Generalized eikonal wave function of a Dirac particle interacting with an arbitrary potential and radiation fields

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A generalized eikonal approximation in the relativistic quantum theory of Dirac particle scattering on an arbitrary electrostatic potential in the field of strong electromagnetic waves is developed. An analytic formula for the particle wave function is obtained. The essence of the approximation is that quadratic scattering potential terms $[\sim U^2(\vec{r})]$ are considered small. [S1050-2947(99)02601-3]

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I. INTRODUCTION

In relativistic quantum theory of elastic scattering of Dirac particles interacting with an arbitrary static potential (atomic, ionic, etc. fields), a so-called generalized eikonal approximation (GEA) was developed in Ref. [1]. Further, in Ref. [2] this approximation was developed for inelastic scattering in the presence of an external electromagnetic (EM) radiation field. However, this treatment is used within the scope of nonrelativistic theory based on the solution of the Schrödinger equation. The generalized eikonal wave function obtained enables us to leave the framework of the ordinary eikonal approximation in stimulated bremsstrahlung (SB), which is not applicable beyond the interaction region ($z \ll |\vec{p}|a^2/\hbar$, where z is the coordinate along the direction of initial momentum of the particle $\vec{p}$, a is the range of the interaction region, and $\hbar$ is the Planck constant). Knowledge of such a time dependence in the eikonal-type wave function becomes especially important for the processes occurring in the strong laser fields. These include laser-assisted electron-atom scattering processes, particularly the above-threshold multiphoton ionization of atoms [3,4]. In addition, in many cases when the condition of the Born approximation is broken, the scattering process is described by the eikonal wave function. Indeed, the Born and low-frequency approximations are appropriate for describing free-free transitions in high-intensity EM radiation fields, but they do not take into account the mutual influence of the scattering and the radiation fields (i.e., the probability of SB is factorized by elastic scattering and photon emission or absorption processes) [5–9]

For high-intensity laser fields the above-mentioned processes require a relativistic treatment. In this work we solve the Dirac equation for the evolution of the particle wave function in the arbitrary electrostatic and plane EM wave fields, which simultaneously takes into account the influence of both the scattering and radiation fields on the state of the particle and the release from the restriction $z \ll |\vec{p}|a^2/\hbar$. In addition, it also takes into account the spin interaction in the scattering process. The GEA wave function so obtained includes the known approximate wave functions of the electron in both short-range and long-range potentials in different limits. Such a wave function allows us to describe the final state of the photoelectron with more accuracy in the above-threshold multiphoton ionization process of atoms. The relativistic description of the latter for high-intensity laser fields taking into account the spin interaction has been developed analytically in Refs. [10–12] with an approximation where the stimulated bremsstrahlung of the emergent electron is neglected. The relativistic consideration of this problem is important as it is generally assumed that the stabilization of atoms in ultraintense laser fields must be solved within the framework of relativistic theory by solving the time-dependent Dirac equation [13]. From this point of view some attempts have been made to solve analytically the Klein-Gordon equation [14,15] or numerically the Dirac equation [13,16] in fields of a static potential and monochromatic EM wave. In these works various simplifications of the issue, using various model potentials of one or two dimensions and various approximations, have been made. The relativistic corrections to the nonrelativistic theory have been given in Refs. [17–19].

The organization of the paper is as follows. In Sec. II we present a solution of the Dirac equation for a charged particle in the fields of an arbitrary electrostatic potential and strong EM radiation. In Sec. III we consider the various limits of the GEA wave function obtained and the conditions of its applicability. In Sec. IV we summarize our conclusions.

II. APPROXIMATE SOLUTION OF THE DIRAC EQUATION IN AN ARBITRARY STATIC POTENTIAL AND A PLANE EM WAVE FIELDS

The problem can be reduced to the investigation of the dynamics of the SB process, which can be described by the Dirac equation for a charged particle in a static potential and in the field of given EM radiation (in natural units $\hbar = c = 1$)

$$\{\gamma[i\partial - eA(\varphi) - e\Lambda(x)] - m\}\Psi(x) = 0, \tag{2.1}$$

where e and m are the Dirac particle charge and mass, respectively, c is the light speed in vacuum, $\Psi(x)$ is the four-component Dirac spinor, $x = (t, \vec{r})$ is the four-component radius vector, $\partial \equiv \partial/\partial x_\mu$ ($\mu = 0,1,2,3$) denotes the first derivative of a function with respect to x,

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$$A(\varphi)=A(kx)=(0,\vec{A}(\omega t-\vec{k}\cdot\vec{r})) \quad (2.2)$$

is the four-vector potential of the plane EM wave with the phase $\varphi=kx$, $k=(\omega,\vec{k})$ is the four-wave vector of the applied EM field of frequency ω, $\Lambda(\Lambda_0(\vec{r}),\vec{0})$ is the four-vector potential of the electrostatic field of an arbitrary scalar potential $\Lambda_0(\vec{r})$, and $\gamma=(\gamma_0,\vec{\gamma})$ are the Dirac matrices.

Introducing a bispinor function $\Phi(x)$, which is connected with the Dirac wave function $\Psi(x)$ by the relation

$$\Psi(x)=\frac{1}{2m}\{\gamma[i\partial-eA(\varphi)-e\Lambda(x)]+m\}\Phi(x), \quad (2.3)$$

we turn Eq. (2.1) into the quadratic equation

$$([i\partial-eA(\varphi)-e\Lambda(x)]^2-m^2-ie\{\gamma\partial[\gamma\Lambda(x)]\} -ie(\gamma k)[\gamma dA(\varphi)/d\varphi])\Phi(x)=0. \quad (2.4)$$

To solve Eq. (2.4) we seek a solution in the form

$$\Phi(x)=f(x)\exp[iS(x)], \quad (2.5)$$

where $f(x)$ is a bispinor function and

$$\Psi_K=\exp[iS(x)] \quad (2.6)$$

is the solution of the Klein-Gordon equation for a charged particle in the static potential and EM wave fields

$$\{[i\partial-eA(\varphi)-e\Lambda(x)]^2-m^2\}\Psi_K=0. \quad (2.7)$$

Substituting the expression (2.5) into Eq. (2.4), we obtain for scalar $S(x)$ and bispinor $f(x)$ functions

$$-i\partial^2S(x)+[\partial S(x)+eA(\varphi)+e\Lambda(x)]^2-m^2=0, \quad (2.8)$$

$$-i\partial^2f(x)+2[\partial S(x)+eA(\varphi)+e\Lambda(x)]\partial f(x) +e\{\gamma\partial[\gamma\Lambda(x)]\}f(x)+e(\gamma k)[\gamma dA(\varphi)/d\varphi]f(x)=0. \quad (2.9)$$

So we have initially represented the Dirac equation (2.1) in the quadratic form [Eq. (2.4)] and then by two equations (2.8) and (2.9), the first of which is the Klein-Gordon equation and the second describes the particle-spin interaction with the given fields.

We look for the solutions of Eqs. (2.8) and (2.9) in the form

$$S(x)=S_V(x)+S_1(x), \quad f(x)=f_V(\varphi)+f_1(x), \quad (2.10)$$

where $S_V(x)$ and $f_V(\varphi)$ are the action and bispinor amplitude of a charged particle in the EM field (Gordon-Volkov state)

$$S_V(x)=-px-\frac{e}{kp}\int_{-\infty}^{\varphi}\left(pA(\varphi')-\frac{e}{2}A^2(\varphi')\right)d\varphi', \quad (2.11)$$

$$f_V(\varphi)=u+\frac{e}{2(kp)}(\gamma k)[\gamma A(\varphi)]u, \quad (2.12)$$

where $p=(\varepsilon,\vec{p})$ and u are the initial four-momentum and bispinor amplitude of a free Dirac particle, respectively ($\overline{u}u=2m$ and $\overline{u}=u^{\dagger}\gamma_0$; $u^{\dagger}$ denotes the transposition and complex conjugation of u).

Let the Oz axis be directed along the initial momentum $\vec{p}$ of the free particle. Then, in accordance with the solution (2.5), we have the initial condition

$$S(z=-\infty,t=-\infty)=-px,$$

corresponding to the asymptotic behavior of the scattering potential at $z=-\infty$ $[\Lambda(z=-\infty)=0]$ and $t=-\infty$ $[\vec{A}(t=-\infty)=\vec{0}]$. We assume that the EM wave is adiabatically switched on at $t=-\infty$ (if necessary, the field must be adiabatically switched off at $t=+\infty$ $[\vec{A}(t=+\infty)=\vec{0}]$).

Substituting the solutions (2.10)–(2.12) into Eqs. (2.8) and (2.9), we have the following equations for $S_1(x)$ and $f_1(x)$, respectively:

$$-i\partial^2S_1(x)+2[\partial S_V(x)+eA(\varphi)]\partial S_1(x) =-2e\Lambda(x)\partial S_V(x)-2e\Lambda(x)\partial S_1(x) -e^2\Lambda^2(x)-[\partial S_1(x)]^2, \quad (2.13)$$

$$-i\partial^2f_1(x)+2[\partial S_V(x)+eA(\varphi)]\partial f_1(x) +2e(\gamma k)[\gamma dA(\varphi)/d\varphi]f_1(x)+2\partial S_1(x)\partial f_V(\varphi) +2e\Lambda(x)\partial f_V(\varphi) =-e\{\gamma\partial[\gamma\Lambda(x)]\}f_V(\varphi)-2\partial S_1(x)\partial f_1(x) -2e\Lambda(x)\partial f_1(x)-e\{\gamma\partial[\gamma\Lambda(x)]\}f_1(x). \quad (2.14)$$

The above-mentioned GEA corresponds to keeping in Eqs. (2.13) and (2.14) only the terms proportional to $U(\vec{r})=e\Lambda_0(\vec{r})$ $[U(\vec{r})$ is the potential energy of the particle in the electrostatic field]; i.e., the terms $\sim\Lambda_0^2$ and $\sim[\partial S_1(x)]^2$ are neglected. Consequently, we shall solve the equations

$$i(\partial_t^2-\vec{\nabla}^2)S_1(t,\vec{r})-2[\partial_tS_V(t,\vec{r})\partial_t-\vec{\nabla}S_V(t,\vec{r})\cdot\vec{\nabla} -e\vec{A}(\varphi)\cdot\vec{\nabla}]S_1(t,\vec{r})=2U(\vec{r})\partial_tS_V(t,\vec{r}), \quad (2.15)$$

$$i(\partial_t^2-\vec{\nabla}^2)f_1(t,\vec{r})-2[\partial_tS_V(t,\vec{r})\partial_t-\vec{\nabla}S_V(t,\vec{r})\cdot\vec{\nabla} -e\vec{A}(\varphi)\cdot\vec{\nabla}]f_1(t,\vec{r})+2e(\gamma k)[\vec{\gamma}\cdot d\vec{A}(\varphi)/d\varphi]f_1(t,\vec{r}) =[\vec{\gamma}\cdot\vec{\nabla}U(\vec{r})]\gamma_0f_V(\varphi)+2[\partial_tS_1(t,\vec{r})\partial_t -\vec{\nabla}S_1(t,\vec{r})\cdot\vec{\nabla}]f_V(\varphi)+2U(\vec{r})\partial_tf_V(\varphi). \quad (2.16)$$

To solve Eqs. (2.15) and (2.16) we turn from variables $t,\vec{r}$ to $\varphi,\vec{\eta}$,

$$\varphi=\omega t-\vec{k}\cdot\vec{r}, \quad \vec{\eta}=\vec{r}, \quad (2.17)$$

and make a Fourier transformation over $\vec{q}$

$$S_1(\varphi,\vec{\eta})=\frac{1}{(2\pi)^3}\int\tilde{S}_1(\varphi,\vec{q})\exp(i\vec{q}\cdot\vec{\eta})d\vec{q}, \quad (2.18)$$

$$f_1(\varphi,\vec{\eta})=\frac{1}{(2\pi)^3}\int \tilde{f}_1(\varphi,\vec{q})\exp(i\vec{q}\cdot\vec{\eta})d\vec{q}. \quad (2.19)$$

Then, using the Lorentz condition

$$kA(\varphi)=0, \quad (2.20)$$

we obtain the equations for the scalar $\tilde{S}(\varphi,\vec{q})$ and bispinor $\tilde{f}(\varphi,\vec{q})$ functions, respectively,

$$i\left(\frac{\vec{q}^2}{2}+\vec{q}\cdot[\vec{\nabla}S_V(t,\vec{r})-e\vec{A}(\varphi)]\right)\tilde{S}_1(\varphi,\vec{q}) +(kp-\vec{k}\cdot\vec{q})\partial_\varphi\tilde{S}_1(\varphi,\vec{q}) =\tilde{U}(\vec{q})\partial_t S_V(t,\vec{r}), \quad (2.21)$$

$$i\left(\frac{\vec{q}^2}{2}+\vec{q}\cdot[\vec{\nabla}S_V(t,\vec{r})-e\vec{A}(\varphi)] -\frac{ie}{2}(\gamma k)[\vec{\gamma}\cdot d\vec{A}(\varphi)/d\varphi]\right)\tilde{f}_1(\varphi,\vec{q}) +(kp-\vec{k}\cdot\vec{q})\partial_\varphi\tilde{f}_1(\varphi,\vec{q}) =\frac{i}{2}(\vec{\gamma}\cdot\vec{q})\gamma_0\tilde{U}(\vec{q})f_V(\varphi)+\tilde{U}(\vec{q})\partial_t f_V(\varphi) +\frac{\vec{k}\cdot\vec{q}}{\omega}\tilde{S}_1(\vec{q},\varphi)\partial_t f_V(\varphi), \quad (2.22)$$

where $\tilde{U}(\vec{q})=\int U(\vec{\eta})\exp(-i\vec{q}\cdot\vec{\eta})d\vec{\eta}$ is the Fourier transform of the function $U(\vec{r})$. We seek the solution of Eq. (2.21) in the form

$$\tilde{S}_1(\varphi,\vec{q})=s_I(\varphi,\vec{q})+s_{II}(\vec{q}), \quad (2.23)$$

where

$$s_I(-\infty,\vec{q})=0 \quad (2.24)$$

and $s_{II}(\vec{q})$ is the action of the particle corresponding to the elastic scattering in the potential field in the absence of an EM wave [the solution of Eq. (2.21) at $\vec{A}(\varphi)=\vec{0}$]

$$s_{II}(\vec{q})=\frac{2i\varepsilon\tilde{U}(\vec{q})}{\vec{q}^2+2\vec{p}\cdot\vec{q}}. \quad (2.25)$$

Then, for $\tilde{S}_1(\varphi,\vec{q})$ we have the expression

$$\tilde{S}_1(\varphi,\vec{q})=\frac{2i\varepsilon\tilde{U}(\vec{q})}{\vec{q}^2+2\vec{p}\cdot\vec{q}}\left[1-ie^{-iB(\varphi,\vec{q})}\int_{-\infty}^{\varphi}e^{iB(\varphi',\vec{q})} \times(\vec{\nabla}S_V(x)-\vec{p}-e\vec{A}(\varphi'))\cdot\vec{q}\,d\varphi'\right] +\frac{e\omega\tilde{U}(\vec{q})}{(kp-\vec{k}\cdot\vec{q})kp}e^{-iB(\varphi,\vec{q})}\int_{-\infty}^{\varphi}e^{iB(\varphi',\vec{q})}[\vec{p}\cdot\vec{A}(\varphi') -e\vec{A}^2(\varphi')/2]d\varphi', \quad (2.26)$$

where the function $B(\varphi,\vec{q})$ is defined as

$$B(\varphi,\vec{q})=\int\left(\frac{\vec{q}^2}{2}+\vec{q}\cdot[\vec{\nabla}S_V(x)-e\vec{A}(\varphi')]\right)\frac{d\varphi'}{kp-\vec{k}\cdot\vec{q}}. \quad (2.27)$$

Making the inverse Fourier transformation of $\tilde{S}_1(\varphi,\vec{q})$ and then turning to the previous variables $(t,\vec{r})$, after simple calculations we obtain the following expression for the scalar part of a particle wave function:

$$S_1(t,\vec{r})=\frac{1}{(2\pi)^3}\int\frac{\tilde{U}(\vec{q})e^{i\vec{q}\cdot\vec{r}}e^{-iB(\varphi,\vec{q})}}{kp-\vec{k}\cdot\vec{q}} \times\int_{-\infty}^{\varphi}e^{iB(\varphi',\vec{q})}\left(-\varepsilon+\frac{e\omega}{kp}[\vec{p}\cdot\vec{A}(\varphi') -e\vec{A}^2(\varphi')/2]\right)d\varphi'd\vec{q}. \quad (2.28)$$

In a similar way, seeking the bispinor function $\tilde{f}_1(\varphi,\vec{q})$ in the form

$$\tilde{f}_1(\varphi,\vec{q})=g_I(\varphi,\vec{q})+g_{II}(\vec{q}), \quad (2.29)$$

where

$$g_I(-\infty,\vec{q})=0 \quad (2.30)$$

and

$$g_{II}(\vec{q})=\frac{(\vec{\gamma}\cdot\vec{q})\gamma_0\tilde{U}(\vec{q})u}{\vec{q}^2+2\vec{p}\cdot\vec{q}} \quad (2.31)$$

[$g_{II}(\vec{q})$ is the spin part of the particle wave function at the elastic scattering in the potential field: the solution of Eq. (2.22) at $\vec{A}(\varphi)=\vec{0}$], we obtain the expression for $\tilde{f}_1(\varphi,\vec{q})$,

$$\widetilde{f}_1(\varphi,\vec{q})=\frac{(\vec{\gamma}\cdot\vec{q})\gamma_0\widetilde{U}(\vec{q})u}{\vec{q}^2+2\vec{p}\cdot\vec{q}}\Bigg[1-\frac{ie^{-iB_1(\varphi,\vec{q})}}{kp-\vec{k}\cdot\vec{q}}\int_{-\infty}^{\varphi}e^{iB_1(\varphi',\vec{q})}\times\Bigg(\vec{q}\cdot[\vec{\nabla}S_V(x)-\vec{p}-e\vec{A}(\varphi')]-\frac{ie}{2}(\gamma k)[\vec{\gamma}\cdot d\vec{A}(\varphi')/d\varphi']\Bigg)d\varphi'\Bigg]+\frac{e^{-iB_1(\varphi,\vec{q})}}{kp-\vec{k}\cdot\vec{q}}\int_{-\infty}^{\varphi}e^{iB_1(\varphi',\vec{q})}\times\Bigg[[\omega\widetilde{U}(\vec{q})+i(\vec{k}\cdot\vec{q})\widetilde{S}_1(\varphi',\vec{q})]\partial_{\varphi'}f_V(\varphi')-\frac{ie(\vec{\gamma}\cdot\vec{q})\gamma_0\widetilde{U}(\vec{q})}{2kp}(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi')]u\Bigg]d\varphi'. \quad (2.32)$$

The function $B_1(\varphi,\vec{q})$ in Eq. (2.32) is defined as

$$B_1(\varphi,\vec{q})=B(\varphi,\vec{q})-\frac{ie(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]}{2(kp-\vec{k}\cdot\vec{q})}. \quad (2.33)$$

As the terms over the first power of $(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]$ are equal to zero [in accordance with the condition (2.20)] (see Ref. [20]), $\exp[iB_1(\vec{q},\varphi)]$ can be written

$$e^{iB_1(\vec{q},\varphi)}=e^{iB(\vec{q},\varphi)}\left(1+\frac{e(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]}{2(kp-\vec{k}\cdot\vec{q})}\right). \quad (2.34)$$

So, after the inverse Fourier transformation and turning to the previous variables we have such an expression for $f_1(t,\vec{r})$,

$$f_1(t,\vec{r})=\frac{i}{16\pi^3}\int\frac{e^{i\vec{q}\cdot\vec{r}}e^{-iB(\varphi,\vec{q})}}{kp-\vec{k}\cdot\vec{q}}\int_{-\infty}^{\varphi}e^{iB(\varphi',\vec{q})}\times\Bigg\{\Bigg[1+\frac{e(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi')-\vec{\gamma}\cdot\vec{A}(\varphi)]}{2(kp-\vec{k}\cdot\vec{q})}\Bigg]\times(\vec{\gamma}\cdot\vec{q})\gamma_0\widetilde{U}(\vec{q})f_V(\varphi')-i2\omega\widetilde{U}(\vec{q})\partial_{\varphi'}f_V(\varphi')+\widetilde{U}(\vec{q})\Bigg[-\varepsilon+\frac{e\omega}{kp}[\vec{p}\cdot\vec{A}(\varphi')-e\vec{A}^2(\varphi')/2]\Bigg]\times\frac{\vec{k}\cdot\vec{q}}{kp(kp-\vec{k}\cdot\vec{q})}[\vec{\gamma}\cdot\vec{A}(\varphi')-\vec{\gamma}\cdot\vec{A}(\varphi)]u\Bigg\}d\varphi'd\vec{q}. \quad (2.35)$$

Then we assume the EM wave to be quasimonochromatic and of an arbitrary polarization with the vector potential

$$\vec{A}(\varphi)=A_0(\varphi)(\hat{\vec{e}}_1\cos\xi\cos\varphi+\hat{\vec{e}}_2\sin\xi\sin\varphi), \quad (2.36)$$

where $A_0(\varphi)$ is the slow varying amplitude of the vector potential $\vec{A}(t,\vec{r})$, $\hat{\vec{e}}_1$ and $\hat{\vec{e}}_2$ are unit vectors perpendicular to each other and to the wave vector $\vec{k}$ ($\hat{\vec{e}}_1\cdot\hat{\vec{e}}_2=0,\hat{\vec{e}}_1\cdot\vec{k}=\hat{\vec{e}}_2\cdot\vec{k}=0$, and $|\hat{\vec{e}}_1|=|\hat{\vec{e}}_2|=1$), and ξ is the polarization angle.

It is useful to introduce a new function $J_n(u,v,\Delta)$, the definition and principal properties of which are given in Appendix A. Utilizing the formula (A9) for the expansion over the functions $J_n(u,v,\Delta)$,

$$\exp[-i\alpha_1\sin(\varphi-\theta_1)+i\alpha_2\sin 2\varphi]=\sum_{n=-\infty}^{\infty}J_n(\alpha_1,-\alpha_2,\theta_1)\exp[-in(\varphi-\theta_1)], \quad (2.37)$$

we carry out the integration over φ' in the expression (2.28). Then we obtain

$$S_1(t,\vec{r})=\frac{i}{4\pi^3}\sum_{n=-\infty}^{\infty}e^{-in\varphi}\int\frac{\widetilde{U}(\vec{q})\{(\varepsilon+\omega Z)D_n-\omega[\alpha(\vec{p})D_{1,n}(\theta(\vec{p}))-Z\cos 2\xi D_{2,n}]\}}{\vec{q}^2+2\vec{p}\cdot\vec{q}+2Z\vec{k}\cdot\vec{q}-2n(kp-\vec{k}\cdot\vec{q})-i0}\times\exp(i\{\vec{q}\cdot\vec{r}+\alpha_1(\vec{q})\sin[\varphi-\theta_1(\vec{q})]-\alpha_2(\vec{q})\sin 2\varphi+\theta_1(\vec{q})n\})d\vec{q}, \quad (2.38)$$

where $\alpha_1(\vec{q})$ is the parameter of the Dirac particle interaction with both scattering and EM wave fields simultaneously

$$\alpha_1(\vec{q})=\frac{e\overline{A_0}\eta(\vec{q})}{kp-\vec{k}\cdot\vec{q}}, \quad (2.39)$$

where $\overline{A_0}$ is the average value of $A_0(\varphi)$ and Z is the relative parameter of the wave intensity defined as

$$Z=\frac{e^2\overline{A_0^2}}{4kp} \quad (2.40)$$

and $\alpha_2(\vec{q})$ has the form

$$\alpha_2(\vec{q})=\frac{\vec{k}\cdot\vec{q}}{2(kp-\vec{k}\cdot\vec{q})}Z\cos 2\xi. \quad (2.41)$$

Then the magnitudes of $\eta(\vec{q})$ and $\theta_1(\vec{q})$ are

$$\eta(\vec{q})=\left\{\left[\left(\frac{(\vec{k}\cdot\vec{q})\vec{p}}{kp}+\vec{q}\right)\cdot\hat{\vec{e}}_1\right]^2\cos^2\xi+\left[\left(\frac{(\vec{k}\cdot\vec{q})\vec{p}}{kp}+\vec{q}\right)\cdot\hat{\vec{e}}_2\right]^2\sin^2\xi\right\}^{1/2}, \quad (2.42)$$

$$\theta_1(\vec{q})=\arctan\left(\frac{\left(\frac{(\vec{k}\cdot\vec{q})\vec{p}}{kp}+\vec{q}\right)\cdot\hat{\vec{e}}_2}{\left(\frac{(\vec{k}\cdot\vec{q})\vec{p}}{kp}+\vec{q}\right)\cdot\hat{\vec{e}}_1}\tan\xi\right) \quad (2.43)$$

and $\alpha(\vec{p})$ is the intensity-dependent amplitude

$$\alpha(\vec{p})=\frac{e\overline{A}_0}{kp}\sqrt{(\vec{p}\cdot\hat{\vec{e}}_1)^2\cos^2\xi+(\vec{p}\cdot\hat{\vec{e}}_2)^2\sin^2\xi}, \quad (2.44)$$

with the phase angle

$$\theta(\vec{p})=\arctan\left(\frac{\vec{p}\cdot\hat{\vec{e}}_2}{\vec{p}\cdot\hat{\vec{e}}_1}\tan\xi\right). \quad (2.45)$$

The functions D_n, $D_{1,n}(\theta(\vec{p}))$, and $D_{2,n}$ are defined by the expressions

$$\sum_{n=-\infty}^{\infty}\exp[-in(\varphi-\theta_1)]J_n(\alpha_1,-\alpha_2,\theta_1)\begin{cases}1\\ \cos[\varphi-\theta(\vec{p})]\\ \cos 2\varphi\end{cases}=\sum_{n=-\infty}^{\infty}\exp[-in(\varphi-\theta_1)]\begin{cases}D_n\\ D_{1,n}(\theta(\vec{p})),\\ D_{2,n}\end{cases} \quad (2.46)$$

so they are satisfied for the relations

$$D_n=J_n(\alpha_1,-\alpha_2,\theta_1),$$

$$D_{1,n}(\theta(\vec{p}))=\frac{1}{2}[J_{n-1}(\alpha_1,-\alpha_2,\theta_1)e^{-i[\theta_1-\theta(\vec{p})]}+J_{n+1}(\alpha_1,-\alpha_2,\theta_1)e^{i[\theta_1-\theta(\vec{p})]}], \quad (2.47)$$

$$D_{2,n}=\frac{1}{2}[J_{n-2}(\alpha_1,-\alpha_2,\theta_1)e^{-i2\theta_1}+J_{n+2}(\alpha_1,-\alpha_2,\theta_1)e^{i2\theta_1}].$$

In the denominator of the integral in expression (2.38) $-i0$ is an imaginary infinitesimal, which shows how the path around the pole in the integrand should be chosen to obtain a certain asymptotic behavior of the wave function, i.e., the outgoing spherical wave [to determine that one must pass to the limit of the Born approximation at $\vec{A}(\varphi)=\vec{0}$].

Using Eq. (2.5), the approximate solution of Eq. (2.1) can be written as

$$\Phi(x)=\frac{1}{\sqrt{2\varepsilon}}[f_V(\varphi)+f_1(x)]\exp[iS_V(x)+iS_1(x)], \quad (2.48)$$

where the spin part $|f_1(x)|\ll|f_V(\varphi)|$ [the final analytic form of $f_1(x)$ is presented in Eq. (B2)] and $S_1(t,\vec{r})$ is presented by Eq. (2.38). Note that the wave function is normalized for the one particle in the unit volume.

Inserting the expression (2.48) for $\Phi(x)$ into Eq. (2.3) and keeping terms to first order of the potential $\Lambda_0(\vec{r})$, we obtain the solution $\Psi(x)$ of the Dirac equation (2.1) in the applied approximation, which coincides with Eq. (2.48). So the bispinor function $\Phi(x)$ is the solution of the Dirac equation in the GEA.

III. DISCUSSION OF THE GEA WAVE FUNCTION IN VARIOUS LIMITS

Formula (2.38) has been obtained in the GEA under the condition that

$$|\vec{\nabla}S_1(\vec{r})|^2\ll|(\varepsilon+\omega Z)U(\vec{r})|. \quad (3.1)$$

To estimate the latter let us evaluate the expression $\vec{\nabla}S_1$ using the formulas (2.39) and (2.42). Then we fix n in the denominator of the expression (2.38) at the most probable value $\overline{n}$ for the action $S_1(\vec{r},t)$. At the circular polarization of the wave the function $J_n(\alpha_1,-\alpha_2,\theta_1)$ turns into the Bessel function $J_n(\alpha_1)$ in accordance with the determination by infinite series representation (A2). Then, to determine the value of $\overline{n}$ we use the argumentation of Choudhury [22] according to which the Bessel function $J_n(z)$ gets its largest value when its index n is roughly equal to its argument

$$\overline{n}(\vec{q})=\langle\alpha_1(\vec{q})\rangle, \quad (3.2)$$

where $\langle\alpha_1(\vec{q})\rangle$ denotes integer value of $\alpha_1(\vec{q})$. This approximative estimation of the Bessel function can be verified by the diagrams of Jahnke and Emde [23]. Then carrying out the summation of n in the formula (2.38), we obtain

$$S_1\approx 2i(\varepsilon+\omega Z)\int\frac{\tilde{U}(\vec{q})e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2+2\vec{p}\cdot\vec{q}+2Z\vec{k}\cdot\vec{q}-2\overline{n}(kp-\vec{k}\cdot\vec{q})-i0}\frac{d\vec{q}}{(2\pi)^3}. \quad (3.3)$$

From the expressions (3.3) and (3.1) the condition of the GEA can be presented in the general form

$$2(\varepsilon+\omega Z)\left|\int\frac{\vec{q}\tilde{U}(\vec{q})e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2+2\vec{p}\cdot\vec{q}+2Z\vec{k}\cdot\vec{q}-2\overline{n}(kp-\vec{k}\cdot\vec{q})-i0}\frac{d\vec{q}}{(2\pi)^3}\right|^2\ll|U(a)|. \tag{3.4}$$

Due to the oscillations of the factor $e^{i\vec{q}\cdot\vec{r}}$ in the integral in Eq. (3.4), the main contribution is in the region where $\vec{q}\cdot\vec{r}\cong 1$, i.e., $|\vec{q}|\simeq|\vec{q}_{eff}|=a^{-1}$, where a is the dimension of the effective range of the scattering potential $\Lambda_0(\vec{r})$. Therefore, the condition (3.4) can be written as

$$\frac{2(\varepsilon+\omega Z)\vec{q}_{eff}^2}{[\vec{q}_{eff}^2+2\vec{p}\cdot\vec{q}_{eff}+2Z\vec{k}\cdot\vec{q}_{eff}-2\overline{n}(kp-\vec{k}\cdot\vec{q}_{eff})]^2}|U(a)|\ll 1. \tag{3.5}$$

The $\overline{n}$ included in the formula (3.5) is the most probable number of photons that is defined by expressions (3.2), (2.39), and (2.42):

$$\overline{n}=\left\langle\frac{e\overline{A}_0\overline{\eta}}{kp-\vec{k}\cdot\vec{q}_{eff}}\left|\frac{(\vec{k}\cdot\vec{q}_{eff})\vec{p}}{kp}+\vec{q}_{eff}\right|\right\rangle, \tag{3.6}$$

$$\overline{\eta}=\sqrt{\left(\frac{\vec{p}'}{p'}\cdot\hat{\vec{e}}_1\right)^2\cos^2\xi+\left(\frac{\vec{p}'}{p'}\cdot\hat{\vec{e}}_2\right)^2\sin^2\xi},$$

$$\vec{p}'=\frac{\vec{p}}{kp}+\frac{\vec{q}_{eff}}{\vec{k}\cdot\vec{q}_{eff}}, \tag{3.7}$$

where $p'=|\vec{p}'|$.

Finally, the condition of applicability of the GEA (3.1) may be written in the form

$$|U(a)|\ll\frac{1}{\Pi_0}\left[\frac{1}{a}+|\vec{\Pi}|-\frac{e\overline{A}_0}{1-v\cos\widehat{\vec{k}\cdot\vec{p}}}\right]^2, \tag{3.8}$$

where $v=|\vec{p}|/\varepsilon$ is the particle velocity and

$$\Pi_0=\varepsilon+\omega Z,\quad \vec{\Pi}=\vec{p}+\vec{k}Z \tag{3.9}$$

are the average values of the particle energy and momentum in the EM field that correspond to the average four-kinetic momentum or ''quasimomentum'' Π of the particle in the wave ($\Pi^2=m_*^2\equiv m^2+e^2\overline{A}^2$, where m_* is the ''effective mass'' of the particle).

The wave function (2.48) in the GEA turns into the wave function in the Born approximation by the scattering potential if

$$|S_1(\vec{r},t)|\ll 1. \tag{3.10}$$

By expanding the second term in the first exponent in the formula (2.48) into a series and keeping only the terms to the first order in $\Lambda_0(\vec{r})$, we obtain

$$\begin{aligned}\Psi_B(x)=\frac{1}{\sqrt{2\varepsilon}}\exp[iS_V(x)]\Bigg\{&f_V(\varphi)+f_1(x)\\&-\frac{1}{4\pi^3}f_V(\varphi)\sum_{n=-\infty}^{\infty}e^{-in\varphi}\int\frac{\tilde{U}(\vec{q})\{(\varepsilon+\omega Z)D_n-\omega[\alpha(\vec{p})D_{1,n}(\theta(\vec{p}))-Z\cos 2\xi D_{2,n}]\}}{\vec{q}^2+2\vec{p}\cdot\vec{q}+2Z\vec{k}\cdot\vec{q}-2n(kp-\vec{k}\cdot\vec{q})-i0}\\&\times\exp(i\{\vec{q}\cdot\vec{r}+\alpha_1(\vec{q})\sin[\varphi-\theta_1(\vec{q})]-i\alpha_2(\vec{q})\sin 2\varphi+\theta_1(\vec{q})n\})d\vec{q}\Bigg\}.\end{aligned} \tag{3.11}$$

The condition when the wave function (3.11) is valid can be written using Eq. (3.10) taking into account Eqs. (3.3) and (3.9):

$$|U(a)|\ll\frac{1}{\Pi_0 a}\left|\frac{1}{a}+|\vec{\Pi}|-\frac{e\overline{A}_0}{1-v\cos\widehat{\vec{k}\cdot\vec{p}}}\right|, \tag{3.12}$$

where $\widehat{\vec{k}\cdot\vec{p}}$ denotes the angle between $\vec{k}$ and $\vec{p}$ vectors. This criterion of validity of the particle wave function of the stimulated scattering in the Born approximation by potential field includes both ''fast'' and ''slow'' particles (in the EM field) cases. Thus, for the fast particles, when $\big||\vec{\Pi}|-e\overline{A}_0/(1-v\cos\widehat{\vec{k}\cdot\vec{p}})\big|a\gg 1$, we have

$$|U(a)|\ll\frac{1}{\Pi_0 a}\left||\vec{\Pi}|-\frac{e\overline{A}_0}{1-v\cos\widehat{\vec{k}\cdot\vec{p}}}\right|. \tag{3.13}$$

From the condition (3.12) for the slow particles, when $\big||\vec{\Pi}|-e\overline{A}_0/(1-v\cos\widehat{\vec{k}\cdot\vec{p}})\big|a\ll 1$, we obtain the strong criterion of Born approximation for SB:

$$|U(a)| \ll \frac{1}{\Pi_0 a^2}. \tag{3.14}$$

Comparing the condition of applicability of the GEA (3.8) and the conditions in the Born approximation (3.13) and (3.14) we see that for the fast particles (in strong laser fields) the wave function obtained in the GEA (2.48) describes the stimulated scattering in regions $||\vec{\Pi}| - e\overline{A}_0/(1-v \times \cos\widehat{\vec{k}\cdot\vec{p}})|a||\vec{\Pi}| - e\overline{A}_0/(1-v\cos\widehat{\vec{k}\cdot\vec{p}})|a \gg 1$ times larger than the wave function in the Born approximation.

Now let us find the asymptote of the electron wave function corresponding to the Born approximation at $r\to+\infty$ and justify the chosen sign at the infinitesimal $i0$ to the path around the pole in the integrals (2.38) and (B2). From the expression (3.11) we have

$$\Psi_B(\vec{r},t) = \frac{\exp[iS_V(x)]}{\sqrt{2\varepsilon}}\left\{ f_V(\varphi) + \frac{1}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\int \frac{\exp[i\vec{q}\cdot\vec{r}]F_n(\varphi,\vec{q})d\vec{q}}{\vec{q}^2 + 2\vec{p}\cdot\vec{q} + 2Z\vec{k}\cdot\vec{q} - 2n(kp - \vec{k}\cdot\vec{q}) - i0}\right\}, \tag{3.15}$$

where the function $F_n(\varphi,\vec{q})$

$$\begin{aligned} F_n(\varphi,\vec{q}) = {}& \tilde{U}(\vec{q})\exp(i\{\alpha_1(\vec{q})\sin[\varphi-\theta_1(\vec{q})] - \alpha_2(\vec{q})\sin 2\varphi + \theta_1(\vec{q})n - n\varphi\}) \\ & \times\Bigg\{\Bigg[D_n\left((\vec{\gamma}\cdot\vec{q})\gamma_0 - \frac{e\omega\left[\vec{q}^2 + 2\vec{p}\cdot\vec{q} - 2\frac{\varepsilon}{\omega}\vec{k}\cdot\vec{q}\right]}{2(kp-\vec{k}\cdot\vec{q})kp}(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]\right) + \frac{e\overline{A}_0(\gamma k)(\vec{\gamma}\cdot\vec{D})_{3,n}(\vec{\gamma}\cdot\vec{q})\gamma_0}{2(kp-\vec{k}\cdot\vec{q})} \\ & + \frac{e\overline{A}_0}{2kp}\left[\frac{\omega\left[\vec{q}^2 + 2\vec{p}\cdot\vec{q} - 2\frac{\varepsilon}{\omega}\vec{k}\cdot\vec{q}\right]}{kp-\vec{k}\cdot\vec{q}} - (\vec{\gamma}\cdot\vec{q})\gamma_0\right](\gamma k)(\vec{\gamma}\cdot\vec{D})_{3,n} + \frac{\omega e\alpha(\vec{q})(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]}{kp-\vec{k}\cdot\vec{q}}D_{1,n}(\theta(\vec{q})) \\ & + \frac{(\vec{k}\cdot\vec{q})(\gamma k)\gamma_0 - \omega(\gamma k)(\vec{\gamma}\cdot\vec{q})}{kp-\vec{k}\cdot\vec{q}}Z(D_n + \cos 2\xi D_{2,n})\Bigg]u \\ & - 2f_V(\varphi)\{(\varepsilon+\omega Z)D_n - \omega[\alpha(\vec{p})D_{1,n}(\theta(\vec{p})) - Z\cos 2\xi D_{2,n}]\}\Bigg\}. \end{aligned} \tag{3.16}$$

To calculate the asymptote of the function (3.15) we temporarily direct the Oq_z coordinate axis along $\vec{r}$ and replace the integration variable $\vec{q}$ by $\vec{p}\,' = \vec{\Pi} + n\vec{k} + \vec{q}$. Turning to spherical coordinates, we carry out the integration over the solid angle by the formula

$$\exp(i\vec{p}\,'\cdot\vec{r})|_{r\to\infty} \Rightarrow \frac{2\pi}{ip'r}[\delta(\hat{\vec{p}}\,' - \hat{\vec{r}})\exp(ip'r) - \delta(\hat{\vec{p}}\,' + \hat{\vec{r}})\exp(-ip'r)], \tag{3.17}$$

where $\hat{\vec{p}}\,'$, $\hat{\vec{r}}$ are unit vectors along $\vec{p}\,'$ and $\vec{r}$, respectively. Then we carry out the integration over p' in the complex plane, passing above the pole $p' = -p_n$ and below the pole $p' = p_n$, where

$$p_n = \sqrt{\vec{\Pi}^2 + n\omega(2\Pi_0 + n\omega)} \tag{3.18}$$

[this path corresponds to chosen sign of the infinitesimal $(-i0)$ in the denominator of the integrand]. As a result, at $r\to\infty$ we obtain

$$\Psi_B(\vec{r},t) = \frac{\exp[iS_V(x)]}{\sqrt{2\varepsilon}} \times\left\{ f_V(\varphi) + \frac{\exp[-i\vec{\Pi}\cdot\vec{r}]}{4\pi r}\sum_{n=n_0}^{\infty} e^{i(p_n\hat{\vec{r}} - n\vec{k})\cdot\vec{r}} \times F_n(\varphi, p_n\hat{\vec{r}} - \vec{\Pi} - n\vec{k})\right\}, \tag{3.19}$$

where $F_n(\varphi, p_n\hat{\vec{r}} - \vec{\Pi} - n\vec{k})$ is defined by Eq. (3.16). Summation of n is carried out with $n_0 = \langle(-\Pi_0 + m_*)/\omega\rangle$.

As it is seen from the expression (3.19), the asymptotic wave function at $n=0$ [if $\vec{A}(\varphi)\equiv\vec{0}$], corresponding to elastic scattering of the electron in the Born approximation, describes the outgoing spherical wave at large distances, according to which the sign of the infinitesimal $i0$ in the poles of the integrals (2.38) and (B2) was chosen. Finally, note that in the nonrelativistic limit the GEA wave function (2.48) obtained reduces to the corresponding one of Ref. [2] and at $\vec{A}(\varphi)\equiv\vec{0}$ it becomes the relativistic wave function of the elastic scattering on an arbitrary electrostatic potential (see Ref. [1]).

IV. CONCLUSION

In the current work the quantum description of relativistic particles induced by bremsstrahlung on an arbitrary electrostatic potential in the field of a strong EM wave is developed. Compared to existing approximations, the wave function (2.48) obtained by solving the Dirac equation has a wide range of applicability. The essence of the approximation is that quadratic scattering potential terms $[\sim U^2(\vec{r})]$ are considered small. Here it also must be taken into account that in reality, in the SB process both the scattering and wave fields are limited from the top by the corresponding values of the fields when the possible process of electron-positron pair production may occur. Consequently, it may be considered that the wave function obtained describes the SB process of a Dirac particle when the spin interaction is also taken into account with high accuracy. In addition, though we obtain a wave function with a so-called general eikonal approximation, in reality even the wave function of the ordinary eikonal approximation for the Dirac particles in the SB process is not known.

From the formula (2.48) we obtain the wave function (3.19) in the Born approximation from the scattering potential $U(\vec{r})$. In boundary cases the GEA wave function obtained reduces to the nonrelativistic GEA wave function (see Ref. [2]) and becomes the relativistic one of Ref. [1] in the absence of an EM wave. Such a wave function (2.48) is important for a more accurate description of the above-threshold ionization of atoms, especially when clarifying the process of stabilization, as considered in Ref. [13]. The calculation of the relativistic multiphoton cross sections of the SB process and the calculation of the probabilities of hydrogenlike atoms in multiphoton ionization in the above threshold regime with this GEA wave function are distinct problems that will be considered in the future.

APPENDIX A: DEFINITION OF THE FUNCTION $J_n(u,v,\Delta)$

A function $J_n(u,v,\Delta)$ may be defined by

$$J_n(u,v,\Delta)=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\exp\{i[u\sin(\theta+\Delta)+v\sin 2\theta-n(\theta+\Delta)]\} \tag{A1}$$

or by an infinite series representation

$$J_n(u,v,\Delta)=\sum_{k=-\infty}^{\infty}e^{-i2k\Delta}J_{n-2k}(u)J_k(v). \tag{A2}$$

Both defining relations are equivalent. From either Eq. (A1) or (A2) it follows that

$$J_n(u,0,\Delta)=J_n(u) \tag{A3}$$

and

$$J_n(0,v,\Delta)=\begin{cases}e^{-i\Delta n}J_{n/2}(v) & \text{for } n \text{ even}\\ 0 & \text{for } n \text{ odd.}\end{cases} \tag{A4}$$

Then we have directly the relative formulas

$$\begin{aligned}J_n(-u,v,\Delta)&=(-1)^nJ_n(u,v,\Delta),\\ J_n(u,-v,\Delta)&=(-1)^nJ_{-n}(u,v,-\Delta),\\ J_n(u,-v,-\Delta)&=(-1)^nJ_{-n}(u,-v,\Delta).\end{aligned} \tag{A5}$$

Then from the well known recurrence relations for the Bessel functions we have

$$J_{n-1}(u,v,\Delta)-J_{n+1}(u,v,\Delta)=2\partial_uJ_n(u,v,\Delta) \tag{A6}$$

and

$$e^{-i2\Delta}J_{n-2}(u,v,\Delta)-e^{i2\Delta}J_{n+2}(u,v,\Delta)=2\partial_vJ_n(u,v,\Delta), \tag{A7}$$

which follow directly from Eq. (A1) or (A2).

An integration by parts in Eq. (A1) yields the relation

$$\begin{aligned}2nJ_n(u,v,\Delta)=&u[J_{n-1}(u,v,\Delta)+J_{n+1}(u,v,\Delta)]\\ &+2v[e^{-i2\Delta}J_{n-2}(u,v,\Delta)\\ &+e^{i2\Delta}J_{n+2}(u,v,\Delta)].\end{aligned} \tag{A8}$$

Other results can be obtained by a combination of Eqs. (A2)–(A8). We perform two important theorems, which can be proved by Eq. (A1). The first is

$$\sum_{n=-\infty}^{\infty}e^{in(\varphi+\Delta)}J_n(u,v,\Delta)=\exp\{i[u\sin(\varphi+\Delta)+v\sin 2\varphi]\} \tag{A9}$$

and the other is

$$\sum_{k=-\infty}^{\infty}J_{n\mp k}(u,v,\Delta)J_k(u',v',\pm\Delta)=J_n(u\pm u',v\pm v',\Delta). \tag{A10}$$

Then the function $J_n(u,v,\Delta)$ at $\Delta=0$ becomes the generalized Bessel function $J_n(u,v)$, which was induced by Reiss (see Ref. [21]).

APPENDIX B: SPIN PART OF THE WAVE FUNCTION

After the integration by parts and simple transformation of $f_1(t,\vec{r})$ in the expression (2.35) we obtain the final form

$$f_1(t,\vec{r})=\frac{i}{16\pi^3}\int \frac{\tilde{U}(\vec{q})\exp\left(i\vec{q}\cdot\vec{r}-iB(\varphi,\vec{q})-\dfrac{e(\gamma k)(\vec{\gamma}\cdot\vec{A}(\varphi))}{2(kp-\vec{k}\cdot\vec{q})}\right)}{kp-\vec{k}\cdot\vec{q}}\int_{-\infty}^{\varphi} e^{iB(\varphi',\vec{q})}$$
$$\times\left\{(\vec{\gamma}\cdot\vec{q})\gamma_0+\frac{e}{2}\left[\frac{(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi')](\vec{\gamma}\cdot\vec{q})\gamma_0}{kp-\vec{k}\cdot\vec{q}}-\frac{(\vec{\gamma}\cdot\vec{q})\gamma_0(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi')]}{kp}\right]\right.$$
$$+\frac{e\omega\left[\vec{q}^2+2\vec{p}\cdot\vec{q}-2\dfrac{\varepsilon}{\omega}\vec{k}\cdot\vec{q}\right]}{2(kp-\vec{k}\cdot\vec{q})kp}\{(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi')]-(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]\}+\frac{\omega e^2(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]}{(kp-\vec{k}\cdot\vec{q})kp}[\vec{q}\cdot\vec{A}(\varphi')]$$
$$\left.+\frac{(\vec{k}\cdot\vec{q})(\gamma k)\gamma_0-\omega(\gamma k)(\vec{\gamma}\cdot\vec{q})}{2(kp-\vec{k}\cdot\vec{q})kp}e^2\vec{A}^2(\varphi')\right\}u\,d\varphi'\,d\vec{q}. \tag{B1}$$

We assume an EM wave to be monochromatic and of an arbitrary polarization with the vector potential in the form (2.36). Using the formula (A9) for the expansion by the functions $J_n(u,v,\theta)$, we carry out the integration over φ' in the expression (B1). Then we obtain

$$f_1(t,\vec{r})=\frac{1}{(2\pi)^3}\sum_{n=-\infty}^{\infty}e^{-in\varphi}\int\frac{\tilde{U}(\vec{q})\exp(i\{\vec{q}\cdot\vec{r}+\alpha_1(\vec{q})\sin[\varphi-\theta_1(\vec{q})]-\alpha_2(\vec{q})\sin 2\varphi+\theta_1(\vec{q})n\})}{\vec{q}^2+2\vec{p}\cdot\vec{q}+2Z\vec{k}\cdot\vec{q}-2n(kp-\vec{k}\cdot\vec{q})-i0}$$
$$\times\left\{D_n\left[(\vec{\gamma}\cdot\vec{q})\gamma_0-\frac{e\omega\left[\vec{q}^2+2\vec{p}\cdot\vec{q}-2\dfrac{\varepsilon}{\omega}\vec{k}\cdot\vec{q}\right]}{2(kp-\vec{k}\cdot\vec{q})kp}(\gamma k)(\vec{\gamma}\cdot\vec{A}(\varphi))\right]\right.$$
$$+\frac{e\overline{A}_0(\gamma k)(\vec{\gamma}\cdot\vec{D})_{3,n}(\vec{\gamma}\cdot\vec{q})\gamma_0}{2(kp-\vec{k}\cdot\vec{q})}+\frac{e\overline{A}_0}{2kp}\left[\frac{\omega\left[\vec{q}^2+2\vec{p}\cdot\vec{q}-2\dfrac{\varepsilon}{\omega}\vec{k}\cdot\vec{q}\right]}{kp-\vec{k}\cdot\vec{q}}-(\vec{\gamma}\cdot\vec{q})\gamma_0\right](\gamma k)(\vec{\gamma}\cdot\vec{D})_{3,n}$$
$$\left.+\frac{\omega e\alpha(\vec{q})(\gamma k)[\vec{\gamma}\cdot\vec{A}(\varphi)]}{kp-\vec{k}\cdot\vec{q}}D_{1,n}(\theta(\vec{q}))+\frac{(\vec{k}\cdot\vec{q})(\gamma k)\gamma_0-\omega(\gamma k)(\vec{\gamma}\cdot\vec{q})}{kp-\vec{k}\cdot\vec{q}}Z(D_n+\cos 2\xi D_{2,n})\right\}u\,d\vec{q}, \tag{B2}$$

where $\alpha_1(\vec{q})$, $\alpha_2(\vec{q})$, Z, $\eta(\vec{q})$, $\theta_1(\vec{q})$, $\alpha(\vec{q})$, $\theta(\vec{q})$, D_n, $D_{1,n}(\theta(\vec{q}))$, and $D_{2,n}$ are determined by the formulas (2.39)–(2.47) and by the definitions

$$(\vec{\gamma}\cdot\vec{D})_{3,n}\equiv\frac{(\vec{\gamma}\cdot\hat{\vec{e}}_1)\cos\xi+i(\vec{\gamma}\cdot\hat{\vec{e}}_2)\sin\xi}{2}J_{n-1}(\alpha_1,-\alpha_2,\theta_1)e^{-i\theta_1(\vec{q})}$$
$$+\frac{(\vec{\gamma}\cdot\hat{\vec{e}}_1)\cos\xi-i(\vec{\gamma}\cdot\hat{\vec{e}}_2)\sin\xi}{2}J_{n+1}(\alpha_1,-\alpha_2,\theta_1)e^{i\theta_1(\vec{q})},$$
$$\alpha(\vec{q})=\frac{e\overline{A}_0}{kp}\sqrt{(\vec{q}\cdot\hat{\vec{e}}_1)^2\cos^2\xi+(\vec{q}\cdot\hat{\vec{e}}_2)^2\sin^2\xi}. \tag{B3}$$

In the denominator of the integral in expression (B2) $-i0$ is an imaginary infinitesimal, chosen to obtain a certain asymptotic behavior of the wave function, i.e., the outgoing spherical wave [to determine that we pass to the limit of the Born approximation at $\vec{A}(\varphi)\equiv\vec{0}$] in accordance with the determination of the scalar part $S_1(\vec{r},t)$ [Eq. (2.38)] of the wave function.

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