COMMENTS

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Comment on "Arrival time in quantum mechanics" and "Time of arrival in quantum mechanics"

Jerzy Kijowski
Centrum Fizyki Teoretycznej, PAN, Aleja Lotników 32/46, 02-668 Warsaw, Poland

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Contrary to claims contained in papers by Grot, Rovelli, and Tate [Phys. Rev. A **54**, 4676 1996)] and Delgado and Muga [Phys. Rev. A **56**, 3425 (1997)], the "time operator," which I have constructed [Rep. Math. Phys. **6**, 361 (1974)] in an axiomatic way, is a self-adjoint operator existing in a *usual Hilbert space* of (nonrelativistic or relativistic) quantum mechanics. [S1050-2947(99)06901-2]

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Earlier I solved the following problem in *standard* quantum mechanics (see [1]). Consider a two-dimensional plane in physical space (e.g., a plane $P_\zeta:=\{z=\zeta\}$, where ζ is a fixed constant) and its measurable subsets (call them windows). Given a window $D\subset P_\zeta$ and the time interval $I:=[t_1,t_2]$, is it possible to define *consistently* the probability $Q_{I\times D}(\psi)$ that a freely moving particle described by a quantum state ψ crosses the window D within the time interval I?

By *consistency* I meant a set of obvious axioms. Some of them were implied by the structure of quantum mechanics: (i) the probability should be given as a 3/2 linear form of the state ψ $Q_{I\times D}=\langle\psi|\hat{q}_{I\times D}|\psi\rangle$, where the operator $\hat{q}_{I\times D}$ must be a projector; (ii) probabilities should sum up for disjointed three-dimensional regions $I\times D$ (we may call them space-time windows); and (iii) probabilities must be normalized. Other axioms were implied by the Galilei invariance of the nonrelativistic quantum mechanics and by the Poincaré invariance in the relativistic case (both cases were considered). It was proved that these axioms define *uniquely* an operator $\hat{t}_\zeta$ that, together with the position operators $\hat{x}$ and $\hat{y}$, give the projector $\hat{q}_{I\times D}$ as an integral of their common spectral measure $d\mathcal{E}(t,x,y)$ over the space-time window $I\times D$.

In a recent publication [2] Grot, Rovelli, and Tate wrote "Kijowski obtained a probability distribution, but not on the usual Hilbert space: thus the interpretation of the wave function in terms of familiar quantities is obscure." Moreover, in [3] Delgado and Muga wrote the following. "Our result turns out to be similar to those previously obtained by Kijowski. However, the approach by Kijowski was based on the definition of a non-conventional wave function which evolves on a family of $x=$const planes (instead of evolving in time according to the Schrödinger equation), and whose relation to the conventional wave function is unclear"

These comments are incorrect. I stress that my construction and the uniqueness proof were performed in the framework of *absolutely conventional* quantum mechanics. The time operator $\hat{t}_\zeta$ on the plane P_ζ was *uniquely obtained* from the axioms.

The main technical device that I have used to simplify the mathematical aspect of the theory was a new representation of wave functions that is little known, although it is perfectly *equivalent* to both the position and the momentum representations. This new representation was obtained from the wave function $\tilde{\psi}(p_x,p_y,p_z)$ in the momentum representation [the Fourier transform of the wave function $\psi(x,y,z)$ in the position representation]. The new representation is obtained by replacing the variable p_z by the signed energy variable

$$s:=E_z\,\mathrm{sgn}(p_z), \tag{1}$$

where E_z is the amount of energy carried by the zth degree of freedom (in the nonrelativistic case it is simply $E_z=(1/2m)p_z^2$; in the relativistic case it is equal to the difference between the actual energy and the energy corresponding to $p_z=0$). The symbol "sgn" stands for "the sign of" and enables us to distinguish between the "right movers" and the "left movers" carrying the same energy. This way, any quantum state may be represented by a square-integrable function of the three variables

$$\tilde{\varphi}(s,p_x,p_y):=\sqrt{\frac{m}{|p_z(s,p_x,p_y)|}}\tilde{\psi}(p_x,p_y,p_z(s,p_x,p_y)) \tag{2}$$

(the square root factor arises because, geometrically, the wave function is a half density and must follow the corresponding transformation law when passing to a new coordinate system). The transformation from the square-integrable functions ψ to the square-integrable functions $\tilde{\varphi}$ is unitary:

$$||\psi||^2=\int|\psi|^2d^3x=\int|\tilde{\psi}|^2d^3p$$
$$=\int|\tilde{\varphi}(s,p_x,p_y)|^2ds\,dp_xdp_y. \tag{3}$$

In this representation the formula for the time operator is the simplest possible: For $\zeta=0$ it turns out to be a momentum canonically conjugate to the parameter s,

$$\hat{t}_0\tilde{\varphi}=-i\hbar\frac{\partial}{\partial s}\tilde{\varphi}, \tag{4}$$

(see p. 373 in [1]), and for other values of ζ it may be obtained from the above operator by a standard translation in the direction of the zth axis. [Actually, I used in [1] the parameter s defined in a slightly different way, namely, $s := E\,\mathrm{sgn}(p_z)$. With respect to Eq. (1), the complete energy E replaces here the quantity E_z. Because both definitions of s differ only by a constant depending on p_x and p_y, formula (4) gives the same result in both descriptions. In the present paper I have chosen the variable s defined by formula (1) because its range is equal to the real axis, without any ''hole'' in the middle, and the entire representation looks more similar, e.g., to the standard formula for the position operator in the momentum representation.]

Still another representation of the quantum state is very useful because it gives the common eigenvector expansion of the three commuting operators $(\hat{t}_\zeta,\hat{x},\hat{y})$. This new representation uses the inverse Fourier transform $\varphi(t,x,y)$ of the wave functions $\tilde{\varphi}(s,p_x,p_y)$. Again, the transformation from ψ to the space of square-integrable wave functions φ is unitary and the probability in question is simply given by the integral over the space-time window

$$Q_{I\times D}(\psi)=\int_{I\times D}|\varphi(t,x,y)|^2 dt\,dx\,dy. \tag{5}$$

The importance of this representation consists in the fact that it gives the generalized eigenfunction expansion of the quantum state ψ with respect to the operator $\hat{t}_\zeta$. Indeed, its eigenfunctions are simply Dirac δ functions in the variable t.

The transition from a plane P_ζ to another plane $P_{\zeta'}$ was also studied within this representation. There is absolutely nothing nonconventional in the fact that such transition operators form a group and the generator of this group is nothing but the momentum operator $\hat{p}_z$. Hence such a transition may be formulated as an ''evolution on a family of $z=\mathrm{const}$ planes'' (as mentioned by Delgado and Muga) with $\hat{p}_z$ being the Hamiltonian of such an evolution. Here the only nonconventional aspect was the use of the above φ representation. To obtain the explicit formula for $\hat{p}_z$ in this representation, we had to express it in terms of momenta $(\hat{s},\hat{p}_x,\hat{p}_y)$, canonically conjugate to the positions (t,x,y). I am afraid that the nonconventionality of my paper, claimed by Delgado and Muga, was based on a misunderstanding of this simple fact.

On the other hand, any quantum state (whether described in my representation as a function φ or in the position representation as the *conventional* wave function ψ) undergoes the *standard* ''chronological evolution'' from time t_1 to t_2, described by the Schrödinger equation. There is absolutely no contradiction between these two different evolutions. They are used to answer *different* physical questions.

In addition, formula (4) may be recalculated from one representation to another (e.g., to the momentum representation or to the position representation). This immediately implies the following formula, relating the above time operator with the position operator $\hat{z}$ and the momentum operator $\hat{p}_z$:

$$\hat{t}_\zeta\psi=-\,\mathrm{sgn}(p_z)m\frac{1}{2}\{(\hat{z}-\zeta)(\hat{p}_z)^{-1}+(\hat{p}_z)^{-1}(\hat{z}-\zeta)\}\psi, \tag{6}$$

valid for sufficiently regular wave functions ψ (such that all the symbols used have an unambiguous meaning). For a beam prepared in such a way that it contains right movers exclusively (i.e., $p_z>0$) this formula may be considered as an analog of the corresponding classical formula for the arrival time defined by the plane P_ζ, expressed in terms of classical observables z and p_z:

$$t_\zeta=-m\frac{z-\zeta}{p_z}. \tag{7}$$

[In formula (6) we obtain the *symmetric* order for the product of noncommuting operators.] For a beam containing left movers exclusively, the arrival time arises here with an opposite sign. Formula (6) was not given in [1], but it is a one-line consequence of Eq. (4). The reason that I do not like such a formula is that it is mathematically ''dangerous'': Without specifying precisely the domain of the operator it is *a priori* not even a self-adjoint operator.

Reference [4] proves indeed that one must be careful in using such formulas: Giannitrapani considers the symmetric version of the classical arrival time (7), i.e., an operator defined in such a way that the sign of p_z in front of Eq. (6) *has been deleted* (cf. also [5]). Using rather complicated arguments he proves that this operator is not self-adjoint. This is not surprising: In our $\tilde{\varphi}(s,p_x,p_y)$ representation this operator is equal to

$$\hat{T}:=-i\,\mathrm{sgn}(s)\hbar\frac{\partial}{\partial s} \tag{8}$$

and one sees immediately that such an operator has *no* self-adjoint extension because its deficiency indices are different (no arguments based on the Pauli theorem are necessary). On the other hand, the axiomatic approach proposed in [1] leads uniquely to a *perfectly self-adjoint* operator (4) (the same techniques were later used in [6] in order to prove the uniqueness of the Newton-Wigner position operator in relativistic quantum mechanics).

Both Refs. [2] and [3] confirm the main theorem proved in [1]: The above construction *is unique*. Indeed, formulas (26)–(33) and (66) of [3] are identical to the definitions proposed in [1]. On the other hand, Ref. [2] constructs an approximation (in a certain sense) of the ''classical'' formula by self-adjoint operators. Again, our $\tilde{\varphi}$ representation simplifies this construction considerably: It is sufficient to *smooth out* the singular vector field (8) within the interval $s\in[-\epsilon,\epsilon]$ and define the operator $\hat{T}_\epsilon$ as a Lie derivative of $\tilde{\varphi}$ with respect to this smooth field. Keeping in mind the fact, that the wave function is *not* a scalar but a half density, we immediately obtain formula (37) of [2].

Although, there is no room to modify the mathematically unique definition of the probability $Q_{I\times D}(\psi)$ within the standard mathematical framework of quantum mechanics (spectral measures, self-adjoint operators, etc.), the problem is still open from the physical point of view. Indeed, the probabilistic interpretation of quantum mechanics is based on short measurements of the type, What is the probability that the particle will be found within the three-dimensional volume V *precisely* at a given time instant t? How to relate them to long measurements, corresponding to time intervals I rather than to time instants t, is by no means obvious. From this point of view the result of Refs. [2–4] are very interesting.

I want to stress that the classification ''nonconventional wave function... whose relation to the conventional wave function is unclear'' could only be conceived by somebody who did not read my paper carefully [1]. Indeed, the composition of the paper is following. In Sec. 1 the Heisenberg energy-time uncertainty principle is discussed. In Sec. 2 I propose to understand the quantity Δt appearing in this principle as the ''average deviation of the time of passing through the plane Q'' (or ''arrival time'' in modern language; see page 363). Moreover, I analyze the physical origin of this uncertainty. Section 3 contains the proof that the three naive ways of defining the above arrival time in *conventional* wave mechanics are incorrect. In Sec. 4 I show how the arrival time can be defined axiomatically in classical statistical mechanics. In Sec. 5 I prove that the same set of axioms may be taken as the definition of the arrival time in *conventional quantum mechanics*. At this point the time operator *is already defined* without any nonconventional tools. The remaining part of the paper is devoted to the analysis of the properties of the quantum mechanical observable $\hat{t}$ defined this way. In Sec. 6 I prove that it possesses the correct classical limit. Finally, in Sec. 7 I show how to diagonalize it (or to find its complete set of eigenvectors). For this technical purpose the nonconventional representations $\tilde{\varphi}$ and φ are introduced. There is no doubt that these are new representations of a *conventional wave function*: In the first formula of this section (p. 370) I define the wave function in this new representation starting from the wave function in the conventional momentum representation. As a conclusion I claim that the complete construction of my arrival time in the context of the absolutely conventional quantum mechanics is contained in the first six sections of Ref. [1].

In the remaining part of the paper I also analyze the relation between arrival times corresponding to different planes (different values of the variable ζ). Delgado and Muga [3] do not like this analysis and remark that the wave function in the φ representation ''evolves on a family of $x=$const planes instead of evolving in time according to the Schrödinger equation.'' I stress, however, that this nonconventional evolution does not contradict the Schrödinger evolution because both evolutions describe different aspects of quantum mechanics.

[1] J. Kijowski, Rep. Math. Phys. **6**, 361 (1974).

[2] N. Grot, C. Rovelli, and R. S. Tate, Phys. Rev. A **54**, 4676 (1996).

[3] V. Delgado and J. G. Muga, Phys. Rev. A **56**, 3425 (1997).

[4] R. Giannitrapani, Int. J. Theor. Phys. **36**, 1601 (1997).

[5] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961); G. R. Allcock, Ann. Phys. (N.Y.) **53**, 253 (1969); **53**, 286 (1969); **53**, 311 (1969).

[6] J. Kijowski and G. Rudolph, Bull. Acad. Pol. Sci., Ser. Sci., Math., Astron. Phys. **24**, 1041 (1976).