side modes have recently been realized in an experiment by Deng *et al.*, where a condensate was subjected to Kapitza-Dirac diffraction by a standing-wave laser field [11]. In order to observe the effects predicted here, this field would need to be replaced by a combination of a strong pump laser and a weak counter-propagating probe sustained by an optical cavity, which does not appear to present any major difficulty.

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