Invariant theory and exact solutions for the quantum Dirac field in a time-dependent spatially homogeneous electric field

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On the basis of the generalized invariant formulation, the invariant-related unitary transformation method is used to study the evolution of the quantum Dirac field in a time-dependent spatially homogeneous electric field. We solve the functional Schrödinger equation for the Dirac field and obtain the exact solutions and corresponding total phase. The total phase includes both the dynamical phase and geometric phase (Aharonov-Anandan phase). [S1050-2947(98)07512-X]

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I. INTRODUCTION

Quantum invariant theory was proposed by Lewis and Riesenfeld in Ref. [1]. This theory is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in Ref. [2] by introducing the concept of basic invariants and used to study the geometric phases [3,4] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. Then, it became more and more recognized that there are actually nothing but different names and attributes given to various parts of the total phase [5] as long as the exact solution of the time-dependent Schrödinger equation with a time-dependent Hamiltonian is concerned. The invariant theory in Refs. [1, 2] for obtaining the exact solutions for systems with time-dependent Hamiltonians is closely related to the study of the total phase (including dynamical phase and geometrical phase); it may then be referred to as the phase formulation. The introduction of the concept of the basic invariants into the invariant theory [2] makes it possible to find a complete set of commuting invariants and generalized time-dependent creation and annihilation operators for some of the time-dependent infinite-dimensional quantum systems—time-dependent quantum fields [6–8]. By using phase formulation and the newly developed invariant-related unitary transformation method [8], the exact solutions and associated phases have been obtained for the charged Klein-Gordon field in a time-dependent spatially homogeneous electric field [7,8]. However, to our knowledge, no work on the quantum Dirac field with a time-dependent Hamiltonian has been seen in the literature. In the present paper, we use the phase formulation and the invariant-related unitary transformation method to obtain the exact solutions and associated phases (including dynamical phases and geometric phases) for the quantum Dirac field in a time-dependent spatially homogeneous electric field.

The present paper is organized as follows. In Sec. II, the phase formulation and the invariant-related unitary transformation method is briefly reviewed. In Sec. III, we calculate the exact solutions and corresponding phases for the Dirac field in a time-dependent spatially homogeneous electric field. In Sec. IV, there are some concluding remarks. Finally, in the Appendix, we present the auxiliary equations which are much more complicated than that obtained in Ref. [7] for scalar field and can only be solved numerically.

II. THE INVARIANT-RELATED UNITARY TRANSFORMATION METHOD

We first outline the Lewis-Riesenfeld invariant theory. Consider a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time dependent. A Hermitian operator $\hat{I}(t)$ is called invariant if it satisfies (using units in which $\hbar = c = 1$)

$$\frac{\partial \hat{I}(t)}{\partial t} - i[\hat{I}(t), \hat{H}(t)] = 0. \tag{2.1}$$

The eigenvalue equation of $\hat{I}(t)$ can be written as

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle,$$
$$\frac{\partial \lambda_n}{\partial t} = 0 \tag{2.2}$$

and the time-dependent Schrödinger equation for the system is

$$i\,\frac{\partial |\Psi(t)\rangle_S}{\partial t} = \hat{H}(t)|\Psi(t)\rangle_S. \tag{2.3}$$

According to Lewis-Riesenfeld quantum-invariant theory [1], the particular solution $|\lambda_n, t\rangle_S$ of Eq. (2.3) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\varphi_n(t)]$; that is,

$$|\lambda_n, t\rangle_S = \exp[i\varphi_n(t)]|\lambda_n, t\rangle. \tag{2.4}$$

Then the general solution of the Schrödinger equation (2.3) can be shown to be

$$|\Psi(t)\rangle_S = \sum_n C_n \exp[i\varphi_n(t)]|\lambda_n, t\rangle, \tag{2.5}$$

where $\varphi_n(t)=\int_0^t\langle\lambda_n,t'|i\partial/\partial t'-\hat{H}(t')|\lambda_n,t'\rangle dt'$, $C_n=\langle\lambda_n,0|\Psi(0)\rangle_S$, $|\lambda_n,t\rangle_S$ $(n=1,2,\ldots)$ are said to form a complete set of the solutions of Eq. (2.3). Note that, in general, $\hat{I}(t)$ is not unique and the complete set changes as the choice of $\hat{I}(t)$ changes.

In Ref. [2], we proposed the generalized invariant theory (the phase formulation) and established the following facts. (i) The formal solution of Eq. (2.1) is $\hat{I}(t)=\hat{U}(t)\hat{I}(0)\hat{U}^+(t)$, where $\hat{U}(t)=P\exp[-i\int_0^t\hat{H}(t')dt']$ is the time-evolution operator for the system and $I(0)$ can be arbitrarily chosen, so that $\hat{I}(t)$ may be Hermitian or non-Hermitian. (ii) There are two basic invariants, $\hat{x}(t)=\hat{U}(t)\hat{x}\hat{U}^+(t)$, $\hat{p}(t)=\hat{U}(t)\hat{p}\hat{U}^+(t)$; any invariant $\hat{I}(t)=\hat{U}(t)\hat{I}(0)\hat{U}^+(t)$ can be expressed as a power series in $\hat{x}(t)$ and $\hat{p}(t)$ as long as $\hat{I}(0)$ can be expressed as a power series in $\hat{x}$ and $\hat{p}$. (iii) In some cases, a chosen non-Hermitian invariant can act as a solution generator, with which one can generate a complete set of solutions of the time-dependent Schrödinger equation (2.3) from one solution of it. (iv) The concept can be generalized to find a complete set of invariants and set up an invariant formulation (representation) for the study of more than one-dimensional time-dependent systems (including infinite-dimensional quantum systems or quantum fields [6–8]).

Now we begin to briefly review the invariant-related unitary transformation method on the basis of the phase formulation. In some cases of physical interest, it is possible to construct a time-dependent unitary transformation $\hat{V}(t)$ for a chosen invariant $\hat{I}(t)$ such that (i) $\hat{I}_0\equiv\hat{V}^+(t)\hat{I}(t)\hat{V}(t)$ is a time-independent operator with

$$\hat{I}_0|\lambda_n\rangle=\lambda_n|\lambda_n\rangle,$$

$$|\lambda_n\rangle=\hat{V}^{-1}(t)|\lambda_n,t\rangle, \tag{2.6}$$

and where the eigenvalue λ_n is the same as that in Eq. (2.2), (ii) the Hamiltonian $\hat{H}(t)$ is changed into $\hat{H}_0(t)$:

$$\hat{H}_0(t)=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}. \tag{2.7}$$

This unitary transformation is easily shown to guarantee that the particular solution $|\lambda_n,t\rangle_{S0}$ of the time-dependent Schrödinger equation [associated with $\hat{H}_0(t)$]

$$i\frac{\partial|\lambda_n,t\rangle_{S0}}{\partial t}=\hat{H}_0(t)|\lambda_n,t\rangle_{S0} \tag{2.8}$$

is different from the eigenfunction $|\lambda_n\rangle$ of $\hat{I}_0$ only by the same phase factor $\exp[i\varphi_n(t)]$ as that in Eq. (2.4):

$$|\lambda_n,t\rangle_{S0}=\exp[i\varphi_n(t)]|\lambda_n\rangle. \tag{2.9}$$

Substitution of $|\lambda_n,t\rangle_{S0}$ in Eq. (2.9) into Eq. (2.8) yields

$$-\dot{\varphi}_n|\lambda_n\rangle=\hat{H}_0(t)|\lambda_n\rangle \tag{2.10}$$

which means that $\hat{H}_0(t)$ differs from $\hat{I}_0$ only by a multiplying c-number factor, depending on the time t. Thus, one is led to the conclusion that if the unitary transformation $\hat{V}(t),\hat{I}_0,\hat{H}_0$, and the eigenfunction $|\lambda_n\rangle$ of $\hat{I}_0$ have been found, the problem of solving the complicated time-dependent Schrödinger equation (2.3) reduces to that of solving the much simplified equation (2.8) since it can be seen from Eq. (2.9) that the solution of Eq. (2.8) can be easily obtained by calculating the phase from Eq. (2.10).

It is worthwhile to emphasize that (a) the above used term ‘‘a chosen invariant’’ implies that the choice of the invariant $\hat{I}(t)$ is not unique and it is usually appropriate to choose $\hat{I}(t)=\hat{U}\hat{I}(0)\hat{U}^+$ as the system is initially in an eigenstate of an operator $I(0)$ [$I(0)$ may be $\hat{H}(t=0)$] and (b) one chosen invariant $\hat{I}(t)$ leads to one definite complete set of the solutions $|\lambda_n,t\rangle$ of Eq. (2.3) regardless of the fact that the unitary transformation $\hat{V}$, which is required to make $\hat{V}^+\hat{I}\hat{V}$ time independent, is only determined up to a time-independent unitary transformation.

III. EXACT SOLUTIONS OF THE TIME-DEPENDENT SCHRÖDINGER EQUATION AND EVOLUTION OF THE QUANTUM DIRAC FIELD IN A TIME-DEPENDENT SPATIALLY HOMOGENEOUS ELECTRIC FIELD

In this section, we study the Dirac field in a time-dependent spatially homogeneous electric field. The Lagrangian density for the Dirac field is [9]

$$L(x)=\overline{\psi}(x)[i\nabla\!\!\!/-eA\!\!\!/-m]\psi(x), \tag{3.1}$$

where

$$\overline{\psi}(x)=\psi^+(x)\gamma_0$$

$$\nabla\!\!\!/=\gamma^\mu\partial_\mu,\quad A\!\!\!/=\gamma^\mu A_\mu$$

$$\gamma^0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \gamma^i=\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$

with σ^i being the Pauli spin matrices. By making use of Eq. (3.1), it is easy to obtain the canonical momentum density π_α in the Weyl gauge $A^0=0$

$$\pi_\alpha=\frac{\partial L}{\partial\dot{\psi}_\alpha}=i\psi^\dagger_\alpha. \tag{3.2}$$

To quantize this field, equal-time anticommutation relations are introduced among the operators $\hat{\pi}$ and $\hat{\psi}$

$$[\hat{\psi}_\alpha(\vec{x},t),\hat{\pi}_\beta(\vec{x}',t)]_+=i\delta_{\alpha\beta}\delta(\vec{x}-\vec{x}'),$$

$$[\hat{\psi}_\alpha(\vec{x},t),\hat{\psi}_\beta(\vec{x}',t)]_+=0,[\hat{\pi}_\alpha(\vec{x},t),\hat{\pi}_\beta(\vec{x}',t)]_+=0. \tag{3.3}$$

We choose to work within the functional Schrödinger picture [6–8,10,11]. From Eqs. (3.1) and (3.2), we get the quantum time-dependent Hamiltonian for the system in the Schrödinger picture

$$\hat{H}(t)=\int\left\{\hat{\pi}(\vec{x},0)\left[-\alpha_i\left(\frac{\partial}{\partial x^i}-eA^i(t)\right)-i\beta m\right]\hat{\psi}(\vec{x},0)\right\}d^3\vec{x}$$
$$=\int\left\{\hat{\psi}^+(\vec{x},0)\left[-i\alpha_i\left(\frac{\partial}{\partial x^i}+ieA^i(t)\right)+\beta m\right]\hat{\psi}(\vec{x},0)\right\}d^3\vec{x}, \tag{3.4}$$

where

$$\alpha_i=\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix},$$
$$\beta=\gamma_0 .$$

Because the spatial sections are flat, we can employ the ''momentum representation'' for the operators [11]

$$\hat{\psi}(\vec{x},0)=\sum_{\pm s}\int\frac{d^3\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,[\hat{b}_s(\vec{p})u_s(\vec{p})e^{i\vec{p}\cdot\vec{x}}+\hat{d}_s^\dagger(\vec{p})v_s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}],$$
$$\hat{\psi}^\dagger(\vec{x},0)=\sum_{\pm s}\int\frac{d^3\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\,[\hat{b}_s^\dagger(\vec{p})\overline{u_s}(\vec{p})\gamma_0e^{-i\vec{p}\cdot\vec{x}}+\hat{d}_s(\vec{p})\overline{v_s}(\vec{p})\gamma_0e^{i\vec{p}\cdot\vec{x}}], \tag{3.5}$$

where $E_p=\sqrt{|\vec{p}|^2+m^2}$, $u_s(\vec{p}),v_s(\vec{p}),\overline{u_s}(\vec{p}),\overline{v_s}(\vec{p})$ are the spinors defined in Ref. [9] $\hat{b}_s^\dagger(\vec{p})(\hat{b}_s(\vec{p}))$ is the creation (annihilation) operator of an ''electron'' with momentum $\vec{p}$ and spin s, $\hat{d}_s^\dagger(\vec{p})(\hat{d}_s(\vec{p}))$ the creation (annihilation) operator of a ''positron'' with $\vec{p}$ and s. It follows from Eqs. (3.3) that the creation and annihilation operators satisfy the anticommutation relations

$$[\hat{b}_s(\vec{p}),\hat{b}_{s'}^\dagger(\vec{p}\,')]_+=\delta_{ss'}\delta(\vec{p}-\vec{p}\,'),$$
$$[\hat{d}_s(\vec{p}),\hat{d}_{s'}^\dagger(\vec{p}\,')]_+=\delta_{ss'}\delta(\vec{p}-\vec{p}\,'),$$
$$[\hat{b}_s(\vec{p}),\hat{b}_{s'}(\vec{p}\,')]_\dagger=[\hat{d}_s(\vec{p}),\hat{d}_{s'}(\vec{p}\,')]_+=[\hat{b}_s^\dagger(\vec{p}),\hat{b}_{s'}^\dagger(\vec{p}\,')]_+=[\hat{d}_s^\dagger(\vec{p}),\hat{d}_{s'}^\dagger(\vec{p}\,')]_+=0,$$
$$[\hat{b}_s(\vec{p}),\hat{d}_{s'}^\dagger(\vec{p}\,')]_+=[\hat{b}_s^\dagger(\vec{p}),\hat{d}_{s'}^\dagger(\vec{p}\,')]_+=[\hat{b}_s(\vec{p}),\hat{d}_{s'}(\vec{p}\,')]_+=[\hat{b}_s^\dagger(\vec{p}),\hat{d}_{s'}(\vec{p}\,')]_+=0. \tag{3.6}$$

Inserting Eq. (3.5) in Eq. (3.4), we get

$$\hat{H}(t)=\int d^3\vec{p}\left\{\sum_{\pm s}[E(t)\hat{b}_s^\dagger\hat{b}_s-E(t)\hat{d}_s\hat{d}_s^\dagger]+\lambda_1(t)\hat{b}_{+s}^\dagger\hat{d}_{+s}^\dagger+\lambda_2(t)\hat{b}_{-s}^\dagger\hat{d}_{-s}^\dagger+\lambda_3(t)\hat{b}_{+s}^\dagger\hat{d}_{-s}^\dagger+\lambda_4(t)\hat{b}_{-s}^\dagger\hat{d}_{+s}^\dagger+\lambda_1^*(t)\hat{d}_{+s}\hat{b}_{+s}+\lambda_2^*(t)\hat{d}_{-s}\hat{b}_{-s}+\lambda_3^*(t)\hat{d}_{-s}\hat{b}_{+s}+\lambda_4^*(t)\hat{d}_{+s}\hat{b}_{-s}\right\}, \tag{3.7}$$

where

$$E(t)=E_p-\frac{e}{E_p}[p_xA_x(t)+p_yA_y(t)+p_zA_z(t)],$$
$$\lambda_1(t)=eA_z(t)+\frac{e}{(E_p+m)E_p}[p_yp_zA_y(t)+p_xp_zA_x(t)-(p_x^2+p_y^2)A_z(t)],$$
$$\lambda_3(t)=eA_x(t)+\frac{e}{(E_p+m)E_p}[p_xp_yA_y(t)+p_xp_zA_z(t)-(p_z^2+p_y^2)A_x(t)]-ie\left\{A_y(t)+\frac{1}{(E_p+m)E_p}\times[p_xp_yA_x(t)+p_yp_zA_z(t)-(p_x^2+p_y^2)A_y(t)]\right\},$$
$$\lambda_2(t)=-\lambda_1(t),\quad \lambda_4(t)=\lambda_3^*(t). \tag{3.8}$$

The time-dependent functional Schrödinger equation for the system is

$$i\frac{\partial}{\partial t}\Psi[\psi(\vec{p});t]=\hat{H}(t)\Psi[\psi(\vec{p});t]. \tag{3.9}$$

It is easy to show that there exists an invariant $\hat{I}(t)$ satisfying

$$\frac{\partial\hat{I}(t)}{\partial t}-i[\hat{I}(t),\hat{H}(t)]=0 \tag{3.10}$$

which is found to be

$$\hat{I}(t)=\sum_{\pm s}\int d^3\vec{p}[\hat{B}_s^\dagger(\vec{p},t)\hat{B}_s(\vec{p},t)-\hat{D}_s(\vec{p},t)\hat{D}_s^\dagger(\vec{p},t)], \tag{3.11}$$

where

$$\hat{B}_{+s}(\vec{p},t)=[\cos\theta_1\cos\theta_2\cos\theta_6+\sin\theta_1\sin\theta_3\sin\theta_6e^{-i(\phi_1+\phi_3+\phi_6)}]\hat{b}_{+s}(\vec{p})+[\cos\theta_1\sin\theta_2\sin\theta_5e^{i(\phi_2+\phi_5)}-\sin\theta_1\cos\theta_3\cos\theta_5e^{-i\phi_1}]\hat{b}_{-s}(\vec{p})+[\cos\theta_1\sin\theta_2\cos\theta_5e^{i\phi_2}+\sin\theta_1\cos\theta_3\sin\theta_5e^{-i(\phi_1+\phi_5)}]\hat{d}_{+s}^\dagger(\vec{p})-[\cos\theta_1\cos\theta_2\sin\theta_6e^{-i\phi_6}-\sin\theta_1\sin\theta_3\cos\theta_6e^{-i(\phi_1+\phi_3)}]\hat{d}_{-s}^\dagger(\vec{p}), \tag{3.12a}$$

$$\hat{B}_{-s}(\vec{p},t)=-[\cos\theta_1\sin\theta_3\sin\theta_6 e^{-i(\phi_3+\phi_6)}-\sin\theta_1\cos\theta_2\cos\theta_6 e^{i\phi_1}]\hat{b}_{+s}(\vec{p})+[\cos\theta_1\cos\theta_3\cos\theta_5 +\sin\theta_1\cos\theta_2\cos\theta_5 e^{i(\phi_1+\phi_2+\phi_5)}]\hat{b}_{-s}(\vec{p})-[\cos\theta_1\cos\theta_3\sin\theta_5 e^{-i\phi_5}-\sin\theta_1\sin\theta_2\cos\theta_5 e^{i(\phi_1+\phi_2)}]\hat{d}^{\dagger}_{+s}(\vec{p}) -[\cos\theta_1\sin\theta_3\cos\theta_6 e^{-i\phi_3}+\sin\theta_1\cos\theta_2\sin\theta_6 e^{i(\phi_1-\phi_6)}]\hat{d}^{\dagger}_{-s}(\vec{p}), \quad (3.12b)$$

$$\hat{D}^{\dagger}_{-s}(\vec{p},t)=[\cos\theta_4\cos\theta_3\sin\theta_6 e^{i\phi_6}+\sin\theta_4\sin\theta_2\cos\theta_6 e^{-i(\phi_2+\phi_4)}]\hat{b}_{+s}(\vec{p})+[\cos\theta_4\sin\theta_3\cos\theta_5 e^{i\phi_3} -\sin\theta_4\cos\theta_2\sin\theta_5 e^{-i(\phi_4-\phi_5)}]\hat{b}_{-s}(\vec{p})-[\cos\theta_4\sin\theta_3\sin\theta_5 e^{i(\phi_3-\phi_5)}+\sin\theta_4\cos\theta_2\cos\theta_5 e^{-i\phi_4}]\hat{d}^{\dagger}_{+s}(\vec{p}) -[\cos\theta_4\cos\theta_3\cos\theta_6-\sin\theta_4\sin\theta_2\sin\theta_6 e^{-i(\phi_2+\phi_4+\phi_6)}]\hat{d}^{\dagger}_{-s}(\vec{p}), \quad (3.12c)$$

$$\hat{D}^{\dagger}_{+s}(\vec{p},t)=-[\cos\theta_4\sin\theta_2\cos\theta_6 e^{-i\phi_2}-\sin\theta_4\cos\theta_3\sin\theta_6 e^{i(\phi_4+\phi_6)}]\hat{b}_{+s}(\vec{p})+[\cos\theta_4\cos\theta_2\sin\theta_5 e^{i\phi_5} +\sin\theta_4\sin\theta_3\cos\theta_5 e^{i(\phi_3+\phi_4)}]\hat{b}_{-s}(\vec{p})+[\cos\theta_4\cos\theta_2\cos\theta_5-\sin\theta_4\sin\theta_3\sin\theta_5 e^{i(\phi_3+\phi_4-\phi_5)}]\hat{d}^{\dagger}_{+s}(\vec{p}) +[\cos\theta_4\sin\theta_2\sin\theta_6 e^{-i(\phi_2+\phi_6)}+\sin\theta_4\cos\theta_3\cos\theta_6 e^{i\phi_4}]\hat{d}^{\dagger}_{-s}(\vec{p}), \quad (3.12d)$$

with θ_m,ϕ_m, $(m=1,2,\ldots 6)$ being the real solutions of the auxiliary equations (see the Appendix). It is easy to show that the operators $\hat{B}^{\dagger}_s(\vec{p},t),\hat{B}_s(\vec{p},t),\hat{D}^{\dagger}_s(\vec{p},t),\hat{D}_s(\vec{p},t)$ satisfy the equal-time anticommutation relations

$$[\hat{B}_s(\vec{p},t),\hat{B}^{\dagger}_{s'}(\vec{p}\,',t)]_+=\delta_{ss'}\delta(\vec{p}-\vec{p}\,'),$$

$$[\hat{D}_s(\vec{p},t),\hat{D}^{\dagger}_{s'}(\vec{p}\,',t)]_+=\delta_{ss'}\delta(\vec{p}-\vec{p}\,'). \quad (3.13)$$

According to the invariant-related unitary transformation method, the unitary transformation $\hat{V}(t)$ can be constructed:

$$\hat{V}(t)=\hat{V}_3(t)\hat{V}_2(t)\hat{V}_1(t) \quad (3.14)$$

$$\hat{V}_1(t)=\exp\int d^3\vec{p}[(-\theta_1 e^{-i\phi_1}\hat{b}^{\dagger}_{+s}\hat{b}_{-s}+\theta_1 e^{i\phi_1}\hat{b}^{\dagger}_{-s}\hat{b}_{+s}) +(-\theta_4 e^{-i\phi_4}\hat{d}_{+s}\hat{d}^{\dagger}_{-s}+\theta_4 e^{i\phi_4}\hat{d}_{-s}\hat{d}^{\dagger}_{+s})],$$

$$\hat{V}_2(t)=\exp\int d^3\vec{p}[(-\theta_2 e^{-i\phi_2}\hat{b}^{\dagger}_{+s}\hat{d}^{\dagger}_{+s}+\theta_2 e^{i\phi_2}\hat{d}_{+s}\hat{b}_{+s}) +(-\theta_3 e^{-i\phi_3}\hat{b}^{\dagger}_{-s}\hat{d}^{\dagger}_{-s}+\theta_3 e^{i\phi_3}\hat{d}_{-s}\hat{b}_{-s})],$$

$$\hat{V}_3(t)=\exp\int d^3\vec{p}[(-\theta_5 e^{-i\phi_5}\hat{b}^{\dagger}_{-s}\hat{d}^{\dagger}_{+s}+\theta_5 e^{i\phi_5}\hat{d}_{+s}\hat{b}_{-s}) +(-\theta_6 e^{-i\phi_6}\hat{b}^{\dagger}_{+s}\hat{d}^{\dagger}_{-s}+\theta_6 e^{i\phi_6}\hat{d}_{-s}\hat{b}_{+s})],$$

where, for simplicity, the argument $\vec{p}$ of $\hat{b}^{\dagger}_s,\hat{b}_s,\hat{d}^{\dagger}_s,\hat{d}_s$ is omitted. With the help of Eq. (3.6), it is easy to show that the $\hat{V}(t)$ in Eq. (3.14) transforms $\hat{I}(t)$ into the time-independent $\hat{I}_V$

$$\hat{I}_V=\hat{V}^{+}(t)\hat{I}(t)\hat{V}(t)=\sum_{\pm s}\int d^3\vec{p}[\hat{b}^{\dagger}_s(\vec{p})\hat{b}_s(\vec{p}) +\hat{d}_s(\vec{p})\hat{d}^{\dagger}_s(\vec{p})]. \quad (3.15)$$

By making use of the unitary operators in Eq. (3.14) and the Backer-Campbell-Hausdorff formula [12], we obtain $\hat{H}_0(t)$ from $\hat{H}(t)$

$$\hat{H}_0(t)=\hat{V}^{+}(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^{+}(t)\frac{\partial\hat{V}(t)}{\partial t}=\int d^3\vec{p}\hat{H}_0(\vec{p},t) \quad (3.16)$$

of which the first term is easily obtained

$$\hat{V}^{+}(t)\hat{H}(t)\hat{V}(t)=\sum_{\pm s}\int d^3\vec{p}[\alpha^{(d)}_s(p,t)\hat{b}^{\dagger}_s(\vec{p})\hat{b}_s(\vec{p})+\beta^{(d)}_s(p,t)\hat{d}_s(\vec{p})\hat{d}^{\dagger}_s(\vec{p})] \quad (3.17)$$

with

$$\alpha^{(d)}_{+s}(p,t)=\sin^2\theta_6\chi^{(d)}_4(p,t)+\cos^2\theta_6\chi^{(d)}_1(p,t)-\sin 2\theta_6[\chi^{(d)}_6(p,t)\sin\phi_6+\chi^{(d)}_5(p,t)\cos\phi_6], \quad (3.18a)$$

$$\alpha^{(d)}_{-s}(p,t)=\sin^2\theta_5\chi^{(d)}_3(p,t)+\cos^2\theta_5\chi^{(d)}_2(p,t)-\sin 2\theta_5[\chi^{(d)}_8(p,t)\sin\phi_5+\chi^{(d)}_7(p,t)\cos\phi_5], \quad (3.18b)$$

$$\beta^{(d)}_{+s}(p,t)=\sin^2\theta_5\chi^{(d)}_2(p,t)+\cos^2\theta_5\chi^{(d)}_3(p,t)+\sin 2\theta_5[\chi^{(d)}_8(p,t)\sin\phi_5+\chi^{(d)}_7(p,t)\cos\phi_5], \quad (3.18c)$$

$$\beta^{(d)}_{-s}(p,t)=\sin^2\theta_6\chi^{(d)}_4(p,t)+\cos^2\theta_6\chi^{(d)}_1(p,t)+\sin 2\theta_6[\chi^{(d)}_6(p,t)\sin\phi_6+\chi^{(d)}_5(p,t)\cos\phi_6], \quad (3.18d)$$

where $\chi_m^{(d)}(p,t)$, $(m=1,2,\ldots,8)$ can be found in the Appendix. By means of the Baker-Hausdorff-Campbell formula [12] and Eq. (3.14), the second term in Eq. (3.16) can be calculated:

$$-i\hat{V}^{+}(t)\frac{\partial\hat{V}(t)}{\partial t}=\sum_{\pm s}\int d^3\vec{p}[\alpha_s^{(g)}(p,t)\hat{b}_s^{\dagger}(\vec{p})\hat{b}_s(\vec{p})+\beta_s^{(g)}(p,t)\hat{d}_s(\vec{p})\hat{d}_s^{\dagger}(\vec{p})] \tag{3.19}$$

with

$$\alpha_{+s}^{(g)}(p,t)=\sin^2\theta_6\chi_4^{(g)}(p,t)+\cos^2\theta_6\chi_1^{(g)}(p,t)-\sin 2\theta_6[\chi_6^{(g)}(p,t)\sin\,\phi_6+\chi_5^{(g)}(p,t)\cos\,\phi_6]-\dot{\phi}_6\sin\,\theta_6, \tag{3.20a}$$

$$\alpha_{-s}^{(g)}(p,t)=\sin^2\theta_5\chi_3^{(g)}(p,t)+\cos^2\theta_5\chi_2^{(g)}(p,t)-\sin 2\theta_5[\chi_8^{(g)}(p,t)\sin\,\phi_5+\chi_7^{(g)}(p,t)\cos\,\phi_5]-\dot{\phi}_5\sin\,\theta_5, \tag{3.20b}$$

$$\beta_{+s}^{(g)}(p,t)=\sin^2\theta_5\chi_2^{(g)}(p,t)+\cos^2\theta_5\chi_3^{(g)}(p,t)+\sin 2\theta_5[\chi_8^{(g)}(p,t)\sin\,\phi_5+\chi_7^{(g)}(p,t)\cos\,\phi_5]-\dot{\phi}_5\sin\,\theta_5, \tag{3.20c}$$

$$\beta_{-s}^{(g)}(p,t)=\sin^2\theta_6\chi_4^{(g)}(p,t)+\cos^2\theta_6\chi_1^{(g)}(p,t)+\sin 2\theta_6[\chi_6^{(g)}(p,t)\sin\,\phi_6+\chi_5^{(g)}(p,t)\cos\,\phi_6]-\dot{\phi}_6\sin\,\theta_6, \tag{3.20d}$$

where $\chi_m^{(g)}(p,t)$ $(m=1,2,\ldots,8)$ can be found in the Appendix. It is clearly seen from Eqs. (3.17),(3.19) that, for each mode $\vec{p}$ in the p space, $\hat{H}_0(\vec{p},t)$ differs from $\hat{I}_V(\vec{p})$ only by a multiplying c-number factor depending on the time t and $p=|\vec{p}|$. $\hat{I}_V$ is time independent and in the discrete notation, it can be regarded as a sum of terms of which each has the form of the Hamiltonian for a simple Fermi oscillator of frequency 1. The solutions to the oscillator eigenvalue problem for $\vec{p}_1,\vec{p}_2,\ldots,$ modes may be characterized by integers $n_1,n_2,\ldots,$ $(n_1,n_2,\ldots,=0,1)$. The ground state of $\hat{I}_V(\vec{p})$ is denoted by $|0\rangle$ and satisfies

$$\hat{b}_s(\vec{p}_m)|0\rangle\equiv\hat{b}_{ms}|0\rangle=0,$$

$$\hat{d}_s(\vec{p}_m)|0\rangle\equiv\hat{d}_{ms}|0\rangle=0. \tag{3.21}$$

By making use of the ground state $|0\rangle$ and the raising operators $\hat{b}_s^{\dagger}(\vec{p}_m)\equiv\hat{b}_{ms}^{\dagger}$, $\hat{d}_s^{\dagger}(\vec{p}_m)\equiv\hat{d}_{ms}^{\dagger}$, we obtain the N-particle excited eigenstate $|N\rangle$ of $\hat{I}_V$ (with the particle number operators being defined to be $\hat{n}_{mbs}=\hat{b}_{ms}^{\dagger}\hat{b}_{ms}$, $\hat{n}_{mds}=\hat{d}_{ms}^{\dagger}\hat{d}_{ms}$, $\hat{N}_{bs}=\Sigma_m\hat{n}_{mbs}$, $\hat{N}_{ds}=\Sigma_m\hat{n}_{mds}$)

$$|N_{bs}\rangle=|n_{1bs},n_{2bs},\ldots,(n_{1bs}+n_{2bs}+\cdots=N_{bs})\rangle$$
$$=\prod_m[\hat{b}_s^{\dagger}(\vec{p}_m)]^{n_{mbs}}|0\rangle, \tag{3.22a}$$

$$|N_{ds}\rangle=|n_{1ds},n_{2ds},\ldots,(n_{1ds}+n_{2ds}+\cdots=N_{ds})\rangle$$
$$=\prod_m[\hat{d}_s^{\dagger}(\vec{p}_m)]^{n_{mbs}}|0\rangle$$

$$(n_{mbs},n_{mds}=0,1) \tag{3.22b}$$

which satisfies

$$\hat{I}_V|N_{bs},N_{ds}\rangle_{I_V}=(N_{b+s}+N_{b-s}+N_{d+s}+N_{d-s})|N_{bs},N_{ds}\rangle_{I_V}, \tag{3.23}$$

where, for convenience, we define $|N_{bs},N_{ds}\rangle_{I_V}\equiv|N_{b+s}\rangle|N_{b-s}\rangle|N_{d+s}\rangle|N_{d-s}\rangle$. According to the invariant-related unitary transformation method [8,2], the solutions $|N_{bs},N_{ds},t\rangle_{S0}$ of the time-dependent functional Schrödinger equation [associated with $\hat{H}_0(t)$] are

$$|N_{bs},N_{ds},t\rangle_{S0}=\exp\left[i\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[i\sum_{\pm s}\vartheta_{ds}(t)\right]|N_{bs},N_{ds}\rangle_{I_V}$$

$$\vartheta_{bs}=-\int_0^t\langle N_{bs}|\hat{H}_0(t')|N_{bs}\rangle dt'=-\int_0^t\langle N_{bs}|\hat{U}^{+}(t')\hat{H}_0(t')\hat{U}(t')-i\hat{U}^{+}(t')\frac{\partial\hat{U}(t')}{\partial t}|N_{bs}\rangle dt'$$
$$=-\sum_m n_{mbs}\int_0^t[\alpha_s^{(d)}(p_m,t')+\alpha_s^{(g)}(p_m,t')]dt',$$

$$\vartheta_{ds}=-\int_0^t\langle N_{ds}|\hat{H}_0(t')|N_{ds}\rangle dt'=-\int_0^t\langle N_{ds}|\hat{U}^{+}(t')\hat{H}_0(t')\hat{U}(t')-i\hat{U}^{+}(t')\frac{\partial\hat{U}(t')}{\partial t}|N_{ds}\rangle dt'$$
$$=-\sum_m n_{mds}\int_0^t[\beta_s^{(d)}(p_m,t')+\beta_s^{(g)}(p_m,t')]dt' \tag{3.24}$$

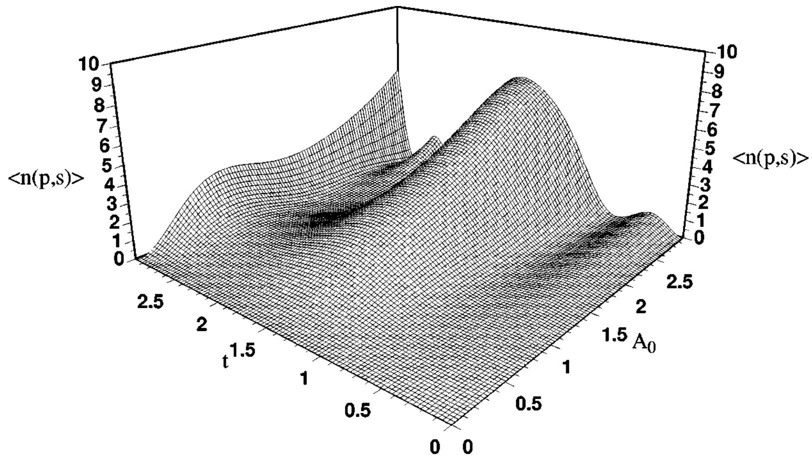

FIG. 1. The time and A_0 dependnece of the expectation value $\langle n(\vec{p},s)\rangle$ of the particle number $\hat{n}(\vec{p},s)\equiv\hat{n}_{b+s}+\hat{n}_{b-s}+\hat{n}_{d+s}+\hat{n}_{d-s}$ for the state $|\Psi_{0,0}(t)\rangle_s$. For the purpose of illustration, we choose the time-dependent electric-magnetic potential field to be of the form $\vec{A}=\vec{A}_0\cos t$, where $\vec{A}_0=(A_0,0,0)$ and A_0 does not depend on time; and set particle mass $m=1$ and the particle momentum $p_x=p_y=p_z=1$.

in which $\vartheta_{bs}(t),\vartheta_{ds}(t)$ are the total phases, including the dynamical phases and geometrical phases. According to the invariant-related unitary transformation method [8,2], the particular exact solutions of the time-dependent Schrödinger equation (3.9) [associated with $\hat{H}(t)$] can be found to be

$$\begin{aligned}|\Psi_{N_{bs},N_{ds}}(t)\rangle_S&=\hat{V}(t)|N_{bs},N_{ds},t\rangle_{S0}\\&=\exp\left[i\sum_{\pm s}\vartheta_{bs}(t)\right]\\&\quad\times\exp\left[i\sum_{\pm s}\vartheta_{ds}(t)\right]\hat{V}(t)|N_{bs},N_{ds}\rangle_{I_V}\\&=\exp\left[i\sum_{\pm s}\vartheta_{bs}(t)\right]\\&\quad\times\exp\left[i\sum_{\pm s}\vartheta_{ds}(t)\right]|N_{bs},N_{ds};t\rangle_I,\end{aligned}\tag{3.25}$$

where $|N_{bs},N_{ds},t\rangle_I$ are the eigenstates of the invariant $\hat{I}(t)$ with particle number $(N_{b+s}+N_{b-s}+N_{d+s}+N_{d-s})$. The particular exact solutions in Eq. (3.25) for all possible n_{bs} and n_{ds} constitute a complete set, this means that the general exact solution of the time-dependent Schrödinger equation (3.9) is a superposition of the particular solutions in Eq. (3.25)

$$\begin{aligned}|\Psi(t)\rangle_S&=\sum_{N_{bs}N_{ds}}C_{N_{bs}N_{ds}}\exp\left[i\sum_{\pm s}\vartheta_{bs}(t)\right]\\&\quad\times\exp\left[i\sum_{\pm s}\vartheta_{ds}(t)\right]|N_{bs},N_{ds},t\rangle_I,\\C_{N_{bs}N_{ds}}&=\langle N_{bs},N_{ds};0|\Psi(0)\rangle_S,\end{aligned}\tag{3.26}$$

where the initial state $|\Psi(0)\rangle_S$ can be chosen arbitrarily.

It is worthwhile to point out that the auxiliary equations obtained in the present paper for the Dirac field are much more complicated than that for a scalar field in Ref. [7], and can only be solved numerically. As an illustration, we choose the initial state at $t=0$ to be the ground state and calculate the time dependence of the expectation value of the particle number $\hat{n}(\vec{p},s)\equiv\hat{n}_{b+s}+\hat{n}_{b-s}+\hat{n}_{d+s}+\hat{n}_{d-s}$ for the state $|\Psi_{0,0}(t)\rangle_s$ (which is the ground state at $t=0$). The result of the calculation is shown in Fig. 1.

IV. CONCLUDING REMARKS

(1) It can be shown that $\hat{b}_s^\dagger(\vec{p})$, $\hat{b}_s(\vec{p})$, $\hat{d}_s^\dagger(\vec{p})$, and $\hat{d}_s(\vec{p})$ in Eq. (3.5), and the quantum Hamiltonian $\hat{H}(t)$ in Eq. (3.7) constitute quasialgebra [13]. It is this algebra that makes it possible to find the unitary transformation $\hat{V}(t)$ in Eq. (3.15). This means that the phase formulation and invariant-related unitary transformation method can only be used to investigate special time-dependent systems with Hamiltonians for which there exist corresponding quasialgebras [13]. The Dirac field studied in this paper is one of such special systems. However, although special, it is interesting, since, to our knowledge, no one has ever obtained exact solutions of the functional Schrödinger equation for a Dirac field with time-dependent Hamiltonian. In addition, the exact solutions obtained are useful as a starting point for relevant time-dependent perturbation quantum field theories as was shown in the case of scalar field [8].

(2) Note that even when we choose $\hat{H}(t=0)$ as $\hat{I}_0$, the ‘‘ground-state’’ solution is the ground eigenstate of the Hamiltonian $\hat{H}(t)$ only at $t=0$, so that, in general, the term ‘‘ground state’’ is without the meaning that it has in the case in which the Hamiltonian is time independent.

(3) It is interesting to point out that the system of auxiliary equations is a Pfaffian system in differential geometry. It is of interest to use the method in differential geometry to discuss the local as well as global properties of this system and its solutions.

(4) The method used in this paper can also be used to discuss the Dirac field in other time-dependent backgrounds, such as the Dirac field in Friedmann-Robertson-Walker flat

spacetimes. Work in this direction has been completed and will be published elsewhere.

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APPENDIX: THE AUXILIARY EQUATIONS

The auxiliary equations for θ_m and ϕ_m $(m=1,\ldots,6)$ in Sec. III are as follows:

$$\dot{\theta}_1=\frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)}\{\sin 2\theta_3\cos\theta_1\cos\theta_4[\lambda_{3r}\sin(\phi_3+\phi_4)+\lambda_{3i}\cos(\phi_3+\phi_4)]+\sin 2\theta_3\sin\theta_1\sin\theta_4 \times[\lambda_{3r}\sin(\phi_3+\phi_4)-\lambda_{3i}\cos(\phi_3+\phi_4)]-\sin 2\theta_2\sin\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_2+\phi_4)-\lambda_{3i}\cos(\phi_2+\phi_4)] -\sin 2\theta_2\cos\theta_1\cos\theta_4[\lambda_{3r}\sin(\phi_1+\phi_2)+\lambda_{3i}\cos(\phi_1+\phi_2)]+\lambda_1\sin 2\theta_2\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_2+\phi_4) +\lambda_1\sin 2\theta_3\sin\theta_1\cos\theta_4\sin\phi_3-\lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_3+\phi_4)-\lambda_1\sin 2\theta_2\sin\theta_1\cos\theta_4\sin\phi_2, \tag{A1}$$

$$\dot{\theta}_2=\cos\theta_4\sin\theta_1[\lambda_{3r}\sin(\phi_1+\phi_2)+\lambda_{3i}\cos(\phi_1+\phi_2)]-\lambda_1\cos\theta_4\cos\theta_1\sin\phi_2-\cos\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_2+\phi_4)+\lambda_{3i}\cos(\phi_2+\phi_4)]-\lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_2+\phi_4), \tag{A2}$$

$$\dot{\theta}_3=\cos\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_3+\phi_4)-\lambda_{3i}\cos(\phi_3+\phi_4)]+\lambda_1\cos\theta_4\cos\theta_1\sin\phi_3-\cos\theta_4\sin\theta_1[\lambda_{3r}\sin(\phi_1+\phi_3)+\lambda_{3i}\cos(\phi_1+\phi_3)]+\lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4), \tag{A3}$$

$$\dot{\theta}_4=\frac{1}{2(\cos^2\theta_2-\sin^2\theta_3)}\{\sin 2\theta_2\cos\theta_1\cos\theta_4[\lambda_{3r}\sin(\phi_2+\phi_4)-\lambda_{3i}\cos(\phi_2+\phi_4)]+\sin 2\theta_2\sin\theta_1\sin\theta_4 \times[\lambda_{3r}\sin(\phi_1+\phi_2)+\lambda_{3i}\cos(\phi_1+\phi_2)]-\sin 2\theta_3\sin\theta_1\sin\theta_4[\lambda_{3r}\sin(\phi_1+\phi_3)+\lambda_{3i}\cos(\phi_1+\phi_3)] -\sin 2\theta_3\cos\theta_1\cos\theta_4[\lambda_{3r}\sin(\phi_3+\phi_4)-\lambda_{3i}\cos(\phi_3+\phi_4)]+\lambda_1\sin 2\theta_2\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_2+\phi_4) +\lambda_1\sin 2\theta_3\sin\theta_4\cos\theta_1\sin\phi_3-\lambda_1\sin 2\theta_3\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4)-\lambda_1\sin 2\theta_2\sin\theta_4\cos\theta_1\sin\phi_2, \tag{A4}$$

$$\dot{\theta}_5=\cos\phi_5[\chi_8^{(d)}(p,t)+\chi_8^{(g)}(p,t)]-\sin\phi_5[\chi_7^{(d)}(p,t)+\chi_7^{(g)}(p,t)], \tag{A5}$$

$$\dot{\theta}_6=\cos\phi_6[\chi_6^{(d)}(p,t)+\chi_6^{(g)}(p,t)]-\sin\phi_6[\chi_5^{(d)}(p,t)+\chi_5^{(g)}(p,t)], \tag{A6}$$

$$\dot{\phi}_1=\frac{\csc 2\theta_1}{2(\cos^2\theta_2-\sin^2\theta_3)}\{\sin 2\theta_3\cos\theta_1\cos\theta_4[\lambda_{3r}\cos(\phi_3+\phi_4)-\lambda_{3i}\sin(\phi_3+\phi_4)]+\sin 2\theta_3\sin\theta_1\sin\theta_4[\lambda_{3r}\cos(\phi_3+\phi_4) +\lambda_{3i}\sin(\phi_3+\phi_4)]-\sin 2\theta_2\sin\theta_1\sin\theta_4[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)]-\sin 2\theta_2\cos\theta_1\cos\theta_4[\lambda_{3r}\cos(\phi_1+\phi_2) -\lambda_{3i}\sin(\phi_1+\phi_2)]+\lambda_1\sin 2\theta_2\cos\theta_1\sin\theta_4\cos(\phi_1+\phi_2+\phi_4)+\lambda_1\sin 2\theta_3\sin\theta_1\cos\theta_4\cos\phi_3 -\lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\cos(\phi_1+\phi_3+\phi_4)-\lambda_1\sin 2\theta_2\sin\theta_1\cos\theta_4\cos\phi_2, \tag{A7}$$

$$\dot{\phi}_2=2[E_b(t)+E_d(t)]-\sin^2\theta_1\dot{\phi}_1-\sin^2\theta_4\dot{\phi}_4-2\cot 2\theta_2\cos\theta_1\sin\theta_4[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)] +2\cot 2\theta_2\cos\theta_4\sin\theta_1[\lambda_{3r}\cos(\phi_1+\phi_2)-\lambda_{3i}\sin(\phi_1+\phi_2)]-2\lambda_1\cot 2\theta_2[\cos\theta_4\cos\theta_1\cos\phi_2+\sin\theta_4\sin\theta_1\cos(\phi_1+\phi_2+\phi_4)], \tag{A8}$$

$$\dot{\phi}_3=-2[E_b(t)+E_d(t)]-\sin^2\theta_1\dot{\phi}_1-\sin^2\theta_4\dot{\phi}_4+2\cot 2\theta_3\cos\theta_1\sin\theta_4[\lambda_{3r}\cos(\phi_3+\phi_4)+\lambda_{3i}\sin(\phi_3+\phi_4)] -2\cot 2\theta_3\cos\theta_4\sin\theta_1[\lambda_{3r}\cos(\phi_1+\phi_3)-\lambda_{3i}\sin(\phi_1+\phi_3)]+2\lambda_1\cot 2\theta_3[\cos\theta_4\cos\theta_1\cos\phi_3+\sin\theta_4\sin\theta_1\cos(\phi_1+\phi_3+\phi_4)], \tag{A9}$$

$$\dot{\phi}_4=\frac{\csc 2\theta_4}{2(\cos^2\theta_2-\sin^2\theta_3)}\{\sin 2\theta_2\cos\ \theta_1\cos\ \theta_4[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)]+\sin 2\theta_2\sin\ \theta_1\ \sin\ \theta_4$$
$$\times[\lambda_{3r}\cos(\phi_1+\phi_2)-\lambda_{3i}\sin(\phi_1+\phi_2)]-\sin 2\theta_3\sin\ \theta_1\sin\ \theta_4[\lambda_{3r}\cos(\phi_1+\phi_3)-\lambda_{3i}\sin(\phi_1+\phi_3)]$$
$$-\sin 2\theta_3\cos\ \theta_1\cos\ \theta_4[\lambda_{3r}\cos(\phi_3+\phi_4)+\lambda_{3i}\sin(\phi_3+\phi_4)]+\lambda_1\sin 2\theta_2\cos\ \theta_4\sin\ \theta_1\cos(\phi_1+\phi_2+\phi_4)$$
$$+\lambda_1\sin 2\theta_3\sin\ \theta_4\cos\ \theta_1\cos\ \phi_3-\lambda_1\sin 2\theta_3\cos\ \theta_4\sin\ \theta_1\cos(\phi_1+\phi_3+\phi_4)-\lambda_1\sin 2\theta_2\sin\ \theta_4\cos\ \theta_1\cos\ \phi_2, \quad \text{(A10)}$$

$$\dot{\phi}_5=[\chi_3^{(d)}(p,t)+\chi_3^{(g)}(p,t)]-[\chi_2^{(d)}(p,t)+\chi_2^{(g)}(p,t)]+2\cot 2\theta_5\{\cos\ \phi_5[\chi_7^{(d)}(p,t)+\chi_7^{(g)}(p,t)]$$
$$-\sin\ \phi_5[\chi_8^{(d)}(p,t)+\chi_8^{(g)}(p,t)]\}, \quad \text{(A11)}$$

$$\dot{\theta}_6=[\chi_4^{(d)}(p,t)+\chi_4^{(g)}(p,t)]-[\chi_1^{(d)}(p,t)+\chi_1^{(g)}(p,t)]+2\cot 2\theta_6\{\cos\ \phi_6[\chi_5^{(d)}(p,t)+\chi_5^{(g)}(p,t)]$$
$$-\sin\ \phi_6[\chi_6^{(d)}(p,t)+\chi_6^{(g)}(p,t)]\}, \quad \text{(A12)}$$

where

$$\chi_1^{(d)}=\frac{1}{2}\cos 2\theta_2[E_b+E_d]-\sin\ \theta_1\cos\ \theta_4\sin 2\theta_2[\lambda_{3r}\cos(\phi_1+\phi_2)-\lambda_{3i}\sin(\phi_1+\phi_2)]+\sin\ \theta_4\cos\ \theta_1\ \sin 2\theta_2$$
$$\times[\lambda_{3r}\cos(\phi_2+\phi_4)+\lambda_{3i}\sin(\phi_2+\phi_4)]+\lambda_1\cos\ \theta_1\cos\ \theta_4\sin 2\theta_2\cos\ \phi_2+\lambda_1\sin\ \theta_1\sin\ \theta_4\sin 2\theta_2$$
$$\times\cos(\phi_1+\phi_2+\phi_4), \quad \text{(A13)}$$

$$\chi_1^{(g)}=-\sin^2\theta_1\cos^2\theta_2\dot{\phi}_1+\sin^2\theta_2\dot{\phi}_2+\sin^2\theta_2\sin^2\theta_4\dot{\phi}_4, \quad \text{(A14)}$$

$$\chi_2^{(d)}=\frac{1}{2}\cos 2\theta_3[E_b+E_d]+\sin\ \theta_4\cos\ \theta_1\sin 2\theta_3[\lambda_{3r}\cos(\phi_3+\phi_4)+\lambda_{3i}\sin(\phi_3+\phi_4)]$$
$$-\sin\ \theta_1\cos\ \theta_4\sin 2\theta_3[\lambda_{3r}\cos(\phi_1+\phi_3)-\lambda_{3i}\sin(\phi_1+\phi_3)]+\lambda_1\cos\ \theta_1\cos\ \theta_4\sin 2\theta_3\cos\ \phi_3$$
$$+\lambda_1\sin\ \theta_1\sin\ \theta_4\sin 2\theta_3\cos(\phi_1+\phi_3+\phi_4), \quad \text{(A15)}$$

$$\chi_2^{(g)}=-\sin^2\theta_1\cos^2\theta_3\dot{\phi}_1+\sin^2\theta_3\dot{\phi}_3+\sin^2\theta_3\sin^2\theta_4\dot{\phi}_4, \quad \text{(A16)}$$

$$\chi_3^{(d)}=-\chi_1^{(d)}, \quad \text{(A17)}$$

$$\chi_3^{(g)}=-\chi_1^{(g)}, \quad \text{(A18)}$$

$$\chi_4^{(d)}=-\chi_2^{(d)}, \quad \text{(A19)}$$

$$\chi_4^{(g)}=-\chi_2^{(g)}, \quad \text{(A20)}$$

$$\chi_5^{(d)}=\sin\ \theta_1\sin\ \theta_2\sin\ \theta_3\sin\ \theta_4[\lambda_{3r}\cos(\phi_2+\phi_4-\phi_1-\phi_3)+\lambda_{3i}\sin(\phi_2+\phi_4-\phi_1-\phi_3)]$$
$$+\lambda_1\sin\ \theta_2\sin\ \theta_3[\sin\ \theta_1\cos\ \theta_4\cos(\phi_1-\phi_2+\phi_3)-\sin\ \theta_4\cos\ \theta_1\cos(\phi_2-\phi_3+\phi_4)]$$
$$+\sin\ \theta_1\cos\ \theta_2\cos\ \theta_3\sin\ \theta_4[\lambda_{3r}\cos(\phi_1-\phi_4)-\lambda_{3i}\sin(\phi_1-\phi_4)]+\cos\ \theta_1\sin\ \theta_2\sin\ \theta_3\cos\ \theta_4[\lambda_{3r}\cos(\phi_2-\phi_3)$$
$$-\lambda_{3i}\sin(\phi_2-\phi_3)]+\lambda_1\cos\ \theta_2\cos\ \theta_3(\sin\ \theta_1\cos\ \theta_4\cos\ \phi_1+\sin\ \theta_4\cos\ \theta_1\cos\ \phi_4)+\lambda_{3r}\cos\ \theta_1\cos\ \theta_2\cos\ \theta_3\cos\ \theta_4, \quad \text{(A21)}$$

$$\chi_5^{(g)}=\dot{\theta}_1\sin(\phi_1+\phi_3)\cos\ \theta_2\sin\ \theta_3+\dot{\theta}_4\sin(\phi_2+\phi_4)\sin\ \theta_2\cos\ \theta_3+\frac{1}{2}[\dot{\phi}_1\cos(\phi_1+\phi_3)\cos\ \theta_2\sin\ \theta_3\sin 2\theta_1$$
$$+\dot{\phi}_4\cos(\phi_2+\phi_4)\cos\ \theta_3\sin\ \theta_2\sin 2\theta_4], \quad \text{(A22)}$$

$$\chi_6^{(d)}=-\sin\,\theta_1\sin\,\theta_2\sin\,\theta_3\sin\,\theta_4[\lambda_{3r}\sin(\phi_2+\phi_4-\phi_1-\phi_3)-\lambda_{3i}\cos(\phi_2+\phi_4-\phi_1-\phi_3)]$$
$$+\lambda_1\sin\,\theta_2\sin\,\theta_3[\sin\,\theta_1\cos\,\theta_4\sin(\phi_1-\phi_2+\phi_3)+\sin\,\theta_4\cos\,\theta_1\sin(\phi_2-\phi_3+\phi_4)]$$
$$+\sin\,\theta_1\cos\,\theta_2\cos\,\theta_3\sin\,\theta_4[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)]-\cos\,\theta_1\sin\,\theta_2\sin\,\theta_3\cos\,\theta_4[\lambda_{3r}\sin(\phi_2-\phi_3)$$
$$+\lambda_{3i}\cos(\phi_2-\phi_3)]+\lambda_1\cos\,\theta_2\cos\,\theta_3(\sin\,\theta_1\cos\,\theta_4\cos\,\phi_1+\sin\,\theta_4\cos\,\theta_1\cos\,\phi_4)-\lambda_{3i}\cos\,\theta_1\cos\,\theta_2\cos\,\theta_3\cos\,\theta_4, \tag{A23}$$

$$\chi_6^{(g)}=-\dot{\theta}_1\cos(\phi_1+\phi_3)\cos\,\theta_2\sin\,\theta_3+\dot{\theta}_4\cos(\phi_2+\phi_4)\sin\,\theta_2\cos\,\theta_3+\frac{1}{2}[\dot{\phi}_1\sin(\phi_1+\phi_3)\cos\,\theta_2\sin\,\theta_3\sin 2\theta_1-\dot{\phi}_4\sin(\phi_2$$
$$+\phi_4)\cos\,\theta_3\sin\,\theta_2\sin 2\theta_4], \tag{A24}$$

$$\chi_7^{(d)}=\sin\,\theta_1\sin\,\theta_2\sin\,\theta_3\sin\,\theta_4[\lambda_{3r}\cos(\phi_1+\phi_2-\phi_3-\phi_4)-\lambda_{3i}\sin(\phi_1+\phi_2-\phi_3-\phi_4)]$$
$$+\lambda_1\sin\,\theta_2\sin\,\theta_3[\sin\,\theta_1\cos\,\theta_4\cos(\phi_1+\phi_2-\phi_3)-\sin\,\theta_4\cos\,\theta_1\cos(\phi_2-\phi_3-\phi_4)]$$
$$+\sin\,\theta_1\cos\,\theta_2\cos\,\theta_3\sin\,\theta_4[\lambda_{3r}\cos(\phi_1-\phi_4)-\lambda_{3i}\sin(\phi_1-\phi_4)]+\cos\,\theta_1\sin\,\theta_2\sin\,\theta_3\cos\,\theta_4[\lambda_{3r}\cos(\phi_2-\phi_3)$$
$$+\lambda_{3i}\sin(\phi_2-\phi_3)]+\lambda_1\cos\,\theta_2\cos\,\theta_3(\sin\,\theta_1\cos\,\theta_4\cos\,\phi_1-\sin\,\theta_4\cos\,\theta_1\cos\,\phi_4)+\lambda_{3r}\cos\,\theta_1\cos\,\theta_2\cos\,\theta_3\cos\,\theta_4, \tag{A25}$$

$$\chi_7^{(g)}=-\dot{\theta}_1\sin(\phi_1+\phi_2)\cos\,\theta_3\sin\,\theta_2-\dot{\theta}_4\sin(\phi_3+\phi_4)\sin\,\theta_3\cos\,\theta_2-\frac{1}{2}[\dot{\phi}_1\cos(\phi_1+\phi_2)\cos\,\theta_3\sin\,\theta_2\sin 2\theta_1+\dot{\phi}_4\cos(\phi_3$$
$$+\phi_4)\cos\,\theta_2\sin\,\theta_3\sin 2\theta_4], \tag{A26}$$

$$\chi_8^{(d)}=-\sin\,\theta_1\sin\,\theta_2\sin\,\theta_3\sin\,\theta_4[\lambda_{3r}\sin(\phi_1+\phi_2-\phi_3-\phi_4)-\lambda_{3i}\cos(\phi_1+\phi_2-\phi_3-\phi_4)]$$
$$-\lambda_1\sin\,\theta_2\sin\,\theta_3[\sin\,\theta_1\cos\,\theta_4\sin(\phi_1+\phi_2-\phi_3)-\sin\,\theta_4\cos\,\theta_1\sin(\phi_2-\phi_3-\phi_4)]$$
$$-\sin\,\theta_1\cos\,\theta_2\cos\,\theta_3\sin\,\theta_4[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)]-\cos\,\theta_1\sin\,\theta_2\sin\,\theta_3\cos\,\theta_4[\lambda_{3r}\sin(\phi_2-\phi_3)$$
$$-\lambda_{3i}\cos(\phi_2-\phi_3)]-\lambda_1\cos\,\theta_2\cos\,\theta_3(\sin\,\theta_1\cos\,\theta_4\cos\,\phi_1+\sin\,\theta_4\cos\,\theta_1\cos\,\phi_4)+\lambda_{3i}\cos\,\theta_1\cos\,\theta_2\cos\,\theta_3\cos\,\theta_4, \tag{A27}$$

$$\chi_8^{(g)}=-\dot{\theta}_1\cos(\phi_1+\phi_2)\cos\,\theta_3\sin\,\theta_2+\dot{\theta}_4\cos(\phi_3+\phi_4)\sin\,\theta_3\cos\,\theta_2+\frac{1}{2}[\dot{\phi}_1\sin(\phi_1+\phi_2)\cos\,\theta_3\sin\,\theta_2\sin 2\theta_1-\dot{\phi}_4\sin(\phi_3$$
$$+\phi_4)\cos\,\theta_2\sin\,\theta_3\sin 2\theta_4], \tag{A28}$$

with $\lambda_{3r},\lambda_{3i}$ being the real and imaginary parts of λ_3, respectively.

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