Levinson's theorem for the Klein-Gordon equation in two dimensions

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In terms of the modified Sturm-Liouville theorem, the two-dimensional Levinson theorem for the Klein-Gordon equation with a cylindrically symmetric potential $V(r)$ is established for an angular momentum m as a relation between the numbers $n_m^{\pm}$ of the particle and antiparticle bound states and the phase shifts $\eta_m(\pm M)$:

$$\eta_m(M)-\eta_m(-M)=\begin{cases}(n_m^{+}-n_m^{-}+1)\pi & \text{when a half-bound state occurs at } E=M \quad \text{for} \quad m=1\\ (n_m^{+}-n_m^{-}-1)\pi & \text{when a half-bound state occurs at } E=-M \quad \text{for} \quad m=1\\ (n_m^{+}-n_m^{-})\pi & \text{the remaining cases.}\end{cases}$$

A solution of the Klein-Gordon equation with the energy M or $-M$ is called a half-bound state if it is finite but does not decay fast enough at infinity to be square integrable. [S1050-2947(99)01702-3]

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I. INTRODUCTION

The Levinson theorem [1], an important theorem in scattering theory, established the relation between the total number of bound states and the phase shift at zero momentum. During the past half-century, the Levinson theorem has been proved by several authors with different methods, and generalized to different fields [1–24]. Roughly speaking, there are three main methods for proving the Levinson theorem. One [1] is based on the elaborate analysis of the Jost function. The second relies on the Green-function method [5]. The third method is used to demonstrate the Levinson theorem by the Sturm-Liouville theorem [6–8]. This simple, intuitive method is readily generalized, and has been applied to many physical problems [6–9,23,24]. Some obstacles and ambiguities, which may occur in the other two methods, disappear in the third method. However, the Sturm-Liouville theorem has to be modified in proving the Levinson theorem for the Klein-Gordon equation [9].

The Klein-Gordon equation, which describes the motion of a relativistic scalar particle, is a second-order differential equation with respect to both space and time. When there exists a potential as the time component of a vector field, the energy eigenvalues it is not necessary for the Klein-Gordon equation to be real, and the eigenfunctions satisfy the orthogonal relations with a weight factor [25,26] such that a parameter ϵ, which is not always real, appears in the normalized relation with a weight factor. As pointed out in Refs. [25,26], after Bose quantization those amplitudes with real and positive ϵ describe particles, and those with real and negative ϵ antiparticles.

Recall that, in three-dimensional spaces, two methods were used to set up the Levinson theorem for the Klein-Gordon equation. One relied on the Green-function method [5,22], where some formulas are valid only for the cases without complex energies. The other was based on a modified Sturm-Liouville theorem [9], by which the Levinson theorem for the Klein-Gordon equation was established for cases even with complex energies.

The reasons we present this paper are as follows. On the one hand, the Levinson theorem in two dimensions has been studied numerically [18] as well as in theory [19–24], in virtue of the wide interest in lower-dimensional field theories. On the other hand, the Levinson theorem for the Klein-Gordon equation in two dimensions has never appeared in the literature, to our knowledge. In our previous works [23,24], Levinson theorems in two dimensions for nonrelativistic and relativistic particles, as well as those with a nonlocal interaction, were established by the Sturm-Liouville theorem. Now we attempt to set up the Levinson theorem for the Klein-Gordon equation in two dimensions for completeness.

This paper is organized as follows. In Sec. II, we review the properties of the Klein-Gordon equation, especially those related to the parameter ϵ. In Sec. III, it is proved that the difference between the number of bound states of a particle and an antiparticle relies only on the changes of the logarith-

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mic derivatives of the wave functions at $E=\pm M$, as the potential $V(r)$ changes from zero to the given value. In Sec. IV, it turns out that these changes are connected with the phase shifts at $E=\pm M$, which then results in the establishment of the two-dimensional Levinson theorem for the Klein-Gordon equation.

II. KLEIN-GORDON EQUATION

Throughout this paper the natural units $\hbar=c=1$ are employed. Consider a relativistic scalar particle satisfying the Klein-Gordon equation

$$(-\nabla^2+M^2)\psi(x)=\{E-V(x)\}^2\psi(x), \tag{1}$$

where the potential $V(x)$ is the time component of a vector field, and M and E denote the mass and the energy of the particle, respectively. Assume that the potential is static and cylindrically symmetric,

$$V(x)=V(r), \tag{2}$$

and satisfies the asymptotic conditions

$$r|V(r)|\to 0 \quad \text{when} \quad r\to 0, \tag{3a}$$

and

$$V(r)=0 \quad \text{when} \quad r\geqslant r_0. \tag{3b}$$

Equation (3a) is required to make the wave function single value at the origin, and Eq. (3b), called the condition of the cutoff potential, is, for the sake of simplicity of discussion, vanishing beyond a sufficiently large radius r_0. Following the method given in Refs. [22,23], the results obtained in the present paper also hold if the potential vanishes faster than r^{-2} at infinity.

Introduce a parameter λ for the potential $V(r)$,

$$V(r,\lambda)=\lambda V(r), \tag{4}$$

which shows that the potential $V(r,\lambda)$ changes from zero to the given potential $V(r)$ when λ increases from zero to 1.

Let

$$\psi(x,\lambda)=r^{-1/2}R_{Em}(r,\lambda)e^{\pm im\varphi}, \quad m=0,1,2,\ldots, \tag{5}$$

where the radial wave equation $R_{Em}(r,\lambda)$ satisfies the radial equation

$$\frac{\partial^2 R_{Em}(r,\lambda)}{\partial r^2}+\left\{(E^2-M^2)-(2EV-V^2)-\frac{m^2-1/4}{r^2}\right\}\times R_{Em}(r,\lambda)=0. \tag{6}$$

Denote by $R_{E_1m}(r,\lambda)$ the solution to Eq. (6) for the energy E_1

$$\frac{\partial^2 R_{E_1m}(r,\lambda)}{\partial r^2}+\left\{(E_1^2-M^2)-(2E_1V-V^2)-\frac{m^2-1/4}{r^2}\right\}\times R_{E_1m}(r,\lambda)=0. \tag{7}$$

Multiplying Eqs. (6) and (7) by $R_{E_1m}(r,\lambda)$ and $R_{Em}(r,\lambda)$, respectively, and calculating their difference, we have

$$\frac{\partial}{\partial r}\{R_{Em}(r,\lambda)R_{E_1m}'^{*}(r,\lambda)-R_{E_1m}^{*}(r,\lambda)R_{Em}'(r,\lambda)\} =-(E_1^*-E)R_{E_1m}^{*}(r,\lambda)(E_1^*+E-2V)R_{Em}(r,\lambda), \tag{8}$$

where the prime denotes the derivative of the radial function with respect to the variable r. It is well known [25,26] that, due to the so-called Klein paradox, the energy eigenvalues are not necessarily real for some potential $V(r)$. Integrating Eq. (8) over the whole space and noting that $R_{Em}(r,\lambda)R_{E_1m}'^{*}(r,\lambda)-R_{E_1m}^{*}(r,\lambda)R_{Em}'(r,\lambda)$ vanishes both at the origin and at infinity for the physically admissible solutions with the different energies E and E_1, we obtain the weighted orthogonality relation for the radial wave function

$$(E_1^*-E)\int_0^\infty R_{E_1m}^{*}(r,\lambda)(E_1^*+E-2V)R_{Em}(r,\lambda)dr=0. \tag{9}$$

As a matter of fact, we are always able to obtain the *real* solutions for the *real* energies. However, it is easy to see from Eq. (9) that the normalized relation for the solutions with real energies are not always positive on account of the weight factor (E_1+E-2V):

$$\int_0^\infty R_{E_1m}(r,\lambda)(E_1+E-2V)R_{Em}(r,\lambda)dr=\begin{cases}\epsilon_E\delta(E_1-E)(E^2-M^2)^{1/2}/|E| & \text{when} \quad |E|>M\\ \epsilon_E\delta_{E_1E} & \text{when} \quad |E|<M.\end{cases} \tag{10}$$

The parameter ϵ_E, which depends on the particular radial wave function $R_{Em}(r,\lambda)$, may be either positive, negative, or vanishing. Normalized factors of the solutions cannot change the sign of ϵ_E. Generally speaking, if $R_{Em}(r,\lambda)$ is a complex solution of Eq. (6) with a complex energy E, then R_{Em}^* is also a solution with a complex energy E^*, and a complex ϵ_E appears for a pair of the complex solutions. It is evident after Bose quantization that those $R_{Em}(r,\lambda)$ with positive ϵ_E describe particles, and those with negative ϵ_E describe antiparticles. The solution with zero ϵ_E can be dealt with as a pair of solutions with infinitesimal $\pm|\epsilon_E|$, which describe a pair of particle and antiparticle bound states. The Hamiltonian and charge operator cannot be written as the diagonal forms for the solutions with complex ϵ_E; therefore, they describe neither particles nor antiparticles. In the present paper, we only count the number of bound states with the real ϵ_E, which are called particle and antiparticle bound states, respectively.

Since we are always able to obtain the *real* solution for the *real* energy, we now solve Eq. (6) in two regions and match two real solutions at r_0. Only one matching condition at r_0 is needed, which is the condition for the logarithmic derivative of the radial wave function

$$A_m(E,\lambda)\equiv\left\{\frac{1}{R_{Em}(r,\lambda)}\frac{\partial R_{Em}(r,\lambda)}{\partial r}\right\}_{r=r_0-}=\left\{\frac{1}{R_{Em}(r)}\frac{\partial R_{Em}(r)}{\partial r}\right\}_{r=r_0+}\equiv B_m(E). \tag{11}$$

Actually, solutions in the region $[0,r_0]$ with $R_{Em}(0,\lambda)=0$ can be obtained in principle. Only one solution is convergent at the origin because of the condition (3a). For example, for the free particle ($\lambda=0$), the solution to Eq. (6) at the region $[0,r_0]$ is proportional to the Bessel function $J_m(x)$,

$$R_{Em}(r,0)=\begin{cases}\sqrt{\dfrac{\pi kr}{2}}J_m(kr) & \text{when}\quad |E|>M \quad\text{and}\quad k=\sqrt{E^2-M^2}\\ e^{-im\pi/2}\sqrt{\dfrac{\pi\kappa r}{2}}J_m(i\kappa r) & \text{when}\quad |E|\leqslant M \quad\text{and}\quad \kappa=\sqrt{M^2-E^2}.\end{cases} \tag{12}$$

The solution $R_{Em}(r,0)$ given in Eq. (12) is a real function. A constant factor in front of the radial wave function $R_{Em}(r,0)$ is not important.

In the region $[r_0,\infty)$, we have $V(r,\lambda)=0$. For $|E|>M$, there are two oscillatory solutions to Eq. (6). Their combination can always satisfy the matching condition (11), so that there is a continuous spectrum for $|E|>M$.

$$R_{Em}(r,\lambda)=\sqrt{\frac{\pi kr}{2}}\{\cos\eta_m(E,\lambda)J_m(kr)-\sin\eta_m(E,\lambda)N_m(kr)\}\sim\cos\left(kr-\frac{m\pi}{2}-\frac{\pi}{4}+\eta_m(E,\lambda)\right)\quad\text{when}\quad r\to\infty, \tag{13}$$

where $N_m(kr)$ is the Neumann function. It is through the matching condition (11) that the radial function $R_{Em}(r,\lambda)$ as well as the phase shift $\eta_m(E,\lambda)$ depend on the parameter λ.

On the other hand, there is only one convergent solution in the region $[r_0,\infty)$ for $|E|\leqslant M$, so that the matching condition (11) is not always satisfied,

$$R_{Em}(r)=e^{i(m+1)\pi/2}\sqrt{\frac{\pi\kappa r}{2}}H_m^{(1)}(i\kappa r)\sim e^{-\kappa r}\quad\text{when}\quad r\to\infty, \tag{14}$$

where $H_m^{(1)}(x)$ is the Hankel function of the first kind. When condition (11) is satisfied, a bound state appears at this energy. This means that there is a discrete spectrum for $|E|\leqslant M$.

For the case with a *real* energy, integrating Eq. (8) in two regions $[0,r_0]$ and $[r_0,\infty)$, respectively, and taking the limit $E_1\to E$, we obtain the following equations in terms of the boundary condition that $R_{Em}(0)=0$ and $R_{Em}(\infty)=0$ for $|E|<M$:

$$\frac{\partial A_m(E,\lambda)}{\partial E}\equiv\frac{\partial}{\partial E}\left(\frac{1}{R_{Em}(r,\lambda)}\frac{\partial R_{Em}(r,\lambda)}{\partial r}\right)_{r=r_0-}=-2R_{Em}(r_0,\lambda)^{-2}\int_0^{r_0}R_{Em}(r,\lambda)^2[E-V(r)]dr \tag{15a}$$

and

$$\frac{dB_m(E)}{dE}\equiv\frac{d}{dE}\left(\frac{1}{R_{Em}(r)}\frac{dR_{Em}(r)}{dr}\right)_{r=r_0+}=2R_{Em}(r_0)^{-2}\int_{r_0}^{\infty}R_{Em}(r)^2E\,dr. \tag{15b}$$

It is demonstrated from Eq. (15) that $A_m(E,\lambda)$ is no longer monotonic with respect to the energy, but $B_m(E)$ is still monotonic with respect to the energy if the energy does not change sign. Furthermore, their difference, $B_m(E)-A_m(E,\lambda)$, is monotonic with respect to the energy for the particle ($\epsilon_E>0$) and for the antiparticle ($\epsilon_E<0$), respectively:

$$\frac{\partial}{\partial E}\{A_m(E,\lambda)-B_m(E)\}=-R_{Em}(r_0,\lambda)^{-2}\epsilon_E. \tag{16}$$

Equation (16) is called the modified Sturm-Liouville theorem. It is owing to the modified Sturm-Liouville theorem that a bound state can be identified as a particle ($\epsilon_E>0$) or an antiparticle one ($\epsilon_E<0$) by whether $A_m(E,\lambda)-B_m(E)$ decreases or increases as the energy E increases.

From the matching condition (11) we have

$$\tan\eta_m(E,\lambda)=\frac{J_m(kr_0)}{N_m(kr_0)}\frac{A_m(E,\lambda)-kJ_m'(kr_0)/J_m(kr_0)-1/(2r_0)}{A_m(E,\lambda)-kN_m'(kr_0)/N_m(kr_0)-1/(2r_0)}, \tag{17}$$

$$\eta_m(E) \equiv \eta_m(E,1), \tag{18}$$

where the prime denotes the derivative of the Bessel function, the Neumann function, and later, the Hankel function with respect to their argument.

The phase shift $\eta_m(E,\lambda)$ is determined from Eq. (17) up to a multiple of π due to the period of the tangent function. Following our previous papers [6,7,22–24], we use the convention for determining the phase shift absolutely that the phase shift $\eta_m(E,0)$ for the free particle ($\lambda=0$) is defined to be zero,

$$\eta_m(E,0)=0 \quad \text{where} \quad \lambda=0. \tag{19}$$

As shown in Eq. (10), the scattering states $|E|>M$ are normalized as the Dirac δ function, where the main contribution to the integration comes from the radial wave functions in the region far away from the origin. Therefore, we may change the integral region in Eq. (10) to $[r_0,\infty)$ where there is no potential. Substituting Eq. (13) into Eq. (10), we obtain

$$\epsilon_E = \pi E \quad \text{when} \quad |E|>M. \tag{20}$$

All scattering states with the positive energy ($E>M$) describe particles, and those with negative energy ($E<-M$) describe antiparticles.

III. NUMBER OF BOUND STATES

In our previous works, Levinson theorems for nonrelativistic and relativistic particles were set up under the help of the Sturm-Liouville theorem. For the Sturm-Liouville problem, the fundamental trick is the definition of a phase angle which is monotonic with respect to the energy [27]. Due to the factor (E_1^*+E-2V) in Eq. (10), the Sturm-Liouville theorem has to be modified for the Klein-Gordon equation. In other words, as shown in Eq. (16), for the Klein-Gordon equation only the difference of the logarithmic derivatives at two sides of $r_0, A_m(E,\lambda)-B_m(E)$, is monotonic with respect to the energy for the particle ($\epsilon_E>0$) and for the antiparticle ($\epsilon_E<0$), respectively.

From Eq. (14), we obtain

$$\begin{aligned} B_m(E) &= \frac{i\kappa H_m^{(1)}(i\kappa r_0)'}{H_m^{(1)}(i\kappa r_0)} - \frac{1}{2r_0} \leqslant B_m(\pm M) \\ &= (-m+1/2)/r_0 \quad \text{when} \quad |E|\leqslant M. \end{aligned} \tag{21}$$

The logarithmic derivative $B_m(E)$ does not depend on λ. On the other hand, when $\lambda=0$ we obtain, from Eq. (12)

$$\begin{aligned} A_m(E,0) &= \frac{i\kappa J_m'(i\kappa r_0)}{J_m(i\kappa r_0)} - \frac{1}{2r_0} \geqslant A_m(\pm M,0) \\ &= (m+1/2)/r_0 \quad \text{when} \quad |E|\leqslant M. \end{aligned} \tag{22}$$

It is evident from Eqs. (21) and (22) that both $B_m(E)$ and $A_m(E,0)$ are continuous curves with respect to the energy which do not intersect with each other except for $m=0$, i.e., the matching condition (11) is not satisfied if $|E|\leqslant M$ and $\lambda=0$. No bound state appears when there is no potential except for $m=0$, where there is a half-bound state at $E=\pm M$, which will be discussed later.

As λ changes from zero to the given potential, $B_m(E)$ does not change, but $A_m(E,\lambda)$ changes continuously except for the points where $R_{Em}(r_0)=0$ and $A_m(E,\lambda)$ tends to infinity. Generally speaking, $A_m(E,\lambda)$ is continuous except for those finite points, and intersects with the curve $B_m(E)$ several times for $|E|\leqslant M$. The bound state will appear only if a point of intersection occurs. The number of points of intersection is the same as the number of bound states. It is shown from Eq. (16) that the relative slope with respect to the energy at the point of intersection decides whether the bound state describes a particle or an antiparticle.

Denote by $n_m^+(\lambda)$ the number of particle bound states, and by $n_m^-(\lambda)$ the number of antiparticle bound states. Their difference is denoted by $N_m(\lambda)$,

$$N_m(\lambda)=n_m^+(\lambda)-n_m^-(\lambda). \tag{23}$$

When the potential $V(r,\lambda)$ changes with λ, the number of points of intersection in the region $|E|\leqslant M$ may change only for the following two reasons. First, the points of intersection move inward or outward at $E=\pm M$. Second, the curve $A_m(E,\lambda)$ intersects with the curve $B_m(E)$ or departs from it through a tangent point. For the second source, owing to the modified Sturm-Liouville theorem (16), a pair of particle and antiparticle bound states will be created or annihilated at the same time, but $N_m(\lambda)$ remains invariant. That is, the difference $N_m(\lambda)$ can change only when a point of intersection moves in or out at $E=\pm M$.

Now we discuss the properties when a point of intersection moves in or out at $E=\pm M$. First we discuss the situation that λ increases across the critical value λ_1 where $A_m(M,\lambda_1)=B_m(M)=(-m+1/2)/r_0$. There are two cases at the critical value

$$\left.\frac{\partial^{n'}}{\partial E^{n'}}A_m(E,\lambda_1)\right|_{E=M} = \left.\frac{\partial^{n'}}{\partial E^{n'}}B_m(E)\right|_{E=M} \quad \text{where} \quad 0\leqslant n'<n,$$

(i)

$$(-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}A_m(E,\lambda_1)\right|_{E=M} > (-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}B_m(E)\right|_{E=M},$$

$$\left.\frac{\partial^{n'}}{\partial E^{n'}}A_m(E,\lambda_1)\right|_{E=M} = \left.\frac{\partial^{n'}}{\partial E^{n'}}B_m(E)\right|_{E=M} \quad \text{where} \quad 0\leqslant n'<n,$$

(ii)

$$(-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}A_m(E,\lambda_1)\right|_{E=M} < (-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}B_m(E)\right|_{E=M},$$

where n is a positive integer. This means that, for the energy $E<M$ but very near M,

$$A_m(E,\lambda_1)>B_m(E) \quad \text{for case (i)}, \tag{24a}$$

$$A_m(E,\lambda_1)<B_m(E) \quad \text{for case (ii)}. \tag{24b}$$

If $A_m(M,\lambda)$ decreases as λ increases across the critical value λ_1, a point of intersection moves in to $E<M$ for case (i) and moves out from $E<M$ for case (ii), and simultaneously, owing to the modified Sturm-Liouville theorem (16), a scattering state with a positive energy becomes a particle bound state for case (i) and an antiparticle bound state becomes a scattering state with a positive energy for case (ii). For both cases $N_m(\lambda)$ increases by 1. Conversely, if $A_m(M,\lambda)$ increases as λ increases across the critical value $\lambda_1, N_m(\lambda)$ decreases by 1 for both cases.

Second, we discuss the situation that λ increases across the critical value λ_2 where $A_m(-M,\lambda_2)=B_m(-M)=(-m+1/2)/r_0$. There are also two cases at the critical value: (i)

$$\left.\frac{\partial^{n'}}{\partial E^{n'}}A_m(E,\lambda_2)\right|_{E=-M}=\left.\frac{\partial^{n'}}{\partial E^{n'}}B_m(E)\right|_{E=-M} \quad \text{where} \quad 0\leqslant n'<n,$$

$$\left.\frac{\partial^{n}}{\partial E^{n}}A_m(E,\lambda_2)\right|_{E=-M}>\left.\frac{\partial^{n}}{\partial E^{n}}B_m(E)\right|_{E=-M}$$

and (ii)

$$\left.\frac{\partial^{n'}}{\partial E^{n'}}A_m(E,\lambda_2)\right|_{E=-M}=\left.\frac{\partial^{n'}}{\partial E^{n'}}B_m(E)\right|_{E=-M} \quad \text{where} \quad 0\leqslant n'<n,$$

$$\left.\frac{\partial^{n}}{\partial E^{n}}A_m(E,\lambda_2)\right|_{E=-M}<\left.\frac{\partial^{n}}{\partial E^{n}}B_m(E)\right|_{E=M}.$$

This means that, for an energy $E>-M$ but very near $-M$,

$$A_m(E,\lambda_2)>B_m(E) \quad \text{for case (i)}, \tag{25a}$$

$$A_m(E,\lambda_2)<B_m(E) \quad \text{for case (ii)}. \tag{25b}$$

If $A_m(-M,\lambda)$ decreases as λ increases across the critical value λ_2, a point of intersection moves in to $E>-M$ for case (i), and moves out from $E>-M$ for case (ii), and simultaneously, owing to the modified Sturm-Liouville theorem (16), a scattering state with a negative energy becomes an antiparticle bound state for case (i) and a particle bound state becomes a scattering state with a negative energy for case (ii). For both cases $N_m(\lambda)$ decreases by one. Conversely, if $A_m(M,\lambda)$ increases as λ increases across the critical value λ_2, $N_m(\lambda)$ increases by one for both cases.

Now, as λ increases from zero to one, we denote by $n_m(\pm M)$ the times that $A_m(\pm M,\lambda)$ decreases across the value $B_m(\pm M)=(-m+1/2)/r_0$, and subtract by the times that $A_m(\pm M,\lambda)$ increases across the value $B_m(\pm M)$. Thus we have

$$N_m\equiv N_m(1)=n_m(M)-n_m(-M). \tag{26}$$

Recall that from Eq. (23) $N_m(\lambda)$ is the difference between the numbers of particle and antiparticle bound states:

$$N_m=N_m(1)=n_m^+(1)-n_m^-(1)\equiv n_m^+-n_m^-. \tag{27}$$

IV. PHASE SHIFTS

Now we turn to the scattering states. The solutions in the region $[r_0,\infty)$ for the scattering states have been given in Eq. (13). The phase shift $\eta_m(\pm M,\lambda)$ is the limit of the phase shift $\eta_m(E,\lambda)$ as k tends to zero. Hence what we are interested in is the phase shift $\eta_m(E,\lambda)$ at a sufficiently small momentum k, $k\ll 1/r_0$. Through the matching condition (11), the phase shift $\eta_m(E,\lambda)$ at a sufficiently small momentum k can be calculated by Eq. (17) and convention (19).

First, we obtain from Eq. (17) that

$$\left.\frac{\partial \eta_m(E,\lambda)}{\partial A_m(E,\lambda)}\right|_{k,\lambda}=\frac{-8r_0\cos^2\eta_m(E,\lambda)}{\pi\{2r_0A_m(E,\lambda)N_m(kr_0)-2kr_0N_m'(kr_0)-N_m(kr_0)\}^2}\leqslant 0. \tag{28}$$

This shows that the phase shift $\eta_m(E,\lambda)$ at a sufficiently small momentum k is monotonic with respect to the logarithmic derivative $A_m(E,\lambda)$ as λ increases.

Second, we discuss the noncritical case where

$$A_m(\pm M,1)\neq B_m(\pm M)=(-m+1/2)/r_0. \tag{29}$$

For the small momentum $(k\sim 0)$ we obtain, from Eq. (17),

$$\tan\eta_m(E,\lambda)\sim\begin{cases}\dfrac{-\pi(kr_0)^{2m}}{2^{2m}m!(m-1)!}\,\dfrac{A_m(\pm M,\lambda)-(m+1/2)/r_0}{A_m(\pm M,\lambda)-B_m(\pm M)} & \text{when} \quad m\geqslant 1\\[2ex] \dfrac{\pi}{2\ln(kr_0)}\,\dfrac{A_m(\pm M,\lambda)-ck^2-(2r_0)^{-1}(1-(kr_0)^2)}{A_m(\pm M,\lambda)-ck^2-(2r_0)^{-1}\left(1+\dfrac{2}{\ln(kr_0)}\right)} & \text{when} \quad m=0.\end{cases} \tag{30}$$

It can be seen from Eq. (30) that $\tan\eta_m(E,\lambda)$ tends to zero as k goes to zero, i.e., that $\eta_m(\pm M,\lambda)$ is always equal to the multiple of π. In other words, if the phase shift $\eta_m(E,\lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, its limit $\eta_m(\pm M,\lambda)$ is equal to $n\pi$, where n is an integer. This means that $\eta_m(\pm M,\lambda)$ changes discontinuously.

If $A_m(E,\lambda)$ decreases as λ increases, $\eta_m(E,\lambda)$ increases. Each time $A_m(\pm M,\lambda)$ decreases across the value $B_m(\pm M)$, $\tan\eta_m(E,\lambda)$ changes from positive to negative, and the phase shift $\eta_m(\pm M,\lambda)$ increases by π. Conversely, if $A_m(E,\lambda)$ increases as λ increases, $\eta_m(E,\lambda)$ decreases. Each time $A_m(\pm M,\lambda)$ increases across the value $B_m(\pm M)$, $\tan\eta_m(E,\lambda)$ changes from negative to positive, and the phase shift $\eta_m(\pm M,\lambda)$ decreases by π.

We should pay some attention to the case of S wave ($m=0$). We included the next leading terms for S wave in Eq. (30). Since the next leading terms in the numerator and the denominator of Eq. (30) are different, as $A_0(\pm M,\lambda)$ decreases across the value $(2r_0)^{-1}=B_0(\pm M)$, the numerator changes from positive to negative first, and then the denominator changes from positive to negative. It is in the second step that $\tan\eta_0(E,\lambda)$ changes from positive to negative, and the phase shift $\eta_0(\pm M,\lambda)$ increases by π. Similarly, each time $A_0(\pm M,\lambda)$ increases across the value $B_0(\pm M)$, and the phase shift $\eta_0(\pm M,\lambda)$ decreases by π.

For $\lambda=0$ and $m=0$, the numerator in Eq. (30) is equal to zero, the denominator is positive, and the phase shift $\eta_0(\pm M,0)$ is defined to be zero. If $A_0(\pm M,\lambda)$ decreases as λ increases from zero, the numerator becomes negative first, and then the denominator changes sign from positive to negative, such that the phase shift $\eta_0(\pm M)$ jumps by π and simultaneously a new bound state appears. Note that this is a particle bound state for E near M, and an antiparticle bound state for E near $-M$. If $A_0(\pm M,\lambda)$ increases as λ increases from zero, the numerator becomes positive, and the remaining factor keeps negative, such that the phase shift $\eta_0(\pm M)$ keeps to be zero, and no bound state appears.

In Sec. III, we denoted by $n_m(\pm M)$ the times that $A_m(\pm M,\lambda)$ decreases across the value $B_m(\pm M)=(-m+1/2)/r_0$ as λ increases from zero to 1, and subtracted the times that $A_m(\pm M,\lambda)$ increases across the value $B_m(\pm M)$. Now $n_m(\pm M)$ is nothing but the phase shift $\eta_m(\pm M,1)$ divided by π

$$\eta_m(\pm M)\equiv\eta_m(\pm M,1)=n_m(\pm M)\pi. \tag{31}$$

Thus we draw a conclusion from Eqs. (26) and (31) that for the noncritical cases (29), the Levinson theorem for the Klein-Gordon equation in two dimensions is

$$N_m\pi=\eta_m(M)-\eta_m(-M). \tag{32}$$

Finally, we discuss the critical cases that when $\lambda=1$,

$$A_m(M,1)=B_m(M)=(-m+1/2)/r_0, \tag{33a}$$

and/or

$$A_m(-M,1)=B_m(-M)=(-m+1/2)/r_0. \tag{33b}$$

In the critical case, the following solution with energy M or $-M$ in the region $[r_0,\infty)$ will match $A_m(M,1)$ or $A_m(-M,1)$ at r_0

$$R_{Em}(r)=r^{-m+1/2},\quad E=M \text{ and/or } -M. \tag{34}$$

It is a bound state when $m\geqslant 2$, but called a half-bound state when $m=1$ and 0. A half-bound state is not a bound state, because its wave function is finite but not square integrable.

From the modified Sturm-Liouville theorem (16) we obtain for the critical case,

$$\frac{d}{dE}\{A_m(E,1)-B_m(E)\}=-c^2\epsilon_E,$$
$$E=M \text{ or } -M, \tag{35}$$

where $c^2>0$. It is easy to see from Eq. (13) that for $m\geqslant 2$

$$\frac{d}{dE}B_m(E)=\frac{d}{dE}\{kN_m'(kr_0)/N_m(kr_0)+1/(2r_0)\}. \tag{36}$$

Therefore, the denominator in Eqs. (17) and (30) for $m\geqslant 2$ becomes

$$A_m(M,1)-B_m(M)-c^2\epsilon_M k^2/(2M^2),\quad m\geqslant 2, \tag{37a}$$

when Eq. (33a) holds, and

$$A_m(-M,1)-B_m(-M)+c^2\epsilon_{-M}k^2/(2M^2),\quad m\geqslant 2, \tag{37b}$$

when Eq. (33b) holds.

For definiteness we discuss the critical case where Eq. (33a) holds. If $A_m(M,\lambda)$ *decreases* to the value $B_m(M)$ as λ increases to 1, the denominator in Eq. (30) changes from positive to negative for $\epsilon_M>0$, and stays positive for $\epsilon_M<0$. That is, the phase shift $\eta_m(M)$ increases an additional π for $\epsilon_M>0$, and does not increase for $\epsilon_M<0$. On the other hand, a scattering state with a positive energy becomes a particle bound state of energy M for $\epsilon_M>0$, and the energy of an antiparticle bound state increases to M for $\epsilon_M<0$. That is, N_m increases by an additional 1 for $\epsilon_M>0$, and N_m does not increase for $\epsilon_M<0$. In both cases the Levinson theorem (32) still holds for $m\geqslant 2$.

Conversely, if $A_m(M,\lambda)$ *increases* to the value $B_m(M)$ as λ increases to 1, the denominator in Eq. (30) remains negative for $\epsilon_M>0$, and changes from negative to positive for $\epsilon_M<0$. That is, the phase shift $\eta_m(M)$ does not decrease for $\epsilon_M>0$, and decreases an additional π for $\epsilon_M<0$. Simultaneously, the energy of a particle bound state increases to M for $\epsilon_M>0$, and a scattering state of a positive energy becomes an antiparticle bound state of energy M for $\epsilon_M<0$. That is, N_m does not decrease for $\epsilon_M>0$, and N_m decreases by an additional 1 for $\epsilon_M<0$. In both cases the Levinson theorem (32) also holds for $m\geqslant 2$.

Through a similar discussion, we conclude that the Levinson theorem (32) also holds for the critical case where Eq. (33b) holds and $m\geqslant 2$.

For $m=1$ and 0, Eq. (36) no longer holds. Fortunately, the denominator in Eq. (30) becomes

$$A_1(\pm M,\lambda)-c_1k^2+(2r_0)^{-1}\{1+2(kr_0)^2\ln(kr_0)\} \quad \text{for} \quad m=1, \tag{38a}$$

$$A_0(\pm M,\lambda)-c_0k^2-(2r_0)^{-1}\left\{1+\frac{2}{\ln(kr_0)}\right\} \quad \text{for} \quad m=0, \tag{38b}$$

where the next leading term with $\ln(kr_0)$ dominates for the critical case.

On the other hand, substituting Eq. (14) into Eq. (15b), we find for $m=1$ and 0 that when $E<M$ and E tends to M, $dB_m(E)/dE$ tends to positive infinity, and when $E>-M$ and E tends to $-M$, $dB_m(E)/dE$ tends to negative infinity. Since $dA_m(E,1)/dE$ at $E=\pm M$ is finite for the critical cases, we conclude that $\epsilon_M>0$ and $\epsilon_{-M}<0$ for the critical cases when $m=1$ and 0.

Now we discuss the critical case where Eq. (33a) holds. If $A_m(M,\lambda)$ *decreases* to the value $B_m(M)$ as λ increases to 1, the denominator in Eq. (30) changes from positive to negative for $m=1$, and remains positive for $m=0$. That is, the phase shift $\eta_m(M)$ increases an additional π for $m=1$, and does not increase for $m=0$. On the other hand, both for $m=1$ and 0, a scattering state with a positive energy becomes a half-bound state (not a bound state), i.e., N_m does not increase as λ reaches to 1.

Conversely, if $A_m(M,\lambda)$ *increases* to the value $B_m(M)$ as λ increases to 1, the denominator in Eq. (30) remains negative for $m=1$, and changes from negative to positive for $m=0$. That is, the phase shift $\eta_m(M)$ does not decrease for $m=1$, and decreases an additional π for $m=0$. On the other hand, both for $m=1$ and for $m=0$, a particle bound state disappears, namely, N_m decreases by an additional 1 as λ reaches to 1.

Now, we discuss the critical case where Eq. (33b) holds. If $A_m(-M,\lambda)$ *decreases* to the value $B_m(-M)$ as λ increases to 1, the denominator in Eq. (30) changes from positive to negative for $m=1$, and remains positive for $m=0$. That is, the phase shift $\eta_m(-M)$ increases an additional π for $m=1$, and does not increase for $m=0$. On the other hand, both for $m=1$ and 0, a scattering state with a negative energy becomes a half-bound state (not a bound state), i.e., N_m does not decrease as λ reaches 1.

Conversely, if $A_m(-M,\lambda)$ *increases* to the value $B_m(-M)$ as λ increases to 1, the denominator in Eq. (30) remains negative for $m=1$, and changes from negative to positive for $m=0$. That is, the phase shift $\eta_m(-M)$ does not decrease for $m=1$, and decreases an additional π for $m=0$. On the other hand, both for $m=1$ and 0, an antiparticle bound state disappears, that is, N_m increases an additional 1 as λ reaches 1.

In summary, the Levinson theorem (32) for the Klein-Gordon equation in two dimensions has to be modified for the critical cases when $m=1$:

$$\eta_m(M)-\eta_m(-M)=\begin{cases}(N_m+1)\pi & \text{when a half-bound state occurs at } E=M \quad \text{for} \quad m=1\\ (N_m-1)\pi & \text{when a half-bound state occurs at } E=-M \quad \text{for} \quad m=1\\ N_m\pi & \text{the remaining cases.}\end{cases} \tag{39}$$

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