Revival and fractional revival in the quantum dynamics of SU(1,1) coherent states

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We have used a generic two-mode Hamiltonian with two associated time scales to study the evolution of generalized SU(1,1) coherent states, in particular the pair and Perelomov coherent states, which have been realized in many systems such as radiation fields, trapped ions, and phonons. We have found that their dynamics does not depend on the ratio of these time scales but is instead determined crucially by the difference in photon numbers of the two modes. This is in stark contrast to the previously studied harmonic-oscillator coherent states and can be attributed to the different nature of the underlying algebra. We provide analytical results for their revival and fractional revival along with numerical plots of the autocorrelation function and the quadrature distribution and demonstrate the formation of Schrödinger cats. The results are extremely sensitive to the Casimir invariant and the complex parameters characterizing SU(1,1) coherent states.
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I. INTRODUCTION

Coherent states, introduced by Schrödinger to describe nonspreading wave packets for harmonic oscillators, have many interesting properties, chief among which is that these are minimum uncertainty states and therefore most classical within the framework of quantum theory. In recent years, the nonlinear quantum dynamics of these states have revealed some striking features. It was found that under the action of a Hamiltonian that is a *nonlinear* function of the photon number operator(s) *only*, an initial coherent state loses its coherent structure quickly due to quantum dephasing induced by the nonlinearity of the Hamiltonian and then regains it (revival) after an interval. At fractions of this time interval, the initial coherent state breaks up into a superposition of two or more coherent states that also can have a coherent structure. This is an example of the quantum phenomenon of fractional revival [1–6] or the formation of Schrödinger cat and catlike states [7] that, unlike a coherent state, have many nonclassical properties.

The concept of coherent states itself has been generalized [8] to describe systems other than harmonic oscillators (or radiation fields). From a group-theoretic point of view, the harmonic oscillator (HO) coherent states arise in systems whose dynamical symmetry group is the so-called Heisenberg-Weyl group. Coherent states of other symmetry groups also exist. For example, the atomic, Bloch, or spin coherent states are described by the algebra of spin operators that are generators of SU(2), the *simplest compact* group. Similarly, one can construct coherent states for the *simplest noncompact* group SU(1,1) [9].

A question arises whether revival and fractional revival (Schrödinger cat formation) occur for the coherent states of these groups as well. With the formation of atomic Schrödinger cat states [10], the answer is obtained in the affirmative for the SU(2) group. In this paper we focus our attention on the SU(1,1) coherent states.

The SU(1,1) group plays a very important role in many problems in optics [11]. For example, optical parametric processes that create or destroy photons in pairs can be described by Hamiltonians that are linear in SU(1,1) generators. Two well-studied classes of SU(1,1) coherent states are the so-called pair coherent states [12] and the Perelomov coherent states [8]. These two apparently different sets of states were found to be special cases of what may be called *generalized* SU(1,1) coherent states [13]. In view of their importance it is therefore of considerable interest to investigate their revival features (if any) and to see whether they can form Schrödinger cats. It may also be noted that in recent times the importance of SU(1,1) states in connection with the motion of an ion in a two-dimensional trap has been emphasized.

It is well known that the formation of Schrödinger cats is closely connected with the nonlinearity of the underlying Hamiltonian. For HO coherent states, the Hamiltonian was nonlinear in the photon-number operator whereas for the atomic coherent states the nonlinearity was with respect to the population inversion operator. For our purpose, therefore, we choose a phase-insensitive two-mode generic Hamiltonian that is quadratic in terms of the generators of SU(1,1). The Hamiltonian was generic in the sense that it models a wide range of systems from elliptically polarized light passing through a fiber [14] to binary Bose condensates. We show that under the action of this Hamiltonian revivals and fractional revivals do indeed occur in the nonlinear dynamics of even the generalized SU(1,1) coherent states. Owing to the generic nature of the Hamiltonian and the generality of the states on which it operates, our present study assumes a greater significance in that it not only fills a long-standing void in the quantum dynamics of coherent states but also reveals the inherent SU(1,1) symmetry of many nonlinear systems.

The plan of the paper is as follows. In Sec. II we give a brief but self-contained review of the generalized SU(1,1) coherent states. We also comment on the production and importance of SU(1,1) coherent states in connection with the states of the radiation field, the motion of ions in a trap [15,16], and even squeezed phonons in solids [17,18]. In

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Sec. III we study their dynamics by means of autocorrelation functions and quadrature distributions. Explicit analytical results are provided for their revival and fractional revival. The paper ends with some concluding remarks in Sec. IV.

II. SU(1,1) COHERENT STATES

A. Definition

The SU(1,1) algebra is spanned by the set of operators $\vec{K}\equiv(K_x,K_y,K_z)$ obeying the commutation relations

$$[K_x,K_y]=-iK_z, \quad [K_y,K_z]=iK_x, \quad [K_z,K_x]=iK_y. \tag{1}$$

The Casimir invariant of SU(1,1) is the operator $Q=K_x^2+K_y^2-K_z^2$. The operator K_z is compact and its spectrum is discrete. For a given eigenvalue of Q, the eigenvalues of K_z differ by an integer. The operators K_x and K_y, on the other hand, are noncompact. Their spectra are continuous. The operators $K_+=K_x+iK_y$ and $K_-=K_x-iK_y$ act as the raising and lowering operators, respectively, on the eigenstates of K_z. There are several representations of SU(1,1) [19]. However, we confine our attention to the so-called discrete representation, which is of interest in quantum optics, especially in the study of optical parametric processes. In the two-mode realization of SU(1,1), $K_-=ab$, $K_+=b^\dagger a^\dagger$, $K_z=(a^\dagger a+b^\dagger b+1)/2$, and $Q=K_0(1-K_0)$, where $K_0=(a^\dagger a-b^\dagger b+1)/2$ is a constant. Note that this constraint on the photon numbers was absent in our earlier work [20] for the two-mode harmonic oscillator coherent states that obey the Heisenberg-Weyl algebra. Without any loss of generality, let us assume that $a^\dagger a-b^\dagger b=q\geqslant 0$. Then the state space consists of two-mode Fock states of the type $|m+q,m\rangle$ (where m, $q=0, 1, 2, 3, \dots$), which are simultaneous eigenstates of $a^\dagger a$ and $b^\dagger b$ with eigenvalues $m+q$ and m, respectively. It follows that

$$\begin{aligned}
K_z|m+q,m\rangle&=[m+(q+1)/2]|m+q,m\rangle,\\
K_-|m+q,m\rangle&=\sqrt{m(m+q)}|m-1+q,m-1\rangle,\\
K_+|m+q,m\rangle&=\sqrt{(m+1)(m+1+q)}|m+1+q,m+1\rangle,\\
Q|m+q,m\rangle&=(1-q^2)/4\,|m+q,m\rangle.
\end{aligned}\tag{2}$$

The commutation relations (1) can be written in an instructive compact form [13]

$$[\vec{a}\cdot\vec{K},\vec{b}\cdot\vec{K}]=i(\vec{a}\times\vec{b})\cdot\vec{K}, \tag{3}$$

where the vectors in Eq. (3) are taken to belong to (2+1)-dimensional Minkowski space in which the scalar product is defined as

$$\vec{a}\cdot\vec{b}=a_xb_x+a_yb_y-a_zb_z, \tag{4}$$

whereas the cross product in Eq. (3) is given by

$$\begin{aligned}
(\vec{a}\times\vec{b})_i&=-\sum_{j,k=x,y,z}\epsilon_{ijk}a_jb_k, \quad i\neq z\\
(\vec{a}\times\vec{b})_z&=\sum_{j,k=x,y}\epsilon_{zjk}a_jb_k,
\end{aligned}\tag{5}$$

where ϵ_{xyz} is 1 (-1) for even (odd) permutation of x,y,z and $\epsilon_{ijk}=0$ if any two of i,j,k are the same. In the following all the dot products will be defined in the sense of Eq. (4).

The uncertainty relation corresponding to Eq. (3) reads

$$\Delta(\vec{a}\cdot\vec{K})^2\Delta(\vec{b}\cdot\vec{K})^2\geqslant\frac{1}{4}[(\vec{a}\times\vec{b})\cdot\vec{K}]^2. \tag{6}$$

Recall that the uncertainty relation is satisfied with equality for those states $|\psi\rangle$ that solve the eigenvalue equation

$$[\vec{a}\cdot\vec{K}+i\lambda\vec{b}\cdot\vec{K}]|\psi\rangle=\xi|\psi\rangle \tag{7}$$

and that the variance in the two components is equal if $\lambda=\pm1$. We define [13] the coherent states of SU(1,1) as those minimum-uncertainty states for which the variance in the two orthogonal components of $\vec{K}$ is the same, i.e., the states that satisfy Eq. (7) for $\lambda=\pm1$ and $\vec{a}\cdot\vec{b}=0$.

B. Construction

Since $\vec{a}$ and $\vec{b}$ are mutually orthogonal, we can transform the pair $\vec{a}\cdot\vec{K}$ and $\vec{b}\cdot\vec{K}$ by means of an SU(1,1) transformation to a pair of operators from K_x, K_y, and K_z. Note that a general SU(1,1) transformation U transforms an SU(1,1) operator $\vec{w}\cdot\vec{K}$ to $\vec{w}'\cdot\vec{K}=U(\vec{w}\cdot\vec{K})U^\dagger$ such that $\vec{w}'\cdot\vec{w}'=\vec{w}\cdot\vec{w}$. Since $\vec{w}\cdot\vec{w}$ is 1 for the noncompact operators K_x,K_y and -1 for the compact operator K_z, it follows that the signs of $\vec{a}\cdot\vec{a}$ and $\vec{b}\cdot\vec{b}$ determine to which one of the operators K_x, K_y, and K_z one can transform $\vec{a}\cdot\vec{K}$ and $\vec{b}\cdot\vec{K}$. However, since $\vec{a}\cdot\vec{b}=0$, the signs of $\vec{a}\cdot\vec{a}$ and $\vec{b}\cdot\vec{b}$ cannot be arbitrary and at most one of them can be negative.

In our (2+1)-dimensional Minkowski space, we define unit vectors $\vec{\theta}$, $\vec{\phi}$, and $\vec{r}$ whose Cartesian components are given by

$$\begin{aligned}
\vec{\theta}&=(\cosh\theta\cos\phi,\cosh\theta\sin\phi,\sinh\theta),\\
\vec{\phi}&=(-\sin\phi,\cos\phi,0),\\
\vec{r}&=\vec{\theta}\times\vec{\phi}=(\sinh\theta\cos\phi,\sinh\theta\sin\phi,\cosh\theta)
\end{aligned}\tag{8}$$

so that $\vec{\theta}\cdot\vec{\theta}=\vec{\phi}\cdot\vec{\phi}=1$, whereas $\vec{r}\cdot\vec{r}=-1$. We also construct the SU(1,1) group transformation

$$U(\theta,\phi)=\exp(i\theta\vec{\phi}\cdot\vec{K}). \tag{9}$$

For both $\vec{a}\cdot\vec{a}$ and $\vec{b}\cdot\vec{b}$ positive, we choose

$$\vec{a}=\vec{\theta}\cos\phi-\vec{\phi}\sin\phi,\quad \vec{b}=\vec{\theta}\sin\phi+\vec{\phi}\cos\phi, \tag{10}$$

so that $\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}=1$, and $\vec{a}\cdot\vec{b}=0$. When one of the norms, say $\vec{a}\cdot\vec{a}$, is negative, we choose

$$\vec{a}=-\vec{r},\quad \vec{b}=\vec{\theta}\cos\phi-\vec{\phi}\sin\phi \tag{11}$$

so that $\vec{a}\cdot\vec{a}=-1$, $\vec{b}\cdot\vec{b}=1$, and $\vec{a}\cdot\vec{b}=0$. It can be shown that for the set (10), $U^{\dagger}(\vec{a}\cdot\vec{K})U=K_x$ and $U^{\dagger}(\vec{b}\cdot\vec{K})U=K_y$, whereas for the set (11), $U^{\dagger}(\vec{a}\cdot\vec{K})U=K_z$ and $U^{\dagger}(\vec{b}\cdot\vec{K})U=K_x$. Thus there are two unitarily inequivalent forms into which the eigenvalue problem (7) can be ‘‘transformed’’ [21]:

$$|\psi\rangle=U(\theta,\phi)|\Phi(\xi,q)\rangle, \tag{12}$$

where either

$$K_{\pm}|\Phi\rangle=\xi|\Phi\rangle \tag{13}$$

or

$$(K_z\pm iK_x)|\Phi\rangle=\xi|\Phi\rangle. \tag{14}$$

In what follows we will choose Eq. (13) to solve for $|\Phi\rangle$.

Since there is no normalizable eigenstate of K_+, the solution of Eq. (13) can only be an eigenstate of K_-, which for a two-mode realization of SU(1,1) is a pair coherent state [12]. The pair coherent states can be expressed in terms of the eigenstates of K_z. Thus

$$|\Phi(\xi,q)\rangle=N(\xi,q)\sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!(n+q)!}}|n+q,n\rangle, \tag{15}$$

where

$$N(\xi,q)=\left[\sum_{n=0}^{\infty}\frac{|\xi|^{2n}}{n!(n+q)!}\right]^{-1/2}. \tag{16}$$

Next we use the disentangling theorem to write the operator U in the normal form

$$U(\theta,\phi)=\exp(\eta K_+)\exp(\eta_z K_z)\exp(-\eta^* K_-), \tag{17}$$

where

$$\eta=\exp(-i\phi)\tanh(|\theta|/2),\quad \eta_z=-2\ln\cosh(|\theta|/2),$$
$$|\eta|<1. \tag{18}$$

The operation of U on $|\Phi\rangle$ is now straightforward and one obtains $|\psi\rangle=|\psi(\eta,\xi,q)\rangle$, where

$$|\psi(\eta,\xi,q)\rangle=N(\xi,q)\exp(-\xi\eta^*)\sum_{n'=0}^{\infty}\frac{\eta^{n'}}{\Gamma(n'+1)}\sum_{n=0}^{\infty}\frac{\xi^n(1-|\eta|^2)^{n+\frac{q+1}{2}}\sqrt{(n+n')!(n+n'+q)!}}{n!(n+q)!}|n+n'+q,n+n'\rangle. \tag{19}$$

This is the *generalized SU(1,1) coherent state* whose dynamical evolution will be studied. Note that this state is characterized by two parameters η and ξ, which are in general complex, and an integer parameter q, which is related to the Casimir invariant $(1-q^2)/4$.

If U is trivially the identity transformation ($\eta\to 0$), then $|\psi\rangle$ is equal to $|\Phi\rangle$, which we recall is an eigenstate of K_-. If U is as given by Eq. (17) but $\xi=0$, then $|\psi\rangle$ reduces to

$$|\Psi(\eta,q)\rangle=\frac{(1-|\eta|^2)^{(q+1)/2}}{\sqrt{q!}}\sum_{p=0}^{\infty}\eta^p\sqrt{\frac{(p+q)!}{p!}}|p+q,p\rangle, \tag{20}$$

which is the expression for the SU(1,1) coherent states as defined by Perelomov [8]. In general, however, $|\psi\rangle$ is neither an eigenstate of K_- nor a Perelomov coherent state. This is clearly seen in the initial quadrature distributions. Recall that the quadrature distribution for a state vector $|\psi(t)\rangle$ is defined as $P(x,y,t)=|\langle x,y|\psi(t)\rangle|^2$, where $|x,y\rangle$ is the eigenvector of $(a+a^{\dagger})/\sqrt{2}$ and $(b+b^{\dagger})/\sqrt{2}$ with eigenvalues x and y, respectively. The SU(1,1) coherent states $|\psi(\eta,\xi,q)\rangle$ will have different initial distributions for different values of η, ξ, and q. In Fig. 1 we show a few such cases for $q=0$. The corresponding cases for $q=1$ are shown in Fig. 2. It is clear that the patterns are very sensitive to both the phase and the strength of the variables ξ and η. Even for the same value of ξ, different unitary transformations will yield different dis-

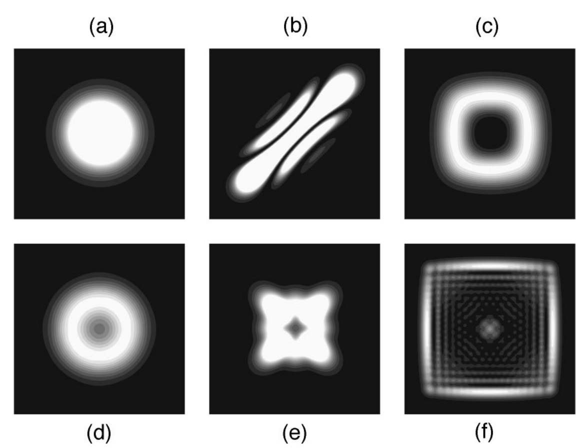

FIG. 1. Contour plots of the *quadrature* distribution $P(x,y,t)$ at $t=0$ for the SU(1,1) coherent state $|\psi(\xi,\eta,0)\rangle$ when (a) $\xi=0$, $\eta=-i\tanh\pi/4$; (b) $\xi=3$, $\eta=0$; (c) $\xi=-3i$, $\eta=0$; (d) $\xi=i$, $\eta=i\tanh\pi/4$; (e) $\xi=3i$, $\eta=-i\tanh 3\pi/20$; and (f) $\xi=3i$, $\eta=-i\tanh 2\pi/5$. Note that case (a) is a Perelomov coherent state, cases (b) and (c) represent pair coherent states, and cases (d)–(f) are more general.

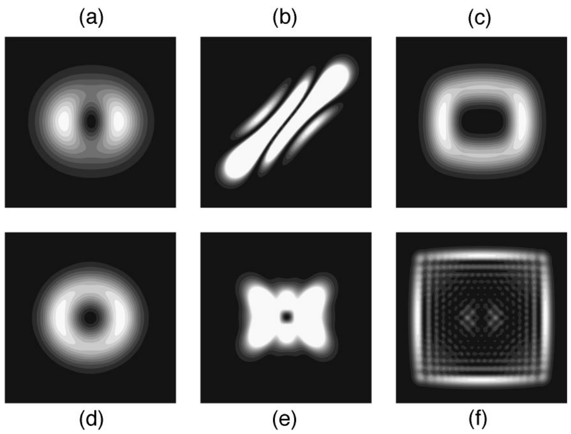

FIG. 2. Same as in Fig. 1, but for $|\psi(\xi,\eta,1)\rangle$.

tributions [see Figs. 1(e),2(e), and 1(f),2(f)]. For the Perelomov coherent states, it can be shown that

$$\langle x,y|\Psi(\eta,q)\rangle=(\pi 2^q q!)^{-1/2}\left(\frac{1-|\eta|^2}{1-\eta^2}\right)^{(q+1)/2}\times\exp\left[\frac{x^2-y^2}{2}-\left(\frac{x-\eta y}{\sqrt{1-\eta^2}}\right)^2\right]H_q\left[\frac{x-\eta y}{\sqrt{1-\eta^2}}\right]. \tag{21}$$

For purely imaginary values of η ($\eta=-i|\eta|$), the above expression (21) yields a Gaussian for even values of q but has a vortex structure [22] for odd values of q.

C. Production

In this section we review briefly the possible methods to produce SU(1,1) coherent states. There are a number of very interesting special cases of Eq. (19) that have been realized. (a) Pair coherent states (15) have been realized for the two modes of the radiation field [12] as well as for the two-dimensional motion of the trapped ion [15]. For the radiation field one considers two-photon absorption and emission processes in a system of three-level atoms under the condition that the response time of the atoms is fast. Theoretical calculations show [12] that under such a condition the field is produced in a pair coherent state of the form (15). For the case of the trapped ion one drives the ion with a laser on resonance and two other lasers with appropriately chosen directions of propagation and tuned to the second lower vibrational side band. Calculations [15] show that in the Lamb-Dicke limit, the ion is found in the state (15). We also note in passing that Meekhof *et al.* [16] have demonstrated how to produce a state such as $|q,0\rangle$. (b) After almost two decades of work on squeezed states one understands very well how Perelomov states can be produced in parametric interactions. Parametric amplification with a classical pump is precisely equivalent to the unitary transformation (17) [23]. For one-dimensional motion of a trapped ion Meekhof *et al.* have again demonstrated how Perelomov states can be produced. Further theoretical work in this direction is described by Gou, Steinbach, and Knight [24]. It is possible to generalize the above works to two-dimensional motion.

It is clear from the foregoing that each mathematical transformation involved in the definition of SU(1,1) coherent state (19) is physically realizable, suggesting that this more general class of coherent states can also be produced. We have given examples corresponding to the radiation field and the motional degrees of freedom of a trapped ion. Clearly there are other systems where such states have been (or would be) realized. For example, Perelomov-like SU(1,1) states for phonons have been studied [18] and have been experimentally realized in $KTaO_3$ [17].

III. WAVE-PACKET DYNAMICS

We have studied the wave-packet dynamics of the SU(1,1) coherent states under the action of a generic two-mode Hamiltonian (we use $\hbar=1$)

$$H=c_1[(a^\dagger a)^2+(b^\dagger b)^2]-c_2 a^\dagger a b^\dagger b, \tag{22}$$

which can be expressed in terms of the SU(1,1) operators K_z and K_0 as

$$H=\frac{\pi}{4}[(2K_z-1)^2/T_-+(2K_0-1)^2/T_+], \tag{23}$$

where $T_\pm=\pi/(2c_1\pm c_2)$. Note that K_0 is a constant and thus the second term in the above Hamiltonian will only provide an overall phase factor in the evolution operator. Consequently, T_+ does not play an active role and although the above Hamiltonian has two time scales T_+ and T_-, the revival features of the SU(1,1) coherent states under the action of this Hamiltonian will not depend on the ratio of the time scales. This is in stark contrast with what was observed for the harmonic-oscillator coherent states [20]. In what follows we will suppress overall phase factors containing the constant term $q^2(1+T_-/T_+)$. Since the SU(1,1) coherent states are expressed in terms of the eigenstates of K_z, we start by considering the evolution of the latter set.

A. Revival

If $\mathcal{U}(t)=\exp(-iHt)$ is the corresponding evolution operator, then it is seen that for odd values of q,

$$\mathcal{U}(T_-)|p+q,p\rangle=|p+q,p\rangle, \tag{24}$$

whereas for even values of q,

$$\mathcal{U}(T_-)|p+q,p\rangle=\exp(-i\pi p)|p+q,p\rangle, \tag{25}$$

$$\mathcal{U}(2T_-)|p+q,p\rangle=|p+q,p\rangle. \tag{26}$$

Using these relations in the expression for $|\psi(\eta,\xi,q)\rangle$, we infer the following result: $|\psi(\eta,\xi,q)\rangle$ revives at *all* integer values of $\tau=t/T_-$ if q is odd and at *even* values of τ if q is even. This is a remarkable result in the following sense. A two-mode SU(1,1) coherent state with a large number of photons in each mode will have its revival time changed by a factor of 2 if it were prepared again with only an extra photon in either mode. Using Eq. (25), we find that for even values of q,

$$\mathcal{U}(T_-)|\psi(\eta,\xi,q)\rangle=|\psi(-\eta,-\xi,q)\rangle. \tag{27}$$

B. Fractional revival

In between revivals, let $t=(r/s)T_-$, where r and s are mutually prime with $r<s$. Note that

$$\mathcal{U}(rT_-/s)|p+q,p\rangle=\exp(-i\pi rp[p+q]/s)|p+q,p\rangle. \tag{28}$$

The phase factor that is quadratic in the index p is now expressed in terms of phase factors that are linear in p by means of a discrete Fourier transform [25]

$$\exp(-i\pi rp^2/s)=\sum_{j=0}^{l-1}\alpha_j^{(r,s)}\exp(-2\pi ipj/l), \tag{29}$$

where

$$l=\begin{cases}s & \text{if } r\neq s (\text{mod } 2)\\ 2s & \text{if } r=s=1(\text{mod } 2).\end{cases} \tag{30}$$

The coefficients $\alpha_j^{(r,s)}$ are given by

$$\alpha_j^{(r,s)}=\frac{1}{l}\sum_{p=0}^{l-1}\exp(-i\pi rp^2/s+2i\pi pj/l) \tag{31}$$

and can be evaluated analytically [26]. Substituting in Eq. (28) we get

$$\mathcal{U}(rT_-/s)|p+q,p\rangle=\sum_{j=0}^{l-1}\alpha_j^{(r,s)}\exp(-i\pi p\gamma_j)|p+q,p\rangle, \tag{32}$$

where

$$\gamma_j=qr/s+2j/l. \tag{33}$$

Using the above relation (32), one obtains

$$\mathcal{U}(rT_-/s)|\psi(\eta,\xi,q)\rangle =\sum_{j=0}^{l-1}\alpha_j^{(r,s)}|\psi(\eta\exp(-i\pi\gamma_j),\xi\exp(-i\pi\gamma_j),q)\rangle, \tag{34}$$

which is a superposition of SU(1,1) coherent states [27].

By appropriate changes of the summation index and using the periodicity properties of $\alpha_j^{(r,s)}$, the arguments of ψ can be made independent of the variable r. Thus the j sum in Eq. (34) can be written as

$$\sum_{j=1}^{s}\beta_j^{(r,s)}|\psi(\eta\exp(-i\pi\Gamma_j),\xi\exp(-i\pi\Gamma_j),q)\rangle.$$

For odd values of q,

$$\Gamma_j=(s+1-2j)/s, \tag{35}$$

$$\beta_j^{(r,s)}=\begin{cases}\alpha^{(r,s)}_{[s+1-qr]/2-j} & \text{if } r\neq s(\text{mod } 2)\\ \alpha^{(r,s)}_{s+1-qr-2j} & \text{if } r=s=1(\text{mod } 2).\end{cases} \tag{36}$$

For even values of q,

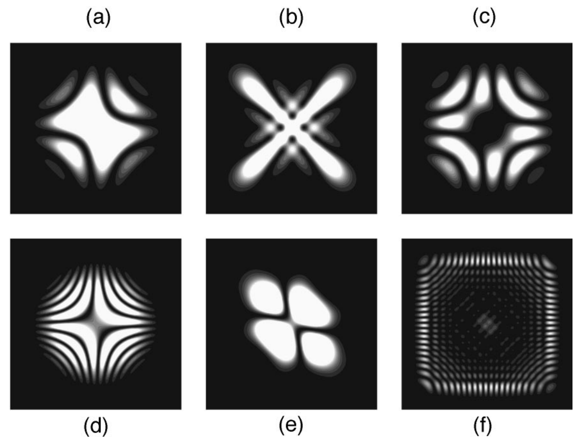

FIG. 3. Same as in Fig. 1, but at $t=T_-/2$.

$$\Gamma_j=\begin{cases}2j/s & \text{if } r\neq s(\text{mod } 2)\\ (2j+1)/s & \text{if } r=s=1(\text{mod } 2),\end{cases} \tag{37}$$

$$\beta_j^{(r,s)}=\begin{cases}\alpha^{(r,s)}_{j-qr/2} & \text{if } r\neq s(\text{mod } 2)\\ \alpha^{(r,s)}_{2j+1-qr} & \text{if } r=s=1(\text{mod } 2).\end{cases} \tag{38}$$

Let us now consider a few simple cases

$$\mathcal{U}(T_-)|\psi(\eta,\xi,0)\rangle=|\psi(-\eta,-\xi,0)\rangle, \tag{39}$$

$$\mathcal{U}(T_-/2)|\psi(\eta,\xi,0)\rangle=\frac{\exp(-i\pi/4)}{\sqrt{2}}[|\psi(\eta,\xi,0)\rangle +i|\psi(-\eta,-\xi,0)\rangle], \tag{40}$$

$$\mathcal{U}(T_-/2)|\psi(\eta,\xi,1)\rangle=\frac{\exp(-i\pi/4)}{\sqrt{2}}[|\psi(-i\eta,-i\xi,1)\rangle +i|\psi(i\eta,i\xi,1)\rangle]. \tag{41}$$

C. Evolution of quadrature distributions

The initial distributions shown in Figs. 1 and 2 are coherent structures. As the system evolves, this coherence is lost rather quickly, but is restored partially (or fully) at times of fractional (or total) revival. Following Eqs. (24)–(26), it is clear that for odd values of q, $P(x,y,T_-)=P(x,y,0)$. For even values of q, on the other hand, $P(x,y,2T_-)=P(x,y,0)$ provided ξ and η are not pure imaginary. Otherwise, one can use Eq. (39) and the fact that the scalar product $\langle x,y|p+q,p\rangle$ is real to show that $P(x,y,T_-)=P(x,y,0)$ for even values of q as well.

We will now seek signatures of fractional revivals in the evolution of $P(x,y,t)$ for various SU(1,1) coherent states as defined in Sec. II B. In Figs. 3 and 4 we have shown the contour plots of the quadrature distribution for the SU(1,1) coherent states $|\psi(\xi,\eta,0)\rangle$ and $|\psi(\xi,\eta,1)\rangle$, respectively, at $t=T_-/2$.

Although the evolution of the two-mode system is effectively governed by one time scale [see comments below Eq. (23)], its structure at times of fractional revival is not necessarily simple, although exact analytical results can be obtained in some cases. Thus for the Perelomov coherent states

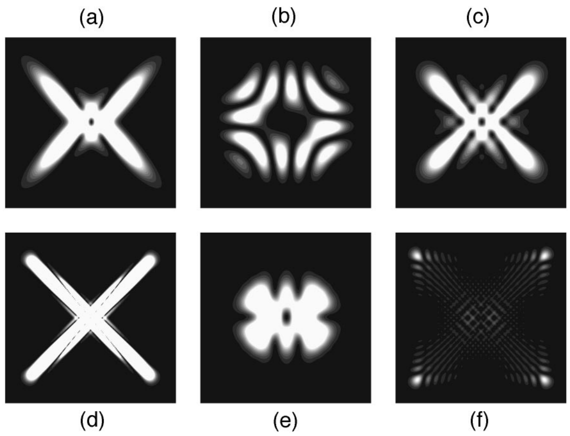

FIG. 4. Same as in Fig. 2, but at $t=T_-/2$.

one can use Eqs. (21) and (40) to show that for $q=0$, $P(x,y,T_-/2)=0$ whenever $xy=n\pi(1+|\eta|^2/8|\eta|)$, where n is an odd integer. The hyperbolic dark fringes are clearly visible in the corresponding plot of Fig. 3. The point to note, however, is that these structures do not seem to contain clearly distinguishable replicas of the initial distribution. The complexity arises due to quantum interference that is augmented not only by the two-mode nature of the system but also by the built-in correlation between the modes via the constraint on the photon number difference q. Nevertheless, we have been able to identify a situation where Schrödinger cats of these SU(1,1) coherent states are formed. This is the case for the pair coherent states when $q=0$. Using Eqs. (39) and (40), we note that at $t=T_-$, $\xi\to-\xi$ so that the initial distribution (Fig. 2, top row, middle column) rotates by $\pi/2$. At $t=T_-/2$, these two distributions add incoherently as the components of the wave function, being $\pi/2$ out of phase, do not interfere with each other. The same is not true for $q=1$ as $\xi\to\pm i\xi$ for the two components [see Eq. (41)], which results in strong interference between them.

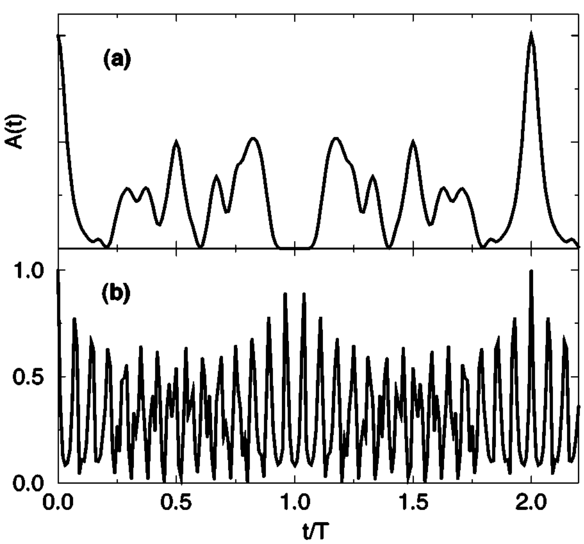

FIG. 5. Autocorrelation function $A(t)=|\langle\psi(0)|\psi(t)\rangle|^2$ of (a) pair coherent states $\psi(0)=\Phi(\xi,q)$ and (b) Perelomov coherent states $\psi(0)=\Psi(\eta,q)$ for $q=0$. We have chosen $\xi=3$ and $\eta=-i\tanh\pi/4$.

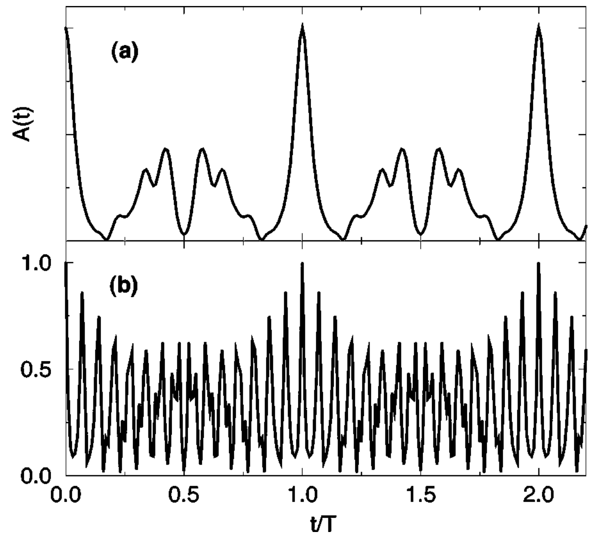

FIG. 6. Same as in Fig. 5, but for $q=1$.

D. Evolution of autocorrelation functions

In Figs. 5 and 6 the autocorrelation function $A(t)=|\langle\psi(0)|\psi(t)\rangle|^2$ of pair and Perelomov coherent states are plotted for $q=0$ and $q=1$, respectively. These plots are in complete agreement with the analytical results presented in the paper. Indeed, for $q=0$, $A(2T_-)=1$ and for $q=1$, $A(T_-)=1$. Furthermore, for pair coherent states, $\langle\Phi(\xi,0)|\Phi(\xi,0)\rangle=1$ and $\langle\Phi(\xi,0)|\Phi(-\xi,0)\rangle=J_0(2|\xi|)/I_0(2|\xi|)$, whereas for Perelomov coherent states, $\langle\Psi(\eta,0)|\Psi(\eta,0)\rangle=1$ and $\langle\Psi(\eta,0)|\Psi(-\eta,0)\rangle=(1-|\eta|^2)/(1+|\eta|^2)$. These scalar products can be used in conjunction with Eqs. (39) and (40) to check the value of the autocorrelation function for these states at $t=T_-$ and $T_-/2$ when $q=0$. A similar analysis can be made for $q=1$ and also for other types of SU(1,1) coherent states. We note in passing that for pair coherent states $\Phi(\xi,0)$, the autocorrelation function can be made to vanish at $t=T_-$ if ξ is chosen such that $2|\xi|$ is a root of the Bessel function $J_0(x)$. In that case, the autocorrelation function will be exactly 1/2 at $t=T_-/2$.

IV. CONCLUSION

In conclusion, we have studied (both analytically and numerically) the evolution of SU(1,1) coherent states, in particular, the pair and Perelomov coherent states, under the action of a phase-insensitive Hamiltonian that is quadratic in terms of the generators of SU(1,1) dynamics. As mentioned in Sec. II, many of these coherent states have been realized for a number of different physical systems. For a generic two-mode Hamiltonian with two associated time scales, we found that the revival dynamics of SU(1,1) coherent states does not depend on the ratio of these time scales. Instead, it depends crucially on the difference in photon numbers of the two modes. These findings are unlike what one obtains for harmonic-oscillator coherent states [20] and can be traced to the different nature of the underlying algebra. However, as in the case of the HO coherent states, we found that the quantum phenomena of revival and fractional revival (or the formation of Schrödinger cats) also occur for the SU(1,1) coherent states. We have provided analytic results for their revival and fractional revival along with numerical plots of the autocorrelation function and the quadrature distribution.

[1] Such revivals have been studied extensively in Rydberg atoms (see Ref. [2]), optical parametric oscillators (see Ref. [3]), the Jaynes-Cummings model (see Ref. [4]), and also, in a general way, the temporal behavior of generic transient signals originating from multilevel quantum systems (see Ref. [5]); see also the recent papers on square-well potentials: D. L. Aronstein and C. Stroud, Phys. Rev. A **55**, 4526 (1997); F. Grobmann, J. M. Rost, and W. P. Schleich, J. Phys. A **30**, L277 (1997).

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