# Phase-space formulation of quantum mechanics and quantum-state reconstruction for physical systems with Lie-group symmetries

C. Brif* and A. Mann†
*Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel*
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We present a detailed discussion of a general theory of phase-space distributions, introduced recently by the authors [J. Phys. A **31**, L9 (1998)]. This theory provides a unified phase-space formulation of quantum mechanics for physical systems possessing Lie-group symmetries. The concept of generalized coherent states and the method of harmonic analysis are used to construct explicitly a family of phase-space functions which are postulated to satisfy the Stratonovich-Weyl correspondence with a generalized tracing condition. The symbol calculus for the phase-space functions is given by means of the generalized twisted product. The phase-space formalism is used to study the problem of the reconstruction of quantum states. In particular, we consider the reconstruction method based on measurements of displaced projectors, which comprises a number of recently proposed quantum-optical schemes and is also related to the standard methods of signal processing. A general group-theoretic description of this method is developed using the technique of harmonic expansions on the phase space. [S1050-2947(99)00702-7]

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## I. INTRODUCTION

The phase-space formulation of quantum mechanics has a long history. In 1932 Wigner [1] introduced his famous function which has found numerous applications in many areas of physics and electronics. In 1949 Moyal [2] discovered that the Weyl correspondence rule [3] can be inverted by the Wigner transform from an operator on the Hilbert space to a function on the phase space. As a result, the quantum expectation value of an operator can be represented by the statistical-like average of the corresponding phase-space function with the statistical density given by the Wigner function associated with the density matrix of the quantum state. In this way quantum mechanics can be formally represented as a statistical theory on classical phase space. It should be emphasized that this phase-space formalism does not replace quantum mechanics by a classical or semiclassical theory. In fact, the phase-space formulation of quantum mechanics (also known as the Moyal quantization) is in principle equivalent to conventional formulations due to Heisenberg, Schrödinger, and Feynman. However, the formal resemblance of quantum mechanics in the Moyal formulation to classical statistical mechanics can yield deeper understanding of differences between the quantum and classical theories. Extensive lists of the literature on this subject can be found in reviews and books [4–9].

The ideas of Moyal were further developed in the late sixties in the works of Cahill and Glauber [10] and Agarwal and Wolf [11]. As mentioned, the Wigner function is related to the Weyl (symmetric) ordering of the position and momentum operators $q$ and $p$ or, equivalently, of the bosonic annihilation and creation operators $a$ and $a^\dagger$. However, there exist other possibilities of ordering. In particular, it was shown [10] that the Glauber-Sudarshan $P$ function [12,13] is associated with the normal ordering and the Husimi $Q$ function [14] with the antinormal ordering of $a$ and $a^\dagger$. Moreover, a whole family of $s$-parametrized functions can be defined on the complex plane which is equivalent to the $q$-$p$ flat phase space. The index $s$ is related to the corresponding ordering procedure of $a$ and $a^\dagger$; the values $+1$, 0, and $-1$ of $s$ correspond to the $P$, $W$, and $Q$ functions, respectively. These phase-space functions are referred to as quasiprobability distributions (QPDs), as they play in the Moyal formulation of quantum mechanics a role similar to that of genuine probability distributions in classical statistical mechanics. Various QPDs have been extensively used in many quantum-optical applications [15,16]. Most recently, there is great interest in the $s$-parametrized distributions because of their role in modern schemes for measuring the quantum state of the radiation field [17].

The mathematical framework and the conceptual background of the Moyal quantization have been essentially enlarged and generalized in two important papers by Bayen *et al.* [18]. Specifically, it was shown that noncommutative deformations of the algebra of classical phase-space functions (defined by the ordinary multiplication) give rise to operator algebras of quantum mechanics. This fact means that introducing noncommutative symbol calculus based on the so-called twisted product (also known as the star or Moyal product), one obtains a completely autonomous reformulation of quantum mechanics in terms of phase-space functions instead of Hilbert-space states and operators. This program of ''quantization by deformation'' has been developed in a number of works [19–23].

For a long time applications of the Moyal formulation were restricted to description of systems like a spinless nonrelativistic quantum particle or a mode of the quantized radiation field (modeled by a quantum harmonic oscillator), i.e., to the case of the flat phase space. Therefore an important problem is the generalization of the standard Moyal quantization for quantum systems possessing an intrinsic

*Electronic address: costya@physics.technion.ac.il
†Electronic address: ady@physics.technion.ac.il

 

group of symmetries, with the phase space being a homogeneous manifold on which the group of transformations acts transitively [20,24]. It has been recently understood that this problem can be solved using the Stratonovich-Weyl (SW) correspondence. The idea of the SW correspondence is that the linear bijective mapping between operators on the Hilbert space and functions on the phase space can be implemented by a kernel which satisfies a number of physically sensible postulates, with covariance and tracing being the two most important ones. This idea first appeared in a paper by Stratonovich [25] in 1956, but it was almost forgotten for decades. The SW correspondence, which has been restated some years ago by Gracia-Bondía and Várilly [26,27], has given a new impulse to the phase-space formulation of quantum theory. The SW method of the Moyal quantization has been applied to a number of important situations: a nonrelativistic free particle with spin, using the extended Galilei group [26]; a relativistic free particle with spin, using the Poincaré group [28]; the spin, using the SU(2) group [27]; compact semisimple Lie groups [29]; one- and two-dimensional kinematical groups [30–33]; the two-dimensional Euclidean group [30,34]; and systems of identical quantum particles [35]. For a review of basic results see Ref. [8].

Notwithstanding the success of the SW method in the Moyal quantization of many important physical systems, the theory suffered from a serious problem. Specifically, it was the absence of a simple and effective method for the construction of the SW kernel which should implement the mapping between Hilbert-space operators and phase-space functions. The construction procedures for the SW kernels, considered during the last decade (see, e.g., Ref. [8]), did not guarantee that the kernel will satisfy all the SW postulates. Only very recently was a general algorithm for constructing the SW kernel for quantum systems possessing Lie-group symmetries proposed [36]. It has been shown that the constructed kernel explicitly satisfies all the desired properties (the SW postulates) and that in the particular cases of the Heisenberg-Weyl group and SU(2) our general expression reduces to the known results.

In the present paper we essentially extend the results of Ref. [36] and present a self-consistent theory of the SW method for the phase-space formulation of quantum mechanics. This theory makes use of the concept of generalized coherent states and of some basic ideas of harmonic analysis. Like the Cahill-Glauber formalism for the Heisenberg-Weyl group, we construct the *s*-parametrized family of functions on the phase space of a quantum system whose dynamical symmetry group is an arbitrary (finite-dimensional) Lie group. Accordingly, we introduce *s*-generalized versions of the tracing condition and the twisted product. The developed phase-space formulation is used for a general group-theoretic description of the quantum-state reconstruction method. This description can be useful not only for measurements of quantum states but also in the field of signal processing.

## II. BASICS OF MOYAL QUANTIZATION

### A. Generalized coherent states and the definition of quantum phase space

Given a specific physical system, the first thing one needs to do for the Moyal quantization (i.e., for constructing phase-space functions) is to determine what the related phase space is. This can often be done by analogy with the corresponding classical problem, thereby providing a direct route for the quantum-classical correspondence. From the technical point of view, the phase space can be conveniently determined using the concept of coherent states [37]. The coherent-state approach is not just a convenient mathematical tool, but it also helps to understand how physical properties of the system are reflected by the geometrical structure of the related phase space. It is possible to say that the concept of coherent states constitutes a bridge between the Moyal phase-space quantization and the Berezin geometric quantization [38].

Let $G$ be a Lie group (connected and simply connected, with finite dimension $n$), which is the dynamical symmetry group of a given quantum system. Let $T$ be a unitary irreducible representation of $G$ acting on the Hilbert space $\mathcal{H}$. By choosing a fixed normalized reference state $|\psi_0\rangle \in \mathcal{H}$, one can define the system of coherent states $\{|\psi_g\rangle\}$:

$$|\psi_g\rangle = T(g)|\psi_0\rangle, \quad g \in G. \tag{2.1}$$

The isotropy subgroup $H \subset G$ consists of all the group elements $h$ that leave the reference state invariant up to a phase factor,

$$T(h)|\psi_0\rangle = e^{i\phi(h)}|\psi_0\rangle, \quad |e^{i\phi(h)}| = 1, \ h \in H. \tag{2.2}$$

For every element $g \in G$, there is a decomposition of $g$ into a product of two group elements, one in $H$ and the other in the coset space $X = G/H$,

$$g = \Omega h, \quad g \in G, \quad h \in H, \quad \Omega \in X. \tag{2.3}$$

It is clear that group elements $g$ and $g'$ with different $h$ and $h'$ but with the same $\Omega$ produce coherent states which differ only by a phase factor: $|\psi_g\rangle = e^{i\delta}|\psi_{g'}\rangle$, where $\delta = \phi(h) - \phi(h')$. Therefore a coherent state $|\Omega\rangle \equiv |\psi_\Omega\rangle$ is determined by a point $\Omega = \Omega(g)$ in the coset space $X$. A very important property is the identity resolution in terms of the coherent states:

$$\int_X d\mu(\Omega)|\Omega\rangle\langle\Omega| = I, \tag{2.4}$$

where $d\mu(\Omega)$ is the invariant integration measure on $X$, the integration is over the whole manifold $X$, and $I$ is the identity operator on $\mathcal{H}$. The natural action of $G$ on $X$ will be denoted by $g \cdot \Omega$.

An important class of coherent-state systems corresponds to the coset spaces $X = G/H$ which are homogeneous Kählerian manifolds. Then $X$ can be considered as the phase space of a classical dynamical system, and the mapping $\Omega \to |\Omega\rangle\langle\Omega|$ is the geometric quantization for this system [38]. The standard (or maximum-symmetry) systems of the coherent states correspond to the cases when an ''extreme'' state of the representation Hilbert space (e.g., the vacuum state of an oscillator or the lowest/highest spin state) is chosen as the reference state. This choice of the reference state leads to systems consisting of states with properties ''closest to those of classical states'' [37,39]. In what follows we will consider the coherent states of maximum symmetry and assume that the phase space of the quantum system is a homogeneous Kählerian manifold $X = G/H$, each point of

which corresponds to a coherent state $|\Omega\rangle$. In particular, the Glauber coherent states of the Heisenberg-Weyl group $H_3$ are defined on the complex plane $\mathbb{C}=H_3/\mathrm{U}(1)$, and the spin coherent states are defined on the unit sphere $\mathbb{S}^2=\mathrm{SU}(2)/\mathrm{U}(1)$. In the more rigorous mathematical language of Kirillov's theory [40], the phase space $X$ is defined as the coadjoint orbit associated with the unitary irreducible representation $T$ of the group $G$ on the Hilbert space $\mathcal{H}$.

### B. The Stratonovich-Weyl correspondence

Once the phase space of a quantum system is determined, the Moyal quantization proceeds in the following way. Let $A$ be an operator on $\mathcal{H}$. Then $A$ can be mapped by a family of functions $F_A(\Omega;s)$ onto the phase space $X$ (the index $s$ labels functions in the family). If $A$ is the density matrix $\rho$ of a quantum system, the corresponding phase-space functions $F_\rho(\Omega;s)\equiv P(\Omega;s)$ are called QPDs. Of course, the phase-space formulation of the quantum theory for a given physical system can be successful only if the functions $F_A(\Omega;s)$ possess some physically motivated properties. These properties were formulated by Stratonovich [25] and are referred to as the SW correspondence.

(0) Linearity: $A\rightarrow F_A(\Omega;s)$ is one-to-one linear map.

(i) Reality:

$$F_{A^\dagger}(\Omega;s)=[F_A(\Omega;s)]^*. \tag{2.5a}$$

(ii) Standardization:

$$\int_X d\mu(\Omega)F_A(\Omega;s)=\mathrm{Tr}\,A. \tag{2.5b}$$

(iii) Covariance:

$$F_{A(g)}(\Omega;s)=F_A(g\cdot\Omega;s), \tag{2.5c}$$

where $A(g)\equiv T(g^{-1})AT(g)$.

(iv) Tracing:

$$\int_X d\mu(\Omega)F_A(\Omega;s)F_B(\Omega;-s)=\mathrm{Tr}(AB). \tag{2.5d}$$

If the function $F_A(\Omega;s)$ satisfies the SW correspondence, it is called the SW symbol of the operator $A$.

The above conditions have a clear physical meaning. The linearity and the tracing conditions are related to the statistical interpretation of the theory. If $B$ is the density matrix (the state operator) of a system, then the tracing condition (2.5d) assures that the statistical average of the phase-space distribution $F_A$ coincides with the quantum expectation value of the operator $A$. O'Connell and Wigner [41] have shown that the tracing condition for density matrices of a spinless quantum particle (there it appears as an overlap relation) is necessary for the uniqueness of the definition of the Wigner function. It has also been shown [29] that the tracing condition is necessary for the uniqueness of the definition of the symbol calculus (twisted or ''star'' products) of the phase-space functions and for the validity of the related noncommutative Fourier analysis. Equation (2.5d) is actually a generalization of the usual tracing condition [25,27,29], as it holds for any $s$ and not only for the Wigner case $s=0$. The reality condition (2.5a) means that if $A$ is self-adjoint, then $F_A(\Omega;s)$ is real. The condition (2.5b) is a natural normalization, which means that the image of the identity operator $I$ is the constant function 1. The covariance condition (2.5c) means that the phase-space formulation must explicitly express the symmetry of the system.

The linearity is taken into account if we implement the map $A\rightarrow F_A(\Omega;s)$ by the generalized Weyl rule

$$F_A(\Omega;s)=\mathrm{Tr}[A\Delta(\Omega;s)], \tag{2.6}$$

where $\{\Delta(\Omega;s)\}$ is a family (labeled by $s$) of operator-valued functions on the phase space $X$. These operators are referred to as the SW kernels. The generalized tracing condition (2.5d) is taken into account if we define the inverse of the generalized Weyl rule (2.6) as

$$A=\int_X d\mu(\Omega)F_A(\Omega;s)\Delta(\Omega;-s). \tag{2.7}$$

Now, the conditions (2.5a)–(2.5c) of the SW correspondence for $F_A(\Omega;s)$ can be translated into the following conditions on the SW kernel $\Delta(\Omega;s)$:

$$\text{(i)}\quad \Delta(\Omega;s)=[\Delta(\Omega;s)]^\dagger \quad \forall\Omega\in X. \tag{2.8a}$$

$$\text{(ii)}\quad \int_X d\mu(\Omega)\Delta(\Omega;s)=I. \tag{2.8b}$$

$$\text{(iii)}\quad \Delta(g\cdot\Omega;s)=T(g)\Delta(\Omega;s)T(g^{-1}). \tag{2.8c}$$

Substituting the inverted maps (2.7) for $A$ and $B$ into the generalized tracing condition (2.5d), we obtain the relation between functions with different values of the index $s$:

$$F_A(\Omega;s)=\int_X d\mu(\Omega')K_{s,s'}(\Omega,\Omega')F_A(\Omega';s'), \tag{2.9}$$

$$K_{s,s'}(\Omega,\Omega')\equiv\mathrm{Tr}[\Delta(\Omega;s)\Delta(\Omega';-s')]. \tag{2.10}$$

If we take $s=s'$ in Eq. (2.9) and take into account the arbitrariness of $A$, we obtain the relation

$$\Delta(\Omega;s)=\int_X d\mu(\Omega')K(\Omega,\Omega')\Delta(\Omega';s), \tag{2.11}$$

where the function

$$K(\Omega,\Omega')=\mathrm{Tr}[\Delta(\Omega;s)\Delta(\Omega';-s)] \tag{2.12}$$

behaves like the $\delta$ function on the manifold $X$.

## III. CONSTRUCTION OF THE STRATONOVICH-WEYL KERNEL

It is clear that the Moyal quantization for a physical system is accomplished by constructing the SW kernel $\Delta(\Omega;s)$ that satisfies the SW postulates. Although the form of the SW kernel has been known for many systems, a general construction method was not known. A procedure that was applied in many works [8,30–32] is as follows. An arbitrary point $\Omega_0\in X$ is fixed and then an ansatz is made for a self-

adjoint operator $\Delta(\Omega_0)$ (usually only the case $s=0$ was considered) that satisfies the standardization condition (2.8b) and the following property:

$$\Delta(\Omega_0)=T(\gamma)\Delta(\Omega_0)T(\gamma^{-1}) \quad \forall \gamma\in H_{\Omega_0}, \tag{3.1}$$

where $H_{\Omega_0}=\{\gamma\in G|\gamma\cdot\Omega_0=\Omega_0\}$ is the isotropy subgroup for $\Omega_0$. For any $\Omega\in X$ there exists $g\in G$ such that $g\cdot\Omega_0=\Omega$, and then the SW kernel is defined by

$$\Delta(\Omega)=\Delta(g\cdot\Omega_0)=T(g)\Delta(\Omega_0)T(g^{-1}). \tag{3.2}$$

This kernel automatically satisfies the covariance condition (2.8c), but the problem is that the tracing is not guaranteed. Of course, in the described procedure the form of the kernel depends on the ansatz and often no kernel satisfying the tracing condition is found.

We propose here a simple and general algorithm for constructing the SW kernels (the whole *s*-parametrized family) which explicitly satisfy all the SW postulates, including both the covariance and the tracing. Our method makes use of Perelomov's concept of coherent states and of only some basic ideas from harmonic analysis. Hopefully, the simplicity and generality of our method will draw more attention to the ideas of the phase-space quantization.

### A. Necessary instruments: harmonic functions, invariant coefficients, and tensor operators

Our problem is to find the explicit form of the SW kernel $\Delta(\Omega;s)$ that satisfies the conditions (2.8a)–(2.8c) and (2.11). In order to accomplish this task, we need three basic ingredients: harmonic functions, invariant coefficients, and tensor operators. The coherent states serve here as the glue that binds them together.

We start by considering the Hilbert space $L^2(X,\mu)$ of square-integrable functions $u(\Omega)$ on $X$ with the invariant measure $d\mu$. The representation $T$ of the Lie group $G$ on $L^2(X,\mu)$ is defined as

$$T(g)u(\Omega)=u(g^{-1}\cdot\Omega). \tag{3.3}$$

The eigenfunctions $Y_\nu(\Omega)$ of the Laplace-Beltrami operator [42] form a complete orthonormal basis in $L^2(X,\mu)$:

$$\sum_\nu Y_\nu^*(\Omega)Y_\nu(\Omega')=\delta(\Omega-\Omega'), \tag{3.4a}$$

$$\int_X d\mu(\Omega)Y_\nu^*(\Omega)Y_{\nu'}(\Omega)=\delta_{\nu\nu'}. \tag{3.4b}$$

The functions $Y_\nu(\Omega)$ are called the harmonic functions, and $\delta(\Omega-\Omega')$ is the delta function in $X$ with respect to the measure $d\mu$. Note that the index $\nu$ is multiple; it has one discrete part, while the other part is discrete for compact manifolds and continuous for noncompact manifolds. In the latter case the summation over $\nu$ includes an integration with the Plancherel measure $d\rho(\nu)$ and the symbol $\delta_{\nu\nu'}$ includes some Dirac delta functions (for more details see Ref. [42]). For conciseness, we omit these details in our formulas. The harmonic functions $Y_\nu(\Omega)$ are linear combinations of matrix elements $T_{\nu\nu'}(g)$. Therefore the transformation rule for the harmonic functions is [42]

$$T(g)Y_\nu(\Omega)=Y_\nu(g^{-1}\cdot\Omega)=\sum_{\nu'} T_{\nu'\nu}(g)Y_{\nu'}(\Omega). \tag{3.5}$$

It should be understood that the summation in Eq. (3.5) is only on the part of $\nu$ that labels functions within an irreducible subspace.

Next, we once again use the coherent states, in order to introduce the concept of invariant coefficients. The positive-valued function $|\langle\Omega|\Omega'\rangle|^2$ is symmetric in $\Omega$ and $\Omega'$. Therefore its expansion in the orthonormal basis must be of the form

$$|\langle\Omega|\Omega'\rangle|^2=\sum_\nu \tau_\nu Y_\nu^*(\Omega)Y_\nu(\Omega')=\sum_\nu \tau_\nu Y_\nu^*(\Omega')Y_\nu(\Omega), \tag{3.6}$$

where $\tau_\nu$ are real positive coefficients. Using the invariance $\langle\Omega|\Omega'\rangle=\langle g\cdot\Omega|g\cdot\Omega'\rangle$ and the unitarity of the representation $T$, we obtain

$$\begin{aligned}|\langle\Omega|\Omega'\rangle|^2&=\sum_\nu \tau_\nu Y_\nu^*(g\cdot\Omega)Y_\nu(g\cdot\Omega')\\&=\sum_{\nu'} Y_{\nu'}^*(\Omega)\sum_\nu \tau_\nu T_{\nu\nu'}(g)Y_\nu(g\cdot\Omega').\end{aligned} \tag{3.7}$$

In order to satisfy this equality, the coefficients $\tau_\nu$ must be invariant under the index transformation of Eq. (3.5): $\tau_\nu=\tau_{\nu'}$. This means that $\tau_\nu$ do not depend on the part of $\nu$ which labels functions within an irreducible subspace. Since the Laplace-Beltrami operator is self-adjoint, one finds that

$$Y_\nu^*(\Omega)=e^{i\phi(\nu)}Y_{\tilde{\nu}}(\Omega), \tag{3.8}$$

where $Y_{\tilde{\nu}}(\Omega)$ is another harmonic function, with the same eigenvalue as $Y_\nu(\Omega)$. Since $|\langle\Omega|\Omega'\rangle|^2$ is real, the coefficients $\tau_\nu$ must be invariant under the index transformation of Eq. (3.8): $\tau_\nu=\tau_{\tilde{\nu}}$.

Next we use the coherent states, harmonic functions, and invariant coefficients for defining the set of operators $\{D_\nu\}$ on $\mathcal{H}$:

$$D_\nu\equiv\tau_\nu^{-1/2}\int_X d\mu(\Omega)Y_\nu(\Omega)|\Omega\rangle\langle\Omega|. \tag{3.9}$$

Using the expression (3.6) and the orthonormality relation (3.4b) for the harmonic functions, we obtain the orthonormality condition for the operators $D_\nu$:

$$\mathrm{Tr}(D_\nu D_{\nu'}^\dagger)=\delta_{\nu\nu'}. \tag{3.10}$$

Note that the factor $\tau_\nu^{-1/2}$ in front of the integral in Eq. (3.9) serves just for the proper normalization. Using Eq. (3.6), we also obtain the relation

$$\tau_\nu^{-1/2}\langle\Omega|D_\nu|\Omega\rangle=Y_\nu(\Omega). \tag{3.11}$$

The invariance of the coefficients $\tau_\nu$ implies that $D_\nu$ are the tensor operators whose transformation rule is the same as for the harmonic functions $Y_\nu(\Omega)$:

$$T(g)D_\nu T(g^{-1})=\sum_{\nu'} T_{\nu'\nu}(g)D_{\nu'}. \tag{3.12}$$

A useful property of the tensor operators is that any operator $A$ on $\mathcal{H}$ can be expanded in the orthonormal basis $\{D_\nu\}$:

$$A=\sum_\nu \mathrm{Tr}\,(AD_\nu^\dagger)D_\nu. \tag{3.13}$$

### B. Explicit form of the kernel

Using the above preliminary results, we are able to find the SW kernel $\Delta(\Omega;s)$ with all the desired properties. Specifically, let us define

$$\Delta(\Omega;s)\equiv\sum_\nu f(s;\tau_\nu)Y_\nu^*(\Omega)D_\nu. \tag{3.14}$$

We will show that the construction of the generalized kernel (3.14) satisfies the SW correspondence. In Eq. (3.14) $f(s;\tau_\nu)$ is a function of $\tau_\nu$ and of the index $s$. We assume that $f$ possesses the invariance properties of $\tau_\nu$.

Using the invariance of $\tau_\nu$ under the index transformation of Eq. (3.8), we see that the reality condition (2.8a) is satisfied if $f(s;\tau_\nu)$ is a real-valued function. Therefore we can consider only real values of the index $s$.

Next we consider the standardization condition (2.8b). Using the definition (3.14), we obtain

$$\int_X d\mu(\Omega)\Delta(\Omega;s)=\sum_\nu f(s;\tau_\nu)D_\nu\int_X d\mu(\Omega)Y_\nu^*(\Omega), \tag{3.15}$$

while Eq. (3.13) can be used to write

$$I=\sum_\nu \mathrm{Tr}(D_\nu^\dagger)D_\nu=\sum_\nu \tau_\nu^{-1/2}D_\nu\int_X d\mu(\Omega)Y_\nu^*(\Omega). \tag{3.16}$$

The standardization condition is satisfied if the expressions (3.15) and (3.16) are equal. Using the identity resolution (2.4) and Eq. (3.6), we can write

$$1=\langle\Omega|\Omega\rangle=\int_X d\mu(\Omega')|\langle\Omega|\Omega'\rangle|^2$$

$$=\sum_\nu \tau_\nu Y_\nu^*(\Omega)\int_X d\mu(\Omega')Y_\nu(\Omega'). \tag{3.17}$$

Multiplying the left and right sides of this equation by $Y_{\nu'}(\Omega)$ and integrating over $d\mu(\Omega)$, we obtain

$$\int_X d\mu(\Omega)Y_\nu(\Omega)=\tau_\nu\int_X d\mu(\Omega)Y_\nu(\Omega). \tag{3.18}$$

Since $\tau_\nu$ is not identically 1, this relation can be satisfied only if there exists some $\nu_0$ such that $\tau_{\nu_0}=1$ and

$$\int_X d\mu(\Omega)Y_\nu(\Omega)\propto\delta_{\nu\nu_0}. \tag{3.19}$$

(As was already mentioned, for noncompact manifolds the symbol $\delta_{\nu\nu'}$ actually includes some Dirac delta functions.) It can be easily seen from Eqs. (3.15), (3.16), and (3.19) that the standardization condition is satisfied if $f(s;\tau_{\nu_0})=\tau_{\nu_0}^{-1/2}$, i.e.,

$$f(s;1)=1 \quad \forall s. \tag{3.20}$$

The covariance condition (2.8c) can be rewritten as

$$\sum_\nu f(s;\tau_\nu)D_\nu Y_\nu^*(g\cdot\Omega)$$

$$=\sum_\nu f(s;\tau_\nu)T(g)D_\nu T(g^{-1})Y_\nu^*(\Omega). \tag{3.21}$$

Using the transformation rules (3.5) and (3.12), Eq. (3.21) can be transformed into

$$\sum_\nu\sum_{\nu'} f(s;\tau_\nu)D_\nu T_{\nu\nu'}(g)Y_{\nu'}^*(\Omega)$$

$$=\sum_\nu\sum_{\nu'} f(s;\tau_\nu)T_{\nu'\nu}(g)D_{\nu'}Y_\nu^*(\Omega). \tag{3.22}$$

Changing the summation indexes $\nu\leftrightarrow\nu'$ on either side of Eq. (3.22), we immediately see that the covariance condition is satisfied by virtue of the invariance of $\tau_\nu$ under the index transformation of Eqs. (3.5) and (3.12).

In order to satisfy the relation (2.11), the function $K(\Omega,\Omega')$ of Eq. (2.12) must be the $\delta$ function in $X$ with respect to the measure $d\mu$,

$$K(\Omega,\Omega')=\sum_\nu Y_\nu^*(\Omega)Y_\nu(\Omega')=\delta(\Omega-\Omega'). \tag{3.23}$$

This result is valid if

$$f(s;\tau_\nu)f(-s;\tau_\nu)=1. \tag{3.24}$$

This property is satisfied only by the exponential function of $s$, i.e.,

$$f(s;\tau_\nu)=[f(\tau_\nu)]^s. \tag{3.25}$$

Note that the standardization condition (3.20) then reads $f(1)=1$.

The exact form of the function $f(\tau_\nu)$ can be determined if we define [43] for $s=-1$

$$\Delta(\Omega;-1)\equiv|\Omega\rangle\langle\Omega|. \tag{3.26}$$

Then we obtain

$$|\Omega\rangle\langle\Omega|=\sum_\nu [f(\tau_\nu)]^{-1}Y_\nu^*(\Omega)D_\nu. \tag{3.27}$$

Multiplying both sides of this equation by $Y_{\nu'}(\Omega)$ and integrating over $d\mu(\Omega)$, we find $f(\tau_\nu)=\tau_\nu^{-1/2}$, i.e.,

$$f(s;\tau_\nu)=\tau_\nu^{-s/2}. \tag{3.28}$$

Obviously, the standardization condition $f(1)=1$ is satisfied. Finally, we obtain

$$\Delta(\Omega;s)=\sum_\nu \tau_\nu^{-s/2}Y_\nu^*(\Omega)D_\nu=\sum_\nu \tau_\nu^{-s/2}Y_\nu(\Omega)D_\nu^\dagger. \tag{3.29}$$

It is evident that this kernel is completely determined by the harmonic functions on the corresponding manifold and by the coherent states which form this manifold. We will see that the SW kernel (3.29) is a generalization of the Cahill-Glauber kernel for a harmonic oscillator [10,11] and of the Agarwal kernel for spin [44].

## IV. PHASE-SPACE FUNCTIONS AND THE SYMBOL CALCULUS

### A. Types of phase-space function

As the explicit form of the SW kernels is known, we can write the SW symbols on the phase space as

$$F_A(\Omega;s)=\sum_\nu \tau_\nu^{-s/2}\mathcal{A}_\nu Y_\nu(\Omega)=\sum_\nu \tau_\nu^{-s/2}\tilde{\mathcal{A}}_\nu Y_\nu^*(\Omega), \tag{4.1}$$

where we have defined

$$\mathcal{A}_\nu\equiv\mathrm{Tr}(AD_\nu^\dagger),\quad \tilde{\mathcal{A}}_\nu\equiv\mathrm{Tr}(AD_\nu). \tag{4.2}$$

For a self-adjoint operator $A$, we get $\tilde{\mathcal{A}}_\nu=\mathcal{A}_\nu^*$. It can be easily verified that substituting expressions (4.1) and (3.29) into the inverse Weyl rule (2.7), one indeed obtains $A=\Sigma_\nu \mathcal{A}_\nu D_\nu$. We also note that the function $K_{s,s'}(\Omega,\Omega')$ of Eq. (2.10) is given by

$$K_{s,s'}(\Omega,\Omega')=\sum_\nu \tau_\nu^{-(s-s')/2}Y_\nu(\Omega)Y_\nu^*(\Omega'), \tag{4.3}$$

and it clearly satisfies Eq. (2.9) which connects the functions with different values of the index $s$. In general, let $F(\Omega)$ and $H(\Omega)$ be two phase-space functions such that

$$F(\Omega)=\sum_\nu F_\nu Y_\nu(\Omega), \tag{4.4}$$

$$H(\Omega)=\sum_\nu H_\nu Y_\nu(\Omega). \tag{4.5}$$

Then they are related through the transformation

$$F(\Omega)=\int_X d\mu(\Omega')K_{FH}(\Omega,\Omega')H(\Omega'), \tag{4.6}$$

$$K_{FH}(\Omega,\Omega')=\sum_\nu \frac{F_\nu}{H_\nu}Y_\nu(\Omega)Y_\nu^*(\Omega'). \tag{4.7}$$

Let $\{|\phi_n\rangle\}$ be a complete orthonormal basis in the Hilbert space $\mathcal{H}$. Using the generalized Weyl rule (2.6) for the operator $A=|\phi_n\rangle\langle\phi_m|$, we obtain

$$F_A(\Omega;s)=\langle\phi_m|\Delta(\Omega;s)|\phi_n\rangle\equiv\Delta_{mn}(\Omega;s). \tag{4.8}$$

Using Eq. (3.29), we find

$$\Delta_{mn}(\Omega;s)=\sum_\nu \tau_\nu^{-s/2}\langle\phi_m|D_\nu^\dagger|\phi_n\rangle Y_\nu(\Omega). \tag{4.9}$$

The standardization and tracing conditions (2.5b) and (2.5d) can be used to show that

$$\int_X d\mu(\Omega)\Delta_{mn}(\Omega;s)=\delta_{mn}, \tag{4.10}$$

$$\int_X d\mu(\Omega)\Delta_{mn}(\Omega;s)\Delta_{kl}(\Omega;-s)=\delta_{ml}\delta_{nk}. \tag{4.11}$$

The functions $\Delta_{mn}(\Omega;s)$ form a useful orthonormal basis in $L^2(X,\mu)$.

The SW symbols obtained for some special values of $s$ are frequently used in numerous applications. In particular, for $s=-1$, we obtain the $Q$ function (Berezin's covariant symbol [38]):

$$Q_A(\Omega)\equiv F_A(\Omega;-1)=\langle\Omega|A|\Omega\rangle. \tag{4.12}$$

Equation (4.12) can be easily obtained by recalling [see Eqs. (3.26) and (3.27)] that

$$\Delta(\Omega;-1)=|\Omega\rangle\langle\Omega|=\sum_\nu \tau_\nu^{1/2}Y_\nu^*(\Omega)D_\nu. \tag{4.13}$$

For $s=1$, we obtain the $P$ function (Berezin's contravariant symbol [38]):

$$P_A(\Omega)\equiv F_A(\Omega;1)=\sum_\nu \tau_\nu^{-1/2}\mathcal{A}_\nu Y_\nu(\Omega), \tag{4.14}$$

whose defining property is

$$A=\int_X d\mu(\Omega)P_A(\Omega)|\Omega\rangle\langle\Omega|. \tag{4.15}$$

The functions $P$ and $Q$ are counterparts in the tracing condition (2.5d). Perhaps the most important SW symbol corresponds to $s=0$, because this function is ''self-conjugate'' in the sense that it is the counterpart of itself in the tracing condition (2.5d). It is natural to call the function with $s=0$ the generalized Wigner function:

$$W_A(\Omega)\equiv F_A(\Omega;0)=\sum_\nu \mathcal{A}_\nu Y_\nu(\Omega). \tag{4.16}$$

The corresponding SW kernel is

$$\Delta(\Omega;0)\equiv\Delta_W(\Omega)=\sum_\nu Y_\nu^*(\Omega)D_\nu. \tag{4.17}$$

### B. The generalized twisted product

The phase-space formulation of quantum mechanics can be made completely autonomous if one introduces a symbol calculus for the functions on the phase space, which replaces the usual manipulations with operators on the Hilbert space. This symbol calculus is based on the so-called twisted product (or Moyal product) which corresponds to the usual product of operators [18,26,27].

Let us first consider the case of the Wigner function ($s=0$). The twisted product of two functions is denoted by $W_A * W_B$ and is determined by the condition

$$W_A(\Omega) * W_B(\Omega) = W_{AB}(\Omega) \tag{4.18}$$

for any two operators $A$ and $B$. Note that the condition (4.18) assures the associativity of the twisted product. On the other hand, this product is, in general, noncommutative. In this way the algebra of operators is mapped onto the algebra of phase-space functions. If one starts from a classical phase-space description, the introduction of the twisted product can be viewed as the quantization realized by a deformation of the algebra of functions [18].

Using the Weyl rule (2.6) and its inverse (2.7), we obtain

$$\begin{aligned} W_{AB}(\Omega) &= \mathrm{Tr}[\Delta_W(\Omega) AB] \\ &= \mathrm{Tr}\Big[ \Delta_W(\Omega) \int_X d\mu(\Omega') W_A(\Omega') \Delta_W(\Omega') \\ &\quad \times \int_X d\mu(\Omega'') W_B(\Omega'') \Delta_W(\Omega'') \Big]. \end{aligned} \tag{4.19}$$

Introducing the function (trikernel)

$$L(\Omega,\Omega',\Omega'') = \mathrm{Tr}[\Delta_W(\Omega)\Delta_W(\Omega')\Delta_W(\Omega'')], \tag{4.20}$$

we obtain the following definition of the twisted product:

$$\begin{aligned} (W_A * W_B)(\Omega) \equiv \int_X \int_X d\mu(\Omega') d\mu(\Omega'') L(\Omega,\Omega',\Omega'') \\ \times W_A(\Omega') W_B(\Omega''). \end{aligned} \tag{4.21}$$

The so-called Moyal bracket is defined as

$$[W_A, W_B]_M = -i(W_A * W_B - W_B * W_A). \tag{4.22}$$

The twisted product can be easily generalized for arbitrary values of $s$. The $s$-parametrized twisted product $(F_A * F_B)(\Omega;s)$ of any two functions $F_A(\Omega;s')$ and $F_B(\Omega;s'')$ is once again determined by the condition

$$F_A(\Omega;s') * F_B(\Omega;s'') = F_{AB}(\Omega;s). \tag{4.23}$$

Analogously to the Wigner function case, this leads to the definition

$$\begin{aligned} (F_A * F_B)(\Omega;s) \equiv \int_X \int_X d\mu(\Omega') d\mu(\Omega'') L_{s,s',s''}(\Omega,\Omega',\Omega'') \\ \times F_A(\Omega';s') F_B(\Omega'';s''), \end{aligned} \tag{4.24}$$

where the generalized trikernel is given by

$$\begin{aligned} L_{s,s',s''}(\Omega,\Omega',\Omega'') &= \mathrm{Tr}[\Delta(\Omega;s)\Delta(\Omega';-s')\Delta(\Omega'';-s'')] \\ &= \sum_{m,n,k} \Delta_{mn}(\Omega;s)\Delta_{nk}(\Omega';-s') \\ &\quad \times \Delta_{km}(\Omega'';-s''). \end{aligned} \tag{4.25}$$

Using the standardization condition (2.8b) and the definition (2.10), we obtain

$$\begin{aligned} \int_X d\mu(\Omega) L_{s,s',s''}(\Omega,\Omega',\Omega'') &= \mathrm{Tr}[\Delta(\Omega';-s')\Delta(\Omega'';-s'')] \\ &= K_{-s',s''}(\Omega',\Omega''). \end{aligned} \tag{4.26}$$

This result together with the relation (2.9) can be used to obtain the so-called tracial identity for the generalized twisted product,

$$\begin{aligned} \int_X d\mu(\Omega)(F_A * F_B)(\Omega;s) = \int_X d\mu(\Omega) F_A(\Omega;s') \\ \times F_B(\Omega;-s'), \end{aligned} \tag{4.27}$$

which holds for any $s$ and $s'$. Equation (4.27) is the phase-space version of the tracial identity for the operators,

$$\mathrm{Tr}(AB) = \sum_\nu A_\nu \tilde{B}_\nu. \tag{4.28}$$

Using the covariance condition (2.8c) and the definition (4.25), we find the invariance property of the trikernel

$$L_{s,s',s''}(g\cdot\Omega, g\cdot\Omega', g\cdot\Omega'') = L_{s,s',s''}(\Omega,\Omega',\Omega''). \tag{4.29}$$

This property implies the equivariance of the twisted product:

$$(F_A * F_B)^g(\Omega;s) = F_A^g(\Omega;s') * F_B^g(\Omega;s''), \tag{4.30}$$

where

$$F_A^g(\Omega;s) \equiv F_A(g^{-1}\cdot\Omega;s). \tag{4.31}$$

## V. EXAMPLES

The general formalism presented above can be understood much better by illustrating it with a number of simple examples. We will consider two simple physical systems: a (nonrelativistic) spinless quantum particle and spin, whose dynamical symmetry groups are the Heisenberg-Weyl group $H_3$ and SU(2), respectively. It should be emphasized that the SW kernels for these basic systems have been known for a long time [45], so the novelty here is not the result itself but the method of derivation. Our aim is to demonstrate how the general algorithm works by applying it to a number of relatively simple and well-known problems. We will show that by identifying harmonic functions, invariant coefficients, and tensor operators for a given system, one can readily derive the explicit form of the SW kernel.

### A. The Heisenberg-Weyl group

First, we consider the Heisenberg-Weyl group $H_3$ which is the dynamical symmetry group for a spinless quantum particle and for a mode of the quantized radiation field (modeled by a quantum harmonic oscillator). The Wigner function [1] and the Moyal quantization [2] were originally introduced for such systems. The kernel implementing the mapping between Hilbert-space operators and $s$-parametrized families of phase-space functions (the SW kernel in our notation) for $H_3$ was introduced by Cahill and Glauber [10]. The generalization of the formalism to the many-dimensional case is straightforward (see, e.g., Ref. [8]).

The nilpotent Lie algebra of $H_3$ is spanned by the basis $\{a,a^\dagger,I\}$, where $a$ and $a^\dagger$ are the boson annihilation and creation operators, satisfying the canonical commutation relation, $[a,a^\dagger]=I$. Group elements can be parametrized in the following way:

$$g=g(\gamma,\varphi),\quad T(g)=e^{\gamma a^\dagger-\gamma^* a}e^{i\varphi I}, \tag{5.1}$$

where $\gamma\in\mathbb{C}$ and $\varphi\in\mathbb{R}$.

The phase space is the complex plane $\mathbb{C}=H_3/\mathrm{U}(1)$, and the (Glauber) coherent states are

$$|\Omega\rangle\equiv|\alpha\rangle=D(\alpha)|0\rangle,\quad \alpha\in\mathbb{C} \tag{5.2}$$

where

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a) \tag{5.3}$$

is the displacement operator. The invariant measure is

$$d\mu(\Omega)\equiv\pi^{-1}d^2\alpha, \tag{5.4}$$

and the corresponding $\delta$ function is

$$\delta(\Omega-\Omega')\equiv\pi\delta^{(2)}(\alpha-\alpha'). \tag{5.5}$$

The harmonic functions on $\mathbb{C}$ are the exponentials:

$$Y_\nu(\Omega)\equiv Y_\xi(\alpha)\equiv Y(\xi,\alpha)=\exp(\xi\alpha^*-\xi^*\alpha). \tag{5.6}$$

Here $\nu\equiv\xi\in\mathbb{C}$ with the Plancherel measure given by $d\rho(\nu)\equiv\pi^{-1}d^2\xi$ and with $\delta_{\nu,\nu'}\equiv\pi\delta^{(2)}(\xi-\xi')$. Note that for the Heisenberg-Weyl group both the phase-space coordinate $\Omega\equiv\alpha$ and the index $\nu\equiv\xi$ are complex numbers, and the Plancherel measure is similar to the invariant measure on $\mathbb{C}$.

The invariant coefficients $\tau_\nu\equiv\tau(\xi)$ can be found in the following way. In the present context Eq. (3.6) takes the form

$$|\langle\alpha|\alpha'\rangle|^2=e^{-|\alpha-\alpha'|^2}=\int_{\mathbb{C}}\frac{d^2\xi}{\pi}\tau(\xi)e^{\xi^*(\alpha-\alpha')-\xi(\alpha-\alpha')^*}.$$

Taking into account that the Fourier transform of a Gaussian function is once again a Gaussian, it is not difficult to obtain

$$\tau(\xi)=\exp(-|\xi|^2). \tag{5.7}$$

Then we deduce that the tensor operator

$$D_\nu\equiv D(\xi)=e^{|\xi|^2/2}\int_{\mathbb{C}}\frac{d^2\alpha}{\pi}e^{\xi\alpha^*-\xi^*\alpha}|\alpha\rangle\langle\alpha| \tag{5.8}$$

is just the displacement operator $D(\xi)=e^{\xi a^\dagger-\xi^* a}$. The natural orthonormal basis in the Hilbert space is the Fock basis $\{|n\rangle\}$, $a^\dagger a|n\rangle=n|n\rangle$ $(n=0,1,2,\ldots)$. The matrix elements of the tensor operator are given by [37]

$$\langle m|D(\xi)|n\rangle=\begin{cases}\sqrt{n!/m!}\,e^{-|\xi|^2/2}\xi^{m-n}L_n^{m-n}(|\xi|^2), & m\geqslant n\\ \sqrt{m!/n!}\;e^{-|\xi|^2/2}(-\xi^*)^{n-m}L_m^{n-m}(|\xi|^2), & m\leqslant n\end{cases}$$

where $L_n^p(x)$ are the associated Laguerre polynomials. Using the parametrization (5.1) of group elements, one can easily find the transformation rule

$$\begin{aligned}T(g)D(\xi)T(g^{-1})&=D(\gamma)D(\xi)D(-\gamma)\\&=\exp(\gamma\xi^*-\gamma^*\xi)D(\xi).\end{aligned} \tag{5.9}$$

Therefore the index $\xi$ does not change under the group transformation, as $D(\xi)$ and $Y(\xi,\alpha)$ are just multiplied by a phase factor. Correspondingly, there is no index transformation, induced by the action of group elements, to which $\tau(\xi)$ should be invariant. On the other hand, $Y^*(\xi,\alpha)=Y(-\xi,\alpha)$, and $\tau(\xi)$ is obviously invariant under the index transformation $\xi\leftrightarrow-\xi$.

Finally, the harmonic functions $Y(\xi,\alpha)$, the invariant coefficients $\tau(\xi)$, and the tensor operators $D(\xi)$ can be substituted into the general formula (3.29). Then one obtains the SW kernel for the Heisenberg-Weyl group:

$$\Delta(\alpha;s)=\int_{\mathbb{C}}\frac{d^2\xi}{\pi}e^{s|\xi|^2/2}e^{\xi^*\alpha-\xi\alpha^*}e^{\xi a^\dagger-\xi^* a}, \tag{5.10}$$

which is exactly the kernel introduced by Cahill and Glauber [10].

### B. The SU(2) group

As another example, we consider SU(2) which is the dynamical symmetry group for the angular momentum or spin and for many other systems, for example, a collection of two-level atoms, the Stokes operators describing the polarization of the quantized light field, two light modes with a fixed total photon number, etc. A number of authors have used different approaches to the construction of the Wigner function for spin [20,23,27,38,44,46–52]. The explicit expressions for the $Q$, $W$, and $P$ functions for arbitrary spin were obtained by Agarwal [44], who used the spin coherent-state representation [37,53,54] and the Fano multipole operators [55]. Várilly and Gracia-Bondía [27] have shown that the spin coherent-state approach is equivalent to the formalism based on the SW correspondence.

The simple Lie algebra of SU(2) is spanned by the basis $\{J_x,J_y,J_z\}$,

$$[J_p,J_r]=i\epsilon_{prt}J_t. \tag{5.11}$$

The unitary irreducible representations are labeled by the index $j$ $(j=0,1/2,1,\ldots)$, and the Hilbert space $\mathcal{H}_j$ is spanned by the orthonormal basis $|j,\mu\rangle$ $(\mu=j,j-1,\ldots,-j)$. Group elements can be parametrized using the Euler angles $\alpha,\beta,\gamma$:

$$g=g(\alpha,\beta,\gamma)=e^{i\alpha J_z}e^{i\beta J_y}e^{i\gamma J_z}. \tag{5.12}$$

The phase space is the unit sphere $\mathbb{S}^2=\mathrm{SU}(2)/\mathrm{U}(1)$, and each coherent state is characterized by the unit vector

$$\mathbf{n}=(\sin\,\theta\,\cos\,\phi,\sin\,\theta\,\sin\,\phi,\cos\,\theta). \tag{5.13}$$

Specifically, the coherent states $|\Omega\rangle\equiv|j;\mathbf{n}\rangle$ are given by the action of the group element

$$g(\Omega)=g(\theta,\phi)=e^{-i\phi J_z}e^{-i\theta J_y} \tag{5.14}$$

on the highest-weight state $|j,j\rangle$:

$$\begin{aligned}|j;\mathbf{n}\rangle=|j;\theta,\phi\rangle&=g(\theta,\phi)|j,j\rangle\\&=\sum_{\mu=-j}^{j}\binom{2j}{j+\mu}^{1/2}\cos^{j+\mu}(\theta/2)\\&\quad\times\sin^{j-\mu}(\theta/2)e^{-i\mu\phi}|j,\mu\rangle.\end{aligned} \tag{5.15}$$

The invariant measure is

$$d\mu(\Omega)\equiv\frac{2j+1}{4\pi}d\mathbf{n}=\frac{2j+1}{4\pi}\sin\,\theta d\theta\,d\phi, \tag{5.16}$$

and the corresponding delta function is

$$\begin{aligned}\delta(\Omega-\Omega')&\equiv\frac{4\pi}{2j+1}\delta(\mathbf{n}-\mathbf{n}')\\&=\frac{4\pi}{2j+1}\delta(\cos\,\theta-\cos\,\theta')\delta(\phi-\phi').\end{aligned} \tag{5.17}$$

The harmonic functions on $\mathbb{S}^2$ are the familiar spherical harmonics:

$$Y_\nu(\Omega)\equiv\sqrt{\frac{4\pi}{2j+1}}Y_{lm}(\theta,\phi). \tag{5.18}$$

In this context $\nu$ is the double discrete index $\{l,m\}$ with $l=0,1,2,\ldots$ and $m=l,l-1,\ldots,-l$. The transformation rule for the spherical harmonics reads

$$g(\alpha,\beta,\gamma)Y_{lm}(\theta,\phi)=\sum_{m'=-l}^{l}\mathcal{D}^{(l)}_{m'm}(\alpha,\beta,\gamma)Y_{lm'}(\theta,\phi), \tag{5.19}$$

where

$$\mathcal{D}^{(l)}_{m'm}(\alpha,\beta,\gamma)=\langle l,m'|g(\alpha,\beta,\gamma)|l,m\rangle \tag{5.20}$$

is the matrix representation of SU(2) elements and $g(\alpha,\beta,\gamma)$ is given by Eq. (5.12). Another property of the spherical harmonics is

$$Y^*_{lm}(\theta,\phi)=(-1)^mY_{l,-m}(\theta,\phi). \tag{5.21}$$

The invariant coefficients can be found using the following expansion [27]:

$$\begin{aligned}|\langle j,\mathbf{n}|j,\mathbf{n}'\rangle|^2&=\left(\frac{1+\mathbf{n}\cdot\mathbf{n}'}{2}\right)^{2j}\\&=\sum_{l=0}^{2j}\frac{2l+1}{2j+1}\langle j,j;l,0|j,j\rangle^2P_l(\mathbf{n}\cdot\mathbf{n}'),\end{aligned} \tag{5.22}$$

where $P_l(x)$ are the Legendre polynomials and

$$\langle j_1,m_1;j_2,m_2|j,m\rangle\equiv C^{j_1j_2j}_{m_1m_2m} \tag{5.23}$$

are the Clebsch-Gordan coefficients. Using the addition formula for the spherical harmonics,

$$\frac{2l+1}{4\pi}P_l(\mathbf{n}\cdot\mathbf{n}')=\sum_{m=-l}^{l}Y^*_{lm}(\mathbf{n})Y_{lm}(\mathbf{n}'), \tag{5.24}$$

Eq. (5.22) can be rewritten as

$$\begin{aligned}|\langle j,\mathbf{n}|j,\mathbf{n}'\rangle|^2&=\frac{4\pi}{2j+1}\sum_{l=0}^{2j}\sum_{m=-l}^{l}\langle j,j;l,0|j,j\rangle^2\\&\quad\times Y^*_{lm}(\mathbf{n})Y_{lm}(\mathbf{n}').\end{aligned} \tag{5.25}$$

Comparing this result with the general formula (3.6), we readily find that the invariant coefficients are given by

$$\tau_\nu\equiv\tau_l=\langle j,j;l,0|j,j\rangle^2=\frac{(2j+1)[(2j)!]^2}{(2j+l+1)!(2j-l)!}. \tag{5.26}$$

Note that $\tau_l=0$ for $l>2j$. The invariance of $\tau_l$ is ensured by the fact that they are independent of $m$.

The tensor operators for spin are the well-known Fano multipole operators [55], which can be written in the form

$$D_{lm}=\sqrt{\frac{2l+1}{2j+1}}\sum_{k,q=-j}^{j}\langle j,k;l,m|j,q\rangle|j,q\rangle\langle j,k|. \tag{5.27}$$

Substituting expressions (5.18), (5.26), and (5.27) into the general formula (3.29), we find that the SW kernel for spin is given by

$$\begin{aligned}\Delta(\theta,\phi;s)&=\sqrt{\frac{4\pi}{2j+1}}\sum_{l=0}^{2j}\langle j,j;l,0|j,j\rangle^{-s}\\&\quad\times\sum_{m=-l}^{l}D_{lm}Y^*_{lm}(\theta,\phi),\end{aligned} \tag{5.28}$$

which coincides for $s=0,\pm1$ with the results by Agarwal [44] and by Várilly and Gracia-Bondía [27].

## VI. RECONSTRUCTION OF QUANTUM STATES

### A. Basic systems and methods

A great amount of work has been devoted in the last few years to the problem of determining the quantum state from information obtained by a set of measurements performed on an ensemble of identically prepared systems. The task is to

reconstruct the density matrix $\rho$ which, according to the principles of quantum physics, contains all available information about the state of a system. Of course, the question arises as to which set of measurements provides information sufficient for the state reconstruction. This question first appeared in early works by Fano [56] and Pauli [57] and was discussed in a number of papers [58–63].

Recently, significant theoretical and experimental progress has been achieved in the reconstruction of quantum states of the light field (see, e.g., a recent book [17]). One of the most successful reconstruction methods in this context is the optical homodyne tomography. A tomographic approach to the Wigner function was discussed by Bertrand and Bertrand [64] and a quantum-optical scheme was proposed by Vogel and Risken [65]. The reconstruction of quantum states of the light field by means of homodyne tomography was realized in a series of intriguing experiments [66,67]. Various methods for data analysis in optical homodyne tomography measurements were recently discussed [68–72]. The tomographic schemes were also generalized for the reconstruction of the joint density matrix for two-mode and multimode optical fields [73–77]. Among other approaches to the reconstruction of quantum states of light we would like to mention the symplectic tomography [78] and the photon counting methods [79–81] (also known as the photon number tomography [82]).

In the case of a single-mode microwave field inside a high-$Q$ cavity, a direct measurement on the system itself is impossible. Instead, one can probe the state of the intracavity field via the detection of atoms after their interaction with the field mode [83–85]. Similar ideas were also applied to the reconstruction of the quantum motional state of a laser-cooled ion trapped in a harmonic potential [84,86–90], including a beautiful experimental realization [91].

State reconstruction procedures were proposed for various quantum systems, for example, one-dimensional wave packets [92,93], harmonic and anharmonic molecular vibrations [94,95], motional states of atom beams [96], Bose-Einstein condensates [97], cyclotron states of a trapped electron [98], atomic Rydberg wave functions [99], etc. State reconstruction methods for systems with a finite-dimensional state space (e.g., for spin) were also discussed [51,52,59,60,100,101]. Experimental reconstructions were also reported for electronic angular-momentum states of hydrogen [102] and for vibrational quantum states of a diatomic molecule [103].

### B. Displaced projectors

It turns out that the majority of schemes used for the reconstruction of quantum states are related to the phase-space formalism. Frequently, the $Q$ function, the Wigner function, or other phase-space QPDs representing the density matrix $\rho$ of the system can be either measured directly or deduced in some way from measured data. In particular, in many proposed and realized schemes the measured quantity is the expectation value

$$p_u(\lambda)=\langle\Gamma_u(\lambda)\rangle=\mathrm{Tr}[\rho\Gamma_u(\lambda)] \tag{6.1}$$

of a self-adjoint operator

$$\Gamma_u(\lambda)=U(\lambda)|u\rangle\langle u|U^\dagger(\lambda), \tag{6.2}$$

which is a transformed projector on a quantum state $|u\rangle$. The unitary operator $U(\lambda)$ represents the corresponding transformation, and the measurements are made for a range of values of the transformation parameter $\lambda$.

We will distinguish here between two possibilities. If $U(\lambda)=T(\Omega)$ is the phase-space displacement operator which represents an element of $X=G/H$, with $G$ being the dynamical symmetry group of a given quantum system, we will call the observable $\Gamma_u(\lambda)=\Gamma_u(\Omega)$ the properly transformed projector (or the displaced projector). Otherwise $\Gamma_u(\lambda)$ will be called the improperly transformed projector.

In order to illustrate these definitions, let us consider a quantum harmonic oscillator which is the model system for a single mode of the quantized radiation field, a laser-cooled ion moving in a harmonic trap, or a harmonic vibrational mode of a diatomic molecule. The corresponding symmetry group is the Heisenberg-Weyl group $H_3$, and the phase space is the complex plane $\mathbb{C}=H_3/\mathrm{U}(1)$ (see Sec. V A). In this context $U(\lambda)=D(\alpha)$ is the Glauber displacement operator, and the expectation value of the displaced projector,

$$p_u(\alpha)=\mathrm{Tr}[\rho\Gamma_u(\alpha)]=\mathrm{Tr}[\rho D(\alpha)\rho_u D^\dagger(\alpha)], \tag{6.3}$$

is called the operational phase-space probability distribution [104–106]. Here, $\rho$ is the density matrix of the quantum state of the system and $\rho_u$ is the density matrix (given by the projector $|u\rangle\langle u|$ for a pure state) of the so-called ''quantum ruler'' state which characterizes the measurement device. For example, displacing the state of the oscillator,

$$\rho\rightarrow\rho(\alpha)=D^\dagger(\alpha)\rho D(\alpha), \quad \alpha\in\mathbb{C}, \tag{6.4}$$

and measuring the probability of finding it in the Fock state $|n\rangle$, one obtains the operational phase-space probability distribution,

$$p_n(\alpha)=\langle n|\rho(\alpha)|n\rangle=\mathrm{Tr}[\rho\Gamma_n(\alpha)]. \tag{6.5}$$

The displaced projector

$$\Gamma_n(\alpha)=D(\alpha)|n\rangle\langle n|D^\dagger(\alpha) \tag{6.6}$$

is obtained for $|u\rangle=|n\rangle$ being the Fock state. In particular, measuring the probability of finding the displaced oscillator in the ground state $|0\rangle$, one obtains the Husimi function $Q(\alpha)=\langle\alpha|\rho|\alpha\rangle$. On the other hand, if one knows the functions $p_n(\alpha)$ for all values of $n$, then the Wigner function can be built as [10]

$$W(\alpha)=2\sum_{n=0}^{\infty}(-1)^n p_n(\alpha). \tag{6.7}$$

This formula can be generalized for QPDs with other values of $s$ [107]:

$$F_\rho(\alpha;s)\equiv P(\alpha;s)=\frac{2}{1-s}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^n p_n(\alpha). \tag{6.8}$$

These methods for determining the Husimi function and the Wigner function (and thus reconstructing the quantum state of the system) were discussed by Royer [92] in 1985.

Recently, such a scheme for measuring the $Q$ function was proposed in the context of trapped ions [87]. Another method for the reconstruction of the motional state of a trapped ion, proposed and experimentally realized by the NIST group [91], employs the interaction between the vibrational mode of the ion and its internal electronic levels. The initial motional state is displaced in the phase space, as in Eq. (6.4), and then the interaction with the two-level internal subsystem is induced for a time $t$. The population $P_{\downarrow}(t,\alpha)$ of the lower internal state $|\downarrow\rangle$ is measured for different values of displacement amplitude $\alpha$ and time $t$ (this measurement can be made with great accuracy by monitoring the fluorescence produced in driving a resonant dipole transition). If $|\downarrow\rangle$ is the internal state at $t=0$, the signal averaged over many measurements is

$$P_{\downarrow}(t,\alpha)=\frac{1}{2}\left[1+\sum_{n=0}^{\infty}p_n(\alpha)\cos(2\Omega_{n,n+1}t)e^{-\gamma_n t}\right], \tag{6.9}$$

where $\Omega_{n,n+1}$ are the Rabi frequencies and $\gamma_n$ are the experimentally determined decay constants. This relation allows one to determine the populations $p_n(\alpha)$ of the displaced motional eigenstates. As one can see from Eq. (6.8), the functions $p_n(\alpha)$ in their turn can be used to calculate the QPDs $P(\alpha;s)$ (e.g., the Wigner function). Alternatively, the density matrix in the Fock representation can be deduced directly from $p_n(\alpha)$.

In the optical domain, the function $p_n(\alpha)$ can be determined in principle as the probability of recording $n$ counts with an ideal photodetector exposed to the displaced light field. In practice, one could use the unbalanced homodyning detection [79–82], in which the signal field is mixed in a beam splitter with the local oscillator of coherent amplitude $\beta$ and the photon statistics of the superimposed field is then counted by a photodetector of quantum efficiency $\zeta$. The resulting counting statistics is denoted by $p_n(\alpha,\eta)$, where $\alpha=-R\beta/T$ is the effective displacement amplitude ($T$ and $R$ are the transmission and reflection coefficients of the beam splitter) and $\eta=\zeta|T|^2$ is the overall quantum efficiency. In this realistic situation formula (6.8) should be replaced by the following result [80]:

$$P(\alpha;s)=\frac{2}{1-s}\sum_{n=0}^{\infty}\left[\frac{2+\eta(s-1)}{\eta(s-1)}\right]^n p_n(\alpha,\eta). \tag{6.10}$$

This method of state reconstruction is sometimes called the photon number tomography.

As an example of measurements with improperly transformed projectors, we mention the optical homodyne tomography [65,66] in which one measures the probability distribution $P(x,\theta)$ for the rotated field quadrature

$$x_\theta=x\cos\theta+p\sin\theta=U(\theta)xU^{\dagger}(\theta). \tag{6.11}$$

The field quadratures $x$ and $p$ can be viewed as the scaled position and momentum operators of the harmonic oscillator, with $a=2^{-1/2}(x+ip)$, and $U(\theta)=\exp(i\theta a^{\dagger}a)$ is the rotation operator (known in optics as the phase shifter) on the phase plane. $U(\theta)$ represents an element of the SO(2)$\sim$U(1) subgroup of the oscillator group $H_4$ whose algebra is spanned by $\{I,a,a^{\dagger},a^{\dagger}a\}$. The improperly transformed projector is given by

$$\Gamma_x(\theta)=U(\theta)|x\rangle\langle x|U^{\dagger}(\theta), \tag{6.12}$$

where $|x\rangle$ are the position eigenstates. The measured distribution $P(x,\theta)$ can be used for determining the Wigner function via the inverse Radon transform [64–66]. Alternatively, the density matrix in some basis (e.g., in the Fock basis) can be deduced directly from $P(x,\theta)$ by averaging a set of pattern functions [68–71]. Another example of measurements with improperly transformed projectors is the symplectic tomography [78], in which the phase-space rotation is accompanied by the squeezing transformation.

In the case of measurements with improperly transformed projectors, a general group-theoretic approach is problematic, because the number of possible transformations is very large and one should consider each situation separately. On the other hand, the method of properly transformed projectors works uniformly for physical systems with different symmetry groups. For example, in the case of the SU(2) symmetry (e.g., spin, two-level atoms, etc.), proposals appeared [100,101] for measuring the $Q$ function,

$$Q(\mathbf{n})=\langle j,\mathbf{n}|\rho|j,\mathbf{n}\rangle=\mathrm{Tr}(\rho|j,\mathbf{n}\rangle\langle j,\mathbf{n}|), \tag{6.13}$$

or, more generally, for measuring the probability

$$p_\mu(\mathbf{n})=\mathrm{Tr}[\rho\Gamma_\mu(\mathbf{n})], \tag{6.14}$$

$$\Gamma_\mu(\mathbf{n})=g(\mathbf{n})|j,\mu\rangle\langle j,\mu|g^{-1}(\mathbf{n}) \tag{6.15}$$

of finding the displaced system

$$\rho(\mathbf{n})=g^{-1}(\mathbf{n})\rho g(\mathbf{n}),\quad \mathbf{n}\in\mathbb{S}^2 \tag{6.16}$$

in the state $|j,\mu\rangle$. These ideas for spin are conceptually very similar to the proposals in the context of optical fields or trapped ions. Therefore it seems natural to apply the phase-space formalism developed above to the general group-theoretic description of the state reconstruction method based on the measurement of displaced projectors.

### C. General reconstruction formalism

From the practical point of view, the reconstruction procedure consists of two steps. First, the system described by the density matrix $\rho$ is displaced in the phase space:

$$\rho\rightarrow\rho(\Omega)=T^{-1}(\Omega)\rho T(\Omega),\quad \Omega\in X. \tag{6.17}$$

The second step is the measurement of the probability to find the (displaced) system in a quantum state $|u\rangle$,

$$p_u(\Omega)=\langle u|\rho(\Omega)|u\rangle. \tag{6.18}$$

Repeating this procedure for a large number of phase-space points $\Omega$, one can, in principle, determine the function $p_u(\Omega)$.

#### 1. More about displaced projectors

The information contained in the function $p_u(\Omega)$ is enough for the reconstruction of the density matrix. It is convenient to analyze this problem with the help of the displaced projector,

$$\Gamma_u(\Omega)=T(\Omega)|u\rangle\langle u|T^{-1}(\Omega), \tag{6.19}$$

whose expectation value gives the measured function $p_u(\Omega)$, as in Eq. (6.1). The displaced projector satisfies a number of useful properties.

(i) It is a self-adjoint operator,

$$\Gamma_u^{\dagger}(\Omega)=\Gamma_u(\Omega) \quad \forall \Omega \in X. \tag{6.20}$$

Since $p_u(\Omega)$ is not only real but also non-negative (this is a probability), $\Gamma_u(\Omega)$ is also a non-negatively defined operator.

(ii) Provided that the state $|u\rangle$ is normalized, $\Gamma_u(\Omega)$ is a trace-class operator of trace one, and the following standardization condition holds:

$$\int_X d\mu(\Omega)\Gamma_u(\Omega)=I. \tag{6.21}$$

This implies the normalization of $p_u(\Omega)$,

$$\int_X d\mu(\Omega)p_u(\Omega)=1. \tag{6.22}$$

(iii) The displaced projector is manifestly covariant,

$$T(g)\Gamma_u(\Omega)T(g^{-1})=\Gamma_u(g\cdot\Omega). \tag{6.23}$$

Consequently, if $p_u(\Omega)$ corresponds to the initial density matrix $\rho$, the function $p_u(g\cdot\Omega)$ will correspond to the transformed density matrix $\rho(g)=T(g^{-1})\rho T(g)$.

Denoting the density matrix of the quantum ruler state by $\rho_u$ (which is $|u\rangle\langle u|$ for a pure state), the operational phase-space probability distribution reads

$$p_u(\Omega)=\mathrm{Tr}[\rho T(\Omega)\rho_u T^{-1}(\Omega)]. \tag{6.24}$$

Using the inverse Weyl rule (2.7) for the density matrix $\rho$, we obtain

$$p_u(\Omega)=\int_X d\mu(\Omega')P(\Omega';s)\mathrm{Tr}[T^{-1}(\Omega) \times\Delta(\Omega';-s)T(\Omega)\rho_u],$$

where $P(\Omega;s)\equiv F_\rho(\Omega;s)$ is the SW symbol of $\rho$. Now, the covariance property (2.8c) can be used to obtain the following expression:

$$p_u(\Omega)=\int_X d\mu(\Omega')P(\Omega\cdot\Omega';s)P_u(\Omega';-s), \tag{6.25}$$

where $P_u(\Omega;s)$ is the SW symbol of $\rho_u$. Therefore the operational phase-space probability distribution $p_u(\Omega)$ is given by a convolution of the two QPDs representing the quantum state of the system and the quantum ruler state of the measurement apparatus. In the particular case of the Heisenberg-Weyl group and for $s=0$, the general expression (6.25) reduces to the known result [105]

$$p_u(\alpha)=\int_{\mathbb{C}}\frac{d^2\alpha'}{\pi}W(\alpha+\alpha')W_u(\alpha'). \tag{6.26}$$

If the quantum ruler state $|u\rangle=|\psi_0\rangle$ is the reference state of the coherent-state basis, then

$$\Gamma_{\psi_0}(\Omega)=|\Omega\rangle\langle\Omega|=\Delta(\Omega;-1) \tag{6.27}$$

is the SW kernel with $s=-1$, and

$$p_{\psi_0}(\Omega)=\langle\Omega|\rho|\Omega\rangle=Q_\rho(\Omega) \tag{6.28}$$

is the $Q$ function. However, except for this coincidence, the displaced projectors are not the SW kernels, as they do not satisfy the tracing condition. On the other hand, the functions $p_u(\Omega)$ differ from the majority of QPDs, as they are positive on the whole phase space (which reflects the fact that they are measurable probabilities). Usually the state $|u\rangle$ is chosen to belong to some complete orthonormal basis $\{|\phi_n\rangle\}$ which consists of energy eigenstates of a natural Hamiltonian of the physical system (e.g., the Fock basis for a harmonic oscillator or $J_z$ eigenstates for spin). Then there exists the relation

$$\sum_n p_{\phi_n}(\Omega)=1, \tag{6.29}$$

which follows from the completeness of the basis.

#### 2. Entropy

A useful quantity for analyzing statistical properties of the system (in particular, the amount of noise) is the entropy. A phase-space version of the entropy can be introduced in the following way:

$$S_u=-\int_X d\mu(\Omega)p_u(\Omega)\ln p_u(\Omega). \tag{6.30}$$

For $|u\rangle=|\psi_0\rangle$, Eq. (6.30) gives

$$S=-\int_X d\mu(\Omega)Q_\rho(\Omega)\ln Q_\rho(\Omega), \tag{6.31}$$

which is a generalization of the Wehrl entropy [108] that was defined originally on the flat phase space of the Weyl-Heisenberg group. The entropy (6.30) can be useful in the reconstruction procedure, as it is a sensitive measure of the noise added to the system during the displacement and detection processes. A similar situation exists also in the field of signal processing [109]. There $|u\rangle$ represents the test signal and $p_u(\Omega)$ is a distribution on the time-frequency space. One can produce various test signals $|u\rangle$ and compute the corresponding entropies $S_u$. Choosing $|u\rangle$ that minimizes the entropy, one obtains the optimal form of pattern analysis (in particular, this method allows one to achieve data compression).

### 3. Harmonic expansions

A useful expression for $p_u(\Omega)$ can be derived in the following way. Using the expansion

$$\rho=\sum_\nu \mathcal{R}_\nu D_\nu, \quad \mathcal{R}_\nu\equiv \mathrm{Tr}(\rho D_\nu^\dagger), \tag{6.32}$$

we obtain

$$p_u(\Omega)=\sum_\nu \mathcal{R}_\nu\langle u|T^{-1}(\Omega)D_\nu T(\Omega)|u\rangle$$

$$=\sum_\nu \mathcal{R}_\nu\sum_{\nu'} T^{-1}_{\nu'\nu}(\Omega)\langle u|D_{\nu'}|u\rangle. \tag{6.33}$$

Expanding $p_u(\Omega)$ in the basis of harmonic functions,

$$p_u(\Omega)=\sum_\nu \kappa_\nu^{(u)}\mathcal{R}_\nu Y_\nu(\Omega), \tag{6.34}$$

we identify the coefficients $\kappa_\nu^{(u)}$ by means of the relation

$$\kappa_\nu^{(u)}Y_\nu(\Omega)=\langle u|T(\Omega^{-1})D_\nu T(\Omega)|u\rangle. \tag{6.35}$$

Formally, we can write

$$\kappa_\nu^{(u)}=\tau_\nu^{-1/2}\int_X\int_X d\mu(\Omega)d\mu(\Omega')Y_\nu^*(\Omega)$$

$$\times Y_\nu(\Omega\cdot\Omega')|\langle u|\Omega'\rangle|^2, \tag{6.36}$$

but actually Eq. (6.35) is more convenient for calculating the coefficients $\kappa_\nu^{(u)}$.

Equation (6.34) for the functions $p_u(\Omega)$ corresponds to the expansion

$$\Gamma_u(\Omega)=\sum_\nu \kappa_\nu^{(u)}Y_\nu^*(\Omega)D_\nu=\sum_\nu \kappa_\nu^{(u)}Y_\nu(\Omega)D_\nu^\dagger \tag{6.37}$$

for the displaced projectors. It follows from the properties of $\Gamma_u(\Omega)$ that the coefficients $\kappa_\nu^{(u)}$ are positive and possess the same invariance properties as $\tau_\nu$. Using the general result (4.6), we obtain the relation between the functions $p_u(\Omega)$ and $p_v(\Omega)$, corresponding to different quantum ruler states $|u\rangle$ and $|v\rangle$,

$$p_u(\Omega)=\int_X d\mu(\Omega')K_{uv}(\Omega,\Omega')p_v(\Omega'), \tag{6.38}$$

$$K_{uv}(\Omega,\Omega')=\sum_\nu \frac{\kappa_\nu^{(u)}}{\kappa_\nu^{(v)}}Y_\nu(\Omega)Y_\nu^*(\Omega'). \tag{6.39}$$

### 4. Deducing the density matrix and quasiprobabilities

Knowledge of the phase-space function $p_u(\Omega)$ allows us to reconstruct the density matrix in a simple way:

$$\mathcal{R}_\nu=[\kappa_\nu^{(u)}]^{-1}\int_X d\mu(\Omega)p_u(\Omega)Y_\nu^*(\Omega). \tag{6.40}$$

Formally, we can also represent the density matrix by means of an integral transform of the displaced projector:

$$\rho=\int_X d\mu(\Omega)r_u(\Omega)\Gamma_u(\Omega). \tag{6.41}$$

This relation gives the density matrix in terms of a phase-space function $r_u(\Omega)$, and in this sense it is the inverse of Eq. (6.1). The function $r_u(\Omega)$ is defined by its harmonic expansion,

$$r_u(\Omega)=\sum_\nu [\kappa_\nu^{(u)}]^{-1}\mathcal{R}_\nu Y_\nu(\Omega). \tag{6.42}$$

We also obtain the following relation between the functions $r_u(\Omega)$ and $p_u(\Omega)$,

$$p_u(\Omega)=\int_X d\mu(\Omega')r_u(\Omega')\mathrm{Tr}[\Gamma_u(\Omega)\Gamma_u(\Omega')], \tag{6.43}$$

where

$$\mathrm{Tr}\,[\Gamma_u(\Omega)\Gamma_v(\Omega')]=|\langle u|T^{-1}(\Omega)T(\Omega')|v\rangle|^2$$

$$=\sum_\nu \kappa_\nu^{(u)}\kappa_\nu^{(v)}Y_\nu(\Omega)Y_\nu^*(\Omega'). \tag{6.44}$$

Certainly, the most convenient way for deducing the density matrix from the measured functions $p_u(\Omega)$ is by calculating the coefficients $\mathcal{R}_\nu$ via Eq. (6.40).

The measured functions $p_u(\Omega)$ can be used also for the reconstruction of various QPDs which represent the density matrix in the phase-space formulation. According to the general expression (4.1), the QPDs for the density matrix $\rho$ are given by the harmonic expansion

$$F_\rho(\Omega;s)\equiv P(\Omega;s)=\sum_\nu \tau_\nu^{-s/2}\mathcal{R}_\nu Y_\nu(\Omega). \tag{6.45}$$

Therefore one can just use the coefficients $\mathcal{R}_\nu$ calculated via Eq. (6.40). On the other hand, Eq. (4.6) can be used to obtain the relation between the QPDs $P(\Omega;s)$ and the measured functions $p_u(\Omega)$:

$$P(\Omega;s)=\int_X d\mu(\Omega')K_{us}^-(\Omega,\Omega')p_u(\Omega'), \tag{6.46}$$

$$p_u(\Omega)=\int_X d\mu(\Omega')K_{us}^+(\Omega,\Omega')P(\Omega';s), \tag{6.47}$$

where the transformation kernels are

$$K_{us}^\pm(\Omega,\Omega')=\sum_\nu [\kappa_\nu^{(u)}\tau_\nu^{s/2}]^{\pm 1}Y_\nu(\Omega)Y_\nu^*(\Omega'). \tag{6.48}$$

It can be easily shown that $K_{us}^+(\Omega,\Omega')=P_u(\Omega^{-1}\cdot\Omega';-s)$ where $P_u(\Omega;s)$ is the SW symbol of $|u\rangle\langle u|$, so Eq. (6.47) is consistent with the relation (6.25).

As was already mentioned, if the state $|u\rangle$ is the reference state $|\psi_0\rangle$ of the coherent-state basis, then $\Gamma_{\psi_0}(\Omega) = |\Omega\rangle\langle\Omega|$ and the measured function $p_{\psi_0}(\Omega)$ coincides with the function $Q_\rho(\Omega) = P(\Omega;-1)$. Comparing the harmonic expansions (6.34) and (6.45) for the case $|u\rangle = |\psi_0\rangle$, we find the following relation:

$$\kappa_\nu^{(\psi_0)} = \tau_\nu^{1/2}. \tag{6.49}$$

In this case we also obtain that the function $r_{\psi_0}(\Omega)$ of Eq. (6.42) is just the $P$ function,

$$r_{\psi_0}(\Omega) = P_\rho(\Omega) = P(\Omega;1). \tag{6.50}$$

Note that in the case of the Heisenberg-Weyl group one can also calculate the QPDs using the formula (6.8).

#### *5. Examples*

We see that the mathematical procedure of the reconstruction of the density matrix $\rho$ and its QPDs $P(\Omega;s)$ from the measured probability $p_u(\Omega)$ actually consists of the simple transformation (6.40). The mathematical tools one needs for this procedure are the harmonic functions $Y_\nu(\Omega)$ and the invariant coefficients $\tau_\nu$ and $\kappa_\nu^{(u)}$. In what follows we compute the explicit form of $\kappa_\nu^{(u)}$ for simple but instructive examples of the Heisenberg-Weyl group (with $|u\rangle$ being a Fock state) and the SU(2) group (with $|u\rangle$ being a $J_z$ eigenstate).

In the case of the Heisenberg-Weyl group, we consider the probability $p_n(\alpha)$ to find the displaced initial state in the Fock state $|n\rangle$ ($n = 0,1,2,\ldots$). Then Eq. (6.35) can be rewritten in the form

$$\kappa^{(n)}(\xi) Y(\xi,\alpha) = \langle n|D(-\alpha)D(\xi)D(\alpha)|n\rangle. \tag{6.51}$$

Using Eq. (5.9), we obtain

$$D(-\alpha)D(\xi)D(\alpha) = Y(\xi,\alpha)D(\xi). \tag{6.52}$$

Therefore the $\kappa$ coefficients are given by

$$\kappa^{(n)}(\xi) = \langle n|D(\xi)|n\rangle = \exp(-\tfrac{1}{2}|\xi|^2) L_n(|\xi|^2). \tag{6.53}$$

Of course, for $n = 0$ one gets $\kappa^{(0)}(\xi) = [\tau(\xi)]^{1/2}$.

In the case of the SU(2) group, we consider the probability $p_\mu(\theta,\phi)$ to find the displaced initial state in the $J_z$ eigenstate $|j,\mu\rangle$ ($\mu = j, j-1, \ldots, -j$). Then Eq. (6.35) takes the form

$$\kappa_{lm}^{(\mu)} Y_{lm}(\theta,\phi) = \sqrt{\frac{2j+1}{4\pi}} \langle j,\mu|g^{-1}(\theta,\phi) D_{lm} g(\theta,\phi)|j,\mu\rangle. \tag{6.54}$$

Using the parametrization (5.14) for $g(\theta,\phi)$ and the transformation rule (5.19), we can write

$$g^{-1}(\theta,\phi) D_{lm} g(\theta,\phi) = \sum_{m'=-l}^{l} \mathcal{D}_{m'm}^{(l)}(0,\theta,\phi) D_{lm'}. \tag{6.55}$$

Since the matrix element of the tensor operator,

$$\langle j,\mu|D_{lm'}|j,\mu\rangle = \sqrt{\frac{2l+1}{2j+1}} \langle j,\mu;l,m'|j,\mu\rangle, \tag{6.56}$$

vanishes unless $m' = 0$, Eq. (6.54) reads

$$\kappa_{lm}^{(\mu)} Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}_{0m}^{(l)}(0,\theta,\phi) \langle j,\mu;l,0|j,\mu\rangle.$$

Taking into account the fact that

$$\mathcal{D}_{0m}^{(l)}(\alpha,\beta,\gamma) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\beta,\gamma), \tag{6.57}$$

we finally obtain that the $\kappa$ coefficients are independent of the index $m$:

$$\kappa_l^{(\mu)} = \langle j,\mu;l,0|j,\mu\rangle. \tag{6.58}$$

For $\mu = j$, one finds $\kappa_l^{(j)} = \tau_l^{1/2}$. Indeed, according to the definition (5.15) of the SU(2) coherent states, the function $Q(\theta,\phi) = \langle j,\mathbf{n}|\rho|j,\mathbf{n}\rangle$ coincides with the probability $p_j(\theta,\phi)$ to find the displaced system in the highest spin state $|j,j\rangle$. It is not difficult to see that the probability $p_{-j}(\theta,\phi)$ to find the displaced system in the lowest spin state $|j,-j\rangle$ is equal to $Q(\theta+\pi,\phi)$. As an application of the relation (6.46), we also obtain the following expression for the SU(2) Wigner function in terms of the measured probability $p_\mu(\mathbf{n})$,

$$W(\mathbf{n}) = \sum_{l=0}^{2j} \frac{(4\pi)^{-1}(2l+1)}{\langle j,\mu;l,0|j,\mu\rangle} \int_X d\mathbf{n}' P_l(\mathbf{n}\cdot\mathbf{n}') p_\mu(\mathbf{n}'), \tag{6.59}$$

where $P_l(x)$ are the Legendre polynomials.

### D. Informational completeness and unsharp measurements

When the question of the state reconstruction arises, it is understood that the set of measurements one makes on an ensemble of identically prepared systems should give complete information about the quantum state. In particular, if one measures expectation values of some observables, it is natural to ask how many such observables are needed to characterize completely the state of the system. In this sense a set of observables, whose expectation values are sufficient to reconstruct the quantum state (or, equivalently, to distinguish between different states), can be considered as informationally complete. A formal definition is as follows [61]: A set of bounded operators $\mathfrak{A} = \{A\}$ on $\mathcal{H}$ is said to be informationally complete if for density matrices $\rho, \rho'$ the equality of expectation values,

$$\mathrm{Tr}(\rho A) = \mathrm{Tr}(\rho' A) \quad \forall A \in \mathfrak{A}, \tag{6.60}$$

implies $\rho = \rho'$.

The informational completeness of positive operator-valued measures covariant with respect to Heisenberg-Weyl, affine, and Galilei groups was recently discussed in Ref. [63]. This subject was shown [63] to be of importance not only in quantum mechanics but also in signal processing where a problem exists of extracting information from nonstationary signals and images. Another interesting feature is

that both in quantum mechanics and in signal processing the phase-space formulation is of great importance for approaching this kind of problems.

It would be interesting to analyze the results of the present paper from the point of view of informational completeness. First, it is evident from the expansion

$$\rho=\sum_{\nu}\ \mathrm{Tr}(\rho D_{\nu}^{\dagger})D_{\nu} \tag{6.61}$$

that the orthonormal set $\{D_{\nu}\}$ of the tensor operators is informationally complete. Correspondingly, the set $\{\Delta(\Omega;s)|\Omega\in X\}$ of the SW kernels for each $s$ is also informationally complete. This fact is reflected by the inverse Weyl rule written as

$$\rho=\int_{X}d\mu(\Omega)\mathrm{Tr}[\rho\Delta(\Omega;s)]\Delta(\Omega;-s). \tag{6.62}$$

In other words, the density matrix can be uniquely reconstructed from its $s$-parametrized QPD $P(\Omega;s)=\mathrm{Tr}\,[\rho\Delta(\Omega;s)]$. From the practical point of view, the important thing is the informational completeness of the set $\{\Gamma_{u}(\Omega)|\Omega\in X\}$ of the displaced projectors for any $|u\rangle\in\mathcal{H}$. This fact was formally proved in Ref. [63]. Here, we presented a simple algorithm (based on the method of harmonic expansion) for the reconstruction of the density matrix from the measurable probabilities $p_{u}(\Omega)=\mathrm{Tr}[\rho\Gamma_{u}(\Omega)]$. This reconstruction procedure clearly implies the informational completeness of the set $\{\Gamma_{u}(\Omega)|\Omega\in X\}$.

One of the useful features of the method of displaced projectors is the ability to take into account in a simple way the unsharpness of a realistic measurement. Of course, it is impossible in practice to make a completely accurate displacement to a specified point $\Omega$ on the phase space. For example, one should take into account the phase and intensity fluctuations of a classical microwave source that displaces the quantum state of the radiation field in a cavity, or instabilities of a classical driving field that displaces the motional state of a trapped ion. Similarly, the so-called coarse graining problem arises in radar analysis due to frequency instabilities of the test signal or uncertainties in timing of signal initiation. As a result, the probabilities $p_{u}(\Omega)$ should be integrated over the variation range. This yields the expectation value of the localization operator defined by [63]

$$Z_{u}(f)=\int_{X}d\mu(\Omega)\ f(\Omega)\Gamma_{u}(\Omega), \tag{6.63}$$

where $f(\Omega)$ is a localization function. $Z_{u}(f)$ has a purely discrete spectrum, is bounded when $f\in L^{k}(X,\mu)$, $k\geq 1$, and is self-adjoint when $f$ is real. In particular, let $B$ be a region (more specifically, a Borel set) in $X$, with the characteristic function $\chi_{B}(\Omega)$ that equals 1 for $\Omega\in B$ and 0 otherwise. Taking $f(\Omega)=\chi_{B}(\Omega)$, one obtains

$$Z_{u}(B)=\int_{B}d\mu(\Omega)\Gamma_{u}(\Omega). \tag{6.64}$$

It is not difficult to see that the informational completeness of the set $\{\Gamma_{u}(\Omega)|\Omega\in X\}$ of displaced projectors implies the informational completeness of the set $\{Z_{u}(B)|B\in\text{Borel sets of }X\}$ of localization operators. Therefore, in the case of realistic unsharp measurements, the localization operators may be conveniently used for analysis and reconstruction of quantum states or electronic signals and images.

## VII. CONCLUSIONS

In the present paper we propose a simple algorithm for constructing the SW kernels which implement the linear bijective mapping between Hilbert-space operators and phase-space functions for physical systems possessing Lie-group symmetries. The constructed kernels are manifestly covariant under the action of the corresponding dynamical symmetry group and satisfy the tracing condition which ensures that quantum expectations can be represented by statistical-like averages over the phase space. Adding the noncommutative twisted product that equips phase-space functions with the algebraic structure of quantum operators, an autonomous phase-space formulation of quantum mechanics is developed.

It turns out that the concept of phase space naturally emerges in the majority of schemes proposed for the reconstruction of quantum states as well as in the standard methods of signal analysis. In particular, we focus on the method based on measurements of displaced projectors and develop its general group-theoretic description. We do so by applying the same technique of harmonic expansions on the phase space that was used for the construction of the SW kernels. The problem of the state reconstruction is also approached using the concept of informational completeness, and the role of localization operators in describing realistic measurements is discussed.

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