BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Detector efficiency in the Greenberger-Horne-Zeilinger paradox: Independent errors

Jan-Åke Larsson*

Matematiska Institutionen, Linköpings Universitet, SE-581 83 Linköping, Sweden

(Received 26 October 1998)

The Greenberger-Horne-Zeilinger (GHZ) paradox is subject to the detector-efficiency ''loophole'' in a similar manner as the Bell inequality. In a paper by J.-Å. Larsson [Phys. Rev. A **57**, R3145 (1998)], the issue is investigated for very general assumptions. Here, the assumptions of constant efficiency and independent errors will be imposed, and it will be shown that the necessary and sufficient efficiency bound is not lowered, but remains at 75%. An explicit local-variable model is constructed in this paper to show the necessity of this bound. In other words, it is not possible to use the independence of experimental nondetection errors to rule out local realism in the GHZ paradox below 75% efficiency. [S1050-2947(99)06706-2]

PACS number(s): 03.65.Bz

I. INTRODUCTION

The most well-known test of local realism is the Bell inequality [1], and it is also well known that it changes in the case of inefficient detectors. Most familiar is the 82.83% bound in [2], where a variant of the Bell inequality, the Clauser-Horne-Shimony-Holt (CHSH) inequality [3], is investigated in the case of independent errors and constant efficiency. The resulting inequality is (in the notation of [2])

$$|E_{13}\pm E_{23}|+|E_{14}\mp E_{24}|\leq 4\eta^{-1}-2$$

[Eq. (1.9) of Ref. [2]]. The left-hand expression above is similar to the usual sum of correlations, but here single and nondetection events have been removed. Quantum-mechanical predictions also violate this inequality, but only if η is larger than the mentioned 82.83%, e.g., in the ideal case ($\eta=1$) the right-hand side equals 2 as in the original CHSH inequality. Furthermore, it is shown that below the bound, there exists a local-variable model that yields correlations that are as large as the quantum-mechanical ones, so the bound is necessary and sufficient. In [4] this result is extended to apply even for dependent errors and nonconstant efficiency.

Another test of local realism is the Greenberger-Horne-Zeilinger (GHZ) paradox [5–7], which is often presented as the final argument against local variables, as it implies that any attempt to construct a local-variable model describing the GHZ setup inevitably results in a contradiction. The stronger result is enabled by the three particles with their three associated measurements, whereas in the Bell inequality there are two particles with two associated measurements. In this three-particle setting, a quantum state is chosen so that a full contradiction is obtained instead of comparing the correlations in an inequality:

$$\frac{1}{\sqrt{2}}(|+++\rangle-|---\rangle)_{ZZ'Z''}, \tag{1}$$

where $|+\rangle_Z$ denotes ''spin up'' in the z direction and $|-\rangle_Z$ denotes ''spin down.'' Below, the measurement results will be denoted X when measuring in the x direction on the first particle, Y' when measuring in the y direction on the second particle, and so on.

The GHZ paradox uses the following four prerequisites.

(*i*) *Realism*. Measurement results can be described by probability theory.

(*ii*) *Locality*. A measurement result should be independent of the detector orientation at the other particles.

(*iii*) *Measurement result restriction*. Only the results ± 1 are allowed.

(*iv*) *GHZ requirement*. The following results should be obtained for the corresponding measurements [from the quantum state in Eq. (1)]:

$$XY'Y''=1,$$

$$YX'Y''=1,$$

$$YY'X''=1,$$

$$XX'X''=-1.$$

If (i)–(iv) hold except on a set of probability zero, the X's and Y's would have values independently of each other and we would have

*Electronic address: jalar@mai.liu.se

$$-1 = XX'X'' = XX'X'' \cdot Y^2 \cdot Y'^2 \cdot Y''^2$$
$$= XY'Y'' \cdot YX'Y'' \cdot YY'X'' = 1, \tag{2}$$

except on the mentioned zero-probability set, which is obviously a contradiction.

II. INEFFICIENCY IN THE GHZ PARADOX

The GHZ paradox is also subject to the inefficiency ''loophole,'' but here there is no inequality, so the derivation of the bounds uses a slightly different technique than in the Bell inequality. Only a brief presentation of that technique will be made here (a more formal exposition is available in [8]).

By (i), the X's and Y's above are random variables (RVs) defined on the ''hidden-variable'' probability space Λ consisting of all possible values of the hidden variable λ. At an inefficient detector, three things may happen: a $+1$ detection, a -1 detection, and a detection error. Often this last event is signalled by letting the numerical result of the experiment be 0, but here another description will be used: *no value will be assigned* to the RV when a measurement error occurs, and thus X is only defined on a *subset* Λ_X of Λ. ''Deterministic local variables'' are used here, but the generalization to the ''stochastic'' case is straightforward.

The prerequisites (i)–(iv) above do not change very much by this generalization. The most notable change is that the products in (iv) quite naturally are required to hold only on the set where all three RVs are defined, e.g., the product $XY'Y''$ is defined on the set

$$\Lambda_{XY'Y''} = \Lambda_X \cap \Lambda_{Y'} \cap \Lambda_{Y''}. \tag{3}$$

This is also reflected in Eq. (2), which is now only valid on the set where the product of all six RVs is defined:

$$\Lambda_{XX'X''YY'Y''} = \Lambda_X \cap \Lambda_{X'} \cap \Lambda_{X''} \cap \Lambda_Y \cap \Lambda_{Y'} \cap \Lambda_{Y''}. \tag{4}$$

If this set has a probability larger than zero, the contradiction remains, but if the probability is zero, the prerequisites [especially (iv)] need not hold on this set, and there is no contradiction. However, the probability of this set cannot be obtained directly from experiment.

To obtain the probability of the above intersection we will use the efficiency of the measurement setup. Because no extra assumptions are made on the properties of the errors, there are four different ways of measuring efficiency.

η_1: The least probability of a detection at a chosen detector.

$\eta_{2,1}$: Given that there is a detection at a chosen detector, the least probability of a detection at another chosen detector.

$\eta_{3,2}$: Given that there are detections at two chosen detectors, the least probability of a detection at the last detector.

$\eta_{3,1}$: Given that there is a detection at a chosen detector, the least probability of a detection at both the other detectors.

There is a slight difference between the first of the above and the remaining three. While the first is often intuitively used as detector efficiency, it may be difficult to estimate it accurately (e.g., by calorimetric methods). The three remaining measures are not as intuitive, but have the additional bonus of being directly available in the coincidence data. When the errors are dependent, all four of the above may be different.

These are used to estimate the probability of the intersection in Eq. (4) by using probabilistic inequalities comparing probabilities of sets and their intersections. The necessary and sufficient bounds obtained from this are

$$\eta_1 > \frac{5}{6} \approx 83.33\%, \tag{5a}$$

$$\eta_{2,1} > \frac{4}{5} = 80\%, \tag{5b}$$

$$\eta_{3,2} > \frac{3}{4} = 75\%, \tag{5c}$$

$$\eta_{3,1} > \frac{3}{5} = 60\%. \tag{5d}$$

The four different bounds are a consequence of the general nature of the result. The result is necessary and sufficient in the sense that if one of the efficiency measures is above the bound, there cannot be a local-variable model that yields the prescribed results, but if all of them are below the bound, an explicit model is available that has the prescribed measurement results of (iv) [and furthermore, the measurement results from the model presented in [8] correspond to that of the quantum state (1) when measuring in any combination of the x and y orientations].

However, the nondetection errors in that model are not independent, because the model is constructed to provide a counterexample in a theorem where no assumptions are made on the properties of the nondetection errors. There is a possibility that the bound would be lowered for the case of independent errors and constant efficiency, and therefore the purpose of the present paper is to examine this case in detail.

III. CONSTANT EFFICIENCY AND INDEPENDENT ERRORS

The two properties we are interested in can be stated in the following way.

(*v*) *Constant detector efficiency*. The probability of a detection at any detector at any orientation is η.

$$\eta = P(\Lambda_X) = P(\Lambda_{X'}) = P(\Lambda_{X''})$$
$$= P(\Lambda_Y) = P(\Lambda_{Y'}) = P(\Lambda_{Y''}).$$

(*vi*) *Independent nondetection errors*. The detection errors are probabilistically independent for detection at different detectors at any orientation, e.g.,

$$P(\Lambda_{YX''}) = P(\Lambda_Y)P(\Lambda_{X''}),$$

$$P(\Lambda_{XY'Y''}) = P(\Lambda_X)P(\Lambda_{Y'})P(\Lambda_{Y''}).$$

With these assumptions, the four different efficiency measures above reduce to one (η) because of the following:

$$\eta_1 = \eta_{2,1} = \eta_{3,2} = \eta, \tag{6a}$$

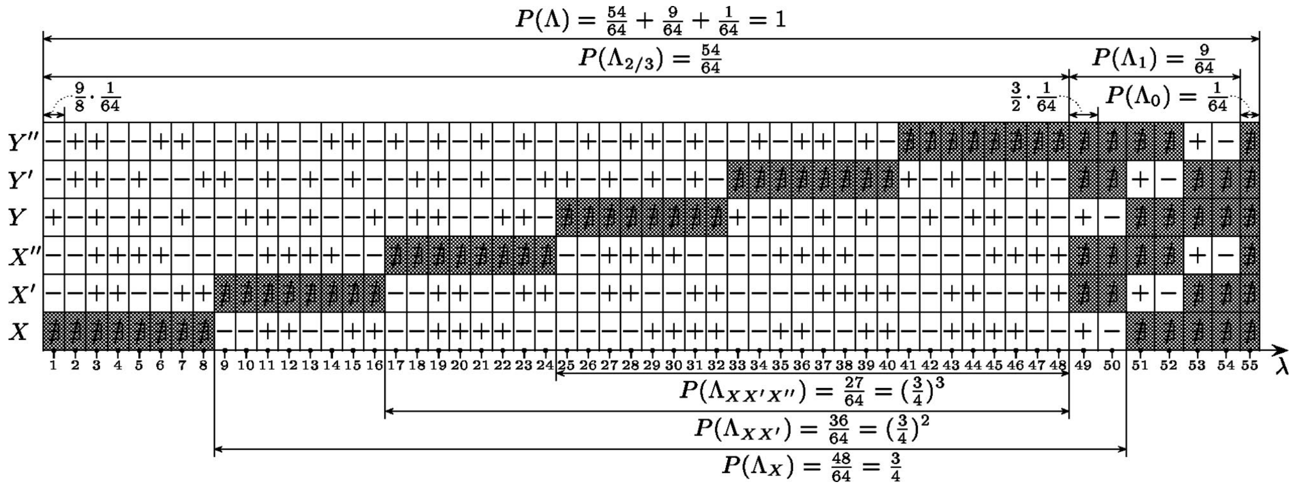

FIG. 1. The model in the case of independent errors at 75% (constant) efficiency. Each point in the sample space is assigned values according to the model (e.g., for the point $\lambda=48$, the measurement results are $X=-1$, $X'=+1$, $X''=+1$, $Y=-1$, $Y'=-1$, and Y'' is undefined at this point, denoting a nondetection error). There are 55 points in the sample space ($\lambda=1,\ldots,55$) along the horizontal axis, and these have different probabilities as shown by the width of the corresponding column [e.g., $P(\lambda=48)=\frac{9}{8}\times\frac{1}{64}$].

$$\eta_{3,1}=\eta^2. \tag{6b}$$

This is a trivial implication of (v) and (vi), and using these assumptions, an immediate corollary of Theorem 2 of Ref. [8] is the following.

Corollary 1. Assuming (i)–(iv) except on a null set, and in addition assuming (v) and (vi), there is a contradiction if the following is satisfied:

$$\eta>\frac{3}{4}=75\%. \tag{7}$$

Furthermore, if Eq. (7) is not satisfied there exists a local-variable model satisfying (i)–(vi) which yields the quantum-mechanical statistics of the GHZ state in the $|\,\rangle_{X^{(i)}}$ and $|\,\rangle_{Y^{(i)}}$ bases except for detector inefficiency.

Proof. The first statement is easily proven using the inequalities (5a)–(5d) and Eqs. (6a) and (6b). There is a violation if one of

$$\eta>\frac{5}{6}\approx 83.33\%, \tag{8a}$$

$$\eta>\frac{4}{5}=80\%, \tag{8b}$$

$$\eta>\frac{3}{4}=75\%, \tag{8c}$$

$$\eta^2>\frac{3}{5}=60\% \tag{8d}$$

is satisfied, i.e., as soon as the lowest bound is. This is Eq. (8c) since the square root of $\frac{3}{5}$ is greater than $\frac{3}{4}$.

The second statement is, as in Theorem 2 of Ref. [8], proved by construction. In [8], the model consists of a sample space Λ of 48 points with equal probability, and these points are assigned measurement results according to two tables in the paper (not repeated here, but see below). The model yields the required results and has constant efficiency, but the nondetection errors are dependent. The probability of a triple coincidence (e.g., at $XX'X''$ orientation) is $\frac{1}{2}$, which is too large, but even so, the model is useful and will be used as a basic building block. This set of 48 points is used as a subset of the full sample space in the construction below, denoted $\Lambda_{2/3}$, the set of double or triple coincidences. The probability of $\Lambda_{2/3}$ should be such that

$$P(\Lambda_{XX'X''})=\left(\frac{3}{4}\right)^3=\frac{27}{64}\quad\left(<\frac{1}{2}=\frac{32}{64}\right), \tag{9}$$

and that is achieved by giving the subset a total probability of $\frac{54}{64}$ (the model is symmetric in such a way that the other triple coincidences have the same probability). In this subset, the value assignment is done in the same way as in [8] (see Fig. 1).

The probability of two or more detections at a particular setup should be the square of $\frac{3}{4}$, and indeed, using the above subset construction (which is symmetric for different setups),

$$P(\Lambda_{XX'})=\frac{2}{3}\times\frac{54}{64}=\frac{9}{16}=\left(\frac{3}{4}\right)^2. \tag{10}$$

The event where at least one detection occurs at a certain setup should now have the probability of $\frac{3}{4}$, but in the construction so far, the probability of this is only

$$P(\Lambda_X\cap\Lambda_{2/3})=\frac{5}{6}\times\frac{54}{64}=\frac{45}{64}\quad\left(<\frac{3}{4}=\frac{48}{64}\right). \tag{11}$$

In addition, there are no purely single detections yet, only double and triple coincidences. To add these events, construct a new subset Λ_1, the set of purely single detections. Let it consist of six points, because we want two points for each of the three particles (allowing for the two results ± 1). Assign equal probability to the points so that

$$P(\Lambda_X\cap\Lambda_1)=\frac{3}{4}\left(1-\frac{3}{4}\right)^2=\frac{3}{64}, \tag{12}$$

which yields

$$P(\Lambda_X)=P(\Lambda_X\cap\Lambda_1)+P(\Lambda_X\cap\Lambda_{2/3})=\frac{3}{4}. \qquad (13)$$

Thus, the total probability of Λ_1 should be $\frac{9}{64}$.

Last, the full sample space should include the set of no detection (Λ_0) at a probability of

$$P(\Lambda_0)=\left(1-\frac{3}{4}\right)^3=\frac{1}{64}, \qquad (14)$$

and adding such a subset (consisting of one point), the total probability of the sample space adds to one.

Group	Points	P(point)	ΣP
$\Lambda_{2/3}$ (double/triple coinc.)	48	$\frac{9}{8}\times\frac{1}{64}$	$\frac{54}{64}$
Λ_1 (purely single detections)	6	$\frac{3}{2}\times\frac{1}{64}$	$\frac{9}{64}$
Λ_0 (no detection)	1	$\frac{1}{64}$	$\frac{1}{64}$

The resulting model is visible in Fig. 1. The statistics are precisely that of the GHZ quantum state using 75% (constant) efficiency and independent errors.

One particle. Detection probability: $\frac{3}{4}$. Measurement at one site at $X^{(i)}$ or $Y^{(i)}$ orientation (six different possibilities) yields two equally probable results ($\pm$).

Two particles. Probability of pair detection: $(\frac{3}{4})^2$. Measurement at two different sites, each at $X^{(i)}$ or $Y^{(i)}$ orientation (12 different possibilities) yields four equally probable results: $++$, $+-$, $-+$, and $--$.

Three particles. Probability of triple detection: $(\frac{3}{4})^3$.

(a) Measurement at three sites at $YY'Y''$, $XX'Y''$, $XY'X''$, or $YX'X''$ orientations yields eight equally probable results: $+++$, $++-$, $+-+$, $\ldots$, $---$.

(b) On measurement at three sites at $XX'X''$ orientation, only four results appear. These four results are equally probable, and are $---$, $++-$, $+-+$, and $-++$, each with an odd number of minus signs as in (iv).

(c) On measurement at three sites at $XY'Y''$, $YX'Y''$, or $YY'X''$ orientations, again only four results appear. In this case, the four equally probable results are $+++$, $+--$, $-+-$, and $--+$, each with an even number of minus signs as in (iv).

The above statements are easily checked in Fig. 1, where the symmetry of the model simplifies the check significantly. This completes the proof.

IV. CONCLUSIONS

The conclusion is that a GHZ experiment with independent errors and constant efficiency refutes local variables if and only if the efficiency is higher than 75%. An experiment at a lower efficiency would not be conclusive, since the model presented in this paper is valid at 75%, and it is easily extended to lower efficiency.

An important observation is that the dependent errors of the model in [8] are not an important feature of a local-variable model mimicking quantum-mechanical behavior. The model in [8] is here modified to the case of independent errors, and thus, there is no test below 75% that can rule out local variables on the basis that the nondetection errors should be independent.

ACKNOWLEDGMENTS

The author would like to thank M. Żukowski and A. Zeilinger for valuable discussions and other support, and also L. Vaidman for a discussion on this subject. This work was financially supported by Civilingenjör Gustaf Sigurd Magnuson's Foundation through the Royal Swedish Academy of Sciences.

[1] J.S. Bell, Physics (Long Island City, NY) **1**, 195 (1964).
[2] A. Garg and N.D. Mermin, Phys. Rev. D **35**, 3831 (1987).
[3] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[4] J.-Å. Larsson, Phys. Rev. A **57**, 3304 (1998).
[5] D. M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, 1989), pp. 69–72.
[6] D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[7] N.D. Mermin, Phys. Today **43** (6), 9 (1990).
[8] J.-Å. Larsson, Phys. Rev. A **57**, R3145 (1998).