Bose-Einstein condensation of a trapped gas in *n* dimensions

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The critical temperature for Bose-Einstein condensation, ground-state fraction, and heat capacity of an ideal gas of Bose particles that are confined by an *n*-dimensional generic power-law potential trap are derived. The conditions for Bose-Einstein condensation and the discontinuous conditions of the heat capacity at critical temperature are obtained. All these quantities and conditions are found to vary markedly with the dimensionality of space, characteristics of particles, and shape of the external potential. The results obtained here are universal. [S1050-2947(99)07306-0]

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At present there is a renewed interest in Bose-Einstein condensation (BEC), particularly after the achievement to create BEC in magnetically trapped alkali gases [1–3]. The constrained role of the external potential for atomic gases may change the performance of gases. It can be said that the external potential creates favorable conditions for controlling degenerate atomic gases and quantitatively investigating their performance. On the other hand, there are different properties for Bose gases in different dimensions and different kinematics characteristics of particles [4–9]. Therefore, it is very important to investigate thoroughly the main characteristics and to obtain some general theoretical conclusions of a trapped Bose gas in any dimension.

In this paper, we investigate the conditions for BEC and the continuity conditions of the heat capacity of an ideal Bose gas with any kinematics characteristic confined by an *n*-dimensional generic power-law potential trap. First, we derive the critical temperature T_c, ground-state fraction, and heat capacity of the system. Then, from these results, we deduce the two conditions mentioned above and make some significant discussions.

We consider an *n*-dimensional ideal Bose gas with the energy spectrum,

$$\varepsilon = a p^s + \sum_{i=1}^{n} \varepsilon_i \left| \frac{x_i}{a_i} \right|^{t_i}, \tag{1}$$

where a, s, a_i, ε_i, and t_i $(i=1,2,...,n)$ are all positive constants, p is the momentum of a particle, and x_i are the components of coordinate in the direction i of a particle. When the particle number in the system is larger and the level spacing is much smaller than the mean kinetic energy of particles (this condition is often satisfied; for example, in the experiment of Anderson *et al.* [1] in 1995 considering BEC, when the frequency of the harmonic potential $\omega = 2\pi \times 200/s$ is adopted and the temperature approaches a condensate temperature $T_c \approx 170\,\text{nK}$, one still has $\hbar\omega/kT \approx 5.6 \times 10^{-3} \ll 1$), the Thomas-Fermi's semiclassical approximation is valid [10]. Thus, the total number of particles N and the total energy U of the system may be expressed, respectively, as

$$N = N_0 + \sum \frac{g}{e^{(\varepsilon-\mu)/kT}-1} = N_0 + \frac{g}{h^n} \int \frac{d^n p\, dx_1 \ldots dx_n}{z^{-1} e^{\varepsilon/kT}-1} \tag{2}$$

and

$$U = \sum \frac{g\varepsilon}{e^{(\varepsilon-\mu)/kT}-1} = \frac{g}{h^n} \int \frac{\varepsilon\, d^n p\, dx_1 \ldots dx_n}{z^{-1} e^{\varepsilon/kT}-1}, \tag{3}$$

where N_0 is the number of particles in the ground state, k and h are, respectively, the Boltzmann and the Plank constants, g is the spin degenerate factor, μ is the chemical potential, and $z = \exp(\mu/kT)$ is the fugacity. Using the Bose integration,

$$g_l(z) = \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}}{z^{-1} e^x - 1} dx, \tag{4}$$

one can obtain

$$N = N_0 + \frac{2^n g C_n \Gamma\left(\frac{n}{s}+1\right) \prod_{i=1}^{n} a_i \Gamma\left(\frac{1}{t_i}+1\right) g_\rho(z) (kT)^\rho}{h^n a^{n/s} \prod_{i=1}^{n} \varepsilon_i^{1/t_i}} \tag{5}$$

and

$$U = \frac{2^n g C_n \rho \Gamma\left(\frac{n}{s}+1\right) \prod_{i=1}^{n} a_i \Gamma\left(\frac{1}{t_i}+1\right) g_{\rho+1}(z) (kT)^{\rho+1}}{h^n a^{n/s} \prod_{i=1}^{n} \varepsilon^{1/t_i}} \tag{6}$$

from Eqs. (2) and (3), respectively, where

$$\rho = \frac{n}{s} + \sum_{i=1}^{n} \frac{1}{t_i}, \tag{7}$$

$$\Gamma(l) = \int_0^\infty y^{l-1} e^{-y} dy, \tag{8}$$

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is the Gamma function, and $C_n=\pi^{n/2}/\Gamma(n/2+1)$.

When $T\to T_c$, there are $\mu\to 0$ and $N_0=0$. Then, we obtain

$$T_c=\frac{1}{k}\left[\frac{Nh^n a^{n/s}\prod_{i=1}^{n}\varepsilon_i^{1/t_i}}{2^n g C_n \Gamma\left(\frac{n}{s}+1\right)\prod_{i=1}^{n} a_i \Gamma\left(\frac{1}{t_i}+1\right)\zeta(\rho)}\right]^{1/\rho} \tag{9}$$

from Eq. (5), where $\zeta(x)$ is the Riemann zeta function which equals $g_x(1)$ when $x\geqslant 1$. At a temperature T below T_c, from Eqs. (5) and (9), one can obtain the ground-state fraction,

$$\frac{N_0}{N}=1-\left(\frac{T}{T_c}\right)^{\rho}. \tag{10}$$

For a three-dimensional nonrelativistic system familiar to us all, $n=3$ and $s=2$. Equations (9) and (10) can be reduced to the corresponding results obtained in Ref. [11]. However, the results derived in the present paper not only are the general forms of some important conclusions obtained in current literature, but also may be used to derive some general conclusions about BEC.

It can be seen from Eq. (9) that the relevant quantity that determines whether or not the system can condense is given by ρ. That is, only when the following equation,

$$\rho=\frac{n}{s}+\sum_{i=1}^{n}\frac{1}{t_i}>1, \tag{11}$$

is satisfied, may BEC take place. Equation (11) shows that the conditions for BEC of a Bose gas depend not only on the dimensionality of space n and the kinematics characteristic of particles s, but also on the shape of the external potential determined by all exponents t_i. For a Bose gas with $n/s\leqslant 1$, an external potential has a decisive effect on the occurrence of BEC. For example, for a two-dimensional nonrelativistic Bose gas, $n=2$ and $s=2$. When the system is free, $t_1\to\infty$ and $t_2\to\infty$ [11]. BEC cannot take place according to Eq. (11). But when the system is trapped in a harmonic potential, $t_1=t_2=2$ and BEC may take place. Equation (11) also shows that when a proper external potential is taken, BEC may take place in any dimensional space.

For a free Bose gas, all $t_i\to\infty$. Then, Eq. (11) may be written as

$$\frac{n}{s}>1. \tag{12}$$

Equation (12) is consistent with the conditions for BEC obtained in Ref. [5]. This shows that the conditions derived in this paper are more general, especially that they can be used to expound the effect of external potential on the occurrence of BEC.

When the external potential is symmetric, i.e., $t_1=t_2=\cdots=t_n=t$, Eq. (11) may be simplified as

$$\frac{n}{s}+\frac{n}{t}>1. \tag{13}$$

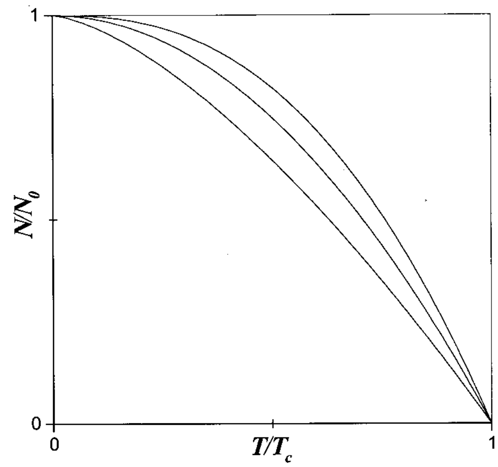

FIG. 1. Ground-state fraction N_0/N as a function of scaled temperature T/T_0 for different ρ.

It is worthwhile to point out that this does not imply that the energy spectrum Eq. (1) may be written as an isotropic form:

$$\varepsilon=ap^s+b\left(\frac{r}{r_0}\right)^t, \tag{14}$$

where both b and r_0 are positive constants and r is the coordinate of a particle. But, it can be shown that when the energy spectrum is isotropic, Eq. (13) is also valid. This also shows that Eq. (11) is quite general.

From Eqs. (10) and (11), one can obtain

$$\left|\frac{dN_0}{dT}\right|_{T_c}>\frac{N}{T_c}. \tag{15}$$

Equation (15) implies that without $|dN_0/dT|>N/T$ at a certain temperature T with $N_0=0$, there cannot be any BEC. This is quite natural, because BEC is a phase transition, which makes a finite fraction of the particles in the system occupy the ground state. Thus, Eq. (15) depicts a common characteristic at the onset of BEC and has universal significance. Therefore, the curves of N_0-T are always convex for Bose systems in BEC phase, even though their curvatures may be different for different Bose systems, as shown in Fig. 1.

Now, we discuss the heat capacity of the trapped Bose gas. It is necessary to point out that the heat capacities discussed here are ones with external potential being kept constant rather than the heat capacities at constant volume or pressure. Under the condition that an external potential is kept constant, from Eqs. (5) and (6), one can obtain the heat capacity of the system at a temperature T above T_c as

$$C_{T>T_c}=\frac{\partial U_{T>T_c}}{\partial T}=Nk\left[\rho(\rho+1)\frac{g_{\rho+1}(z)}{g_\rho(z)}-\rho^2\frac{g_\rho(z)}{g_{\rho-1}(z)}\right]. \tag{16}$$

For a temperature T below T_c, there is $z=1$, so $g_{\rho+1}(z)=\zeta(\rho+1)$ and Eq. (6) may be expressed as

$$U_{T<T_c}=NkT\rho\frac{\zeta(\rho+1)}{\zeta(\rho)}\left(\frac{T}{T_c}\right)^{\rho}. \tag{17}$$

The derivative of Eq. (17) with respect to T gives the heat capacity of the system at a temperature T below T_c as

$$C_{T<T_c}=Nk\rho(\rho+1)\frac{\zeta(\rho+1)}{\zeta(\rho)}\left(\frac{T}{T_c}\right)^{\rho}. \tag{18}$$

From Eqs. (16) and (18), we obtain the difference between the heat capacities at critical temperature T_c as

$$\Delta C_{T_c}=C_{T_c^-}-C_{T_c^+}=Nk\rho^2\frac{g_\rho(1)}{g_{\rho-1}(1)}. \tag{19}$$

Equation (19) shows that depending on the parameter ρ, the system may or may not display a discontinuity in the heat capacity C at T_c. If

$$\rho=\frac{n}{s}+\sum\frac{1}{t_i}>2, \tag{20}$$

C will be discontinuous at T_c. If ρ satisfies

$$1<\rho\leqslant 2, \tag{21}$$

C will be continuous at T_c. It can be seen from Eq. (20) that when $n/s\leqslant 2$, an external potential will decide whether the heat capacity of the system at critical temperature is continuous or not. For example, for a three-dimensional nonrelativistic ideal Bose gas, $n=3$ and $s=2$. Then, $n/s=3/2<2$. When the system is confined in a rigid container of volume V (i.e., $t_1\rightarrow\infty$, $t_2\rightarrow\infty$, and $t_3\rightarrow\infty$), $C(=C_v)$ is continuous at T_c because $\rho=3/2<2$. When the system is confined by a harmonic potential, i.e., $t_1=t_2=t_3=2$, C is discontinuous at T_c because $\rho=3>2$.

Moreover, Eqs. (18) and (19) show that both $C_{T_c^-}$ and ΔC_{T_c} increase monotonically as the parameter ρ increases, while $C_{T_c^+}$ is not a monotonical function of ρ and has a minimum, but the minimum value is close to the value of $C_{T_c^+}$ at $\rho=2$. This indicates that, in general, in the vicinity of T_c, the larger the value of ρ, the larger the energy that is required to raise a Bose system to a higher temperature state, particularly for the system in the BEC phase.

On the other hand, it can be seen from Eq. (18) that when the temperature is quite low, $C_{T<T_c}$ decreases as ρ increases, and the larger the value of ρ, the more quickly that $C_{T<T_c}$ tends to zero. There is no difficulty in understanding it. At very low temperatures, the larger the value of ρ, the larger the fraction of Bosons in the ground state, such that the less the number of Bosons excited when the temperature raises.

In summary, to our knowledge it is the first time for giving the general conditions for Bose-Einstein condensation of a trapped Bose gas in any dimension and for judging whether the heat capacity of the system at critical temperature is continuous or not. Although we have restricted the discussion in this paper to the case of the ideal gas, the essential physics can be revealed. Of course, the inclusion of interactions between the particles profoundly changes the nature of the BEC phase transition [12], and is important for the occurrence of a macroscopic phase. When the interactions between the particles are considered, the two conditions obtained here can be changed but Eq. (15) is still valid. Some calculated results show that the effects of the weak interaction on the two conditions may only change the bounds of ρ. It would be very interesting to study further how the interactions would affect the conditions presented here.

[1] M. H. Anderson *et al.*, Science **269**, 198 (1995).
[2] C. C. Bradely *et al.*, Phys. Rev. Lett. **75**, 1687 (1995).
[3] K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[4] S. R. de Groot *et al.*, Proc. R. Soc. London, Ser. A **203**, 266 (1950).
[5] R. Beckmann *et al.*, Phys. Rev. Lett. **43**, 1277 (1979).
[6] H. P. Rojas, Phys. Lett. A **234**, 13 (1997).
[7] S. H. Kim, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 1053 (1997).
[8] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981).
[9] W. Ketterle and N. J. van Druten, Phys. Rev. A **54**, 656 (1996).
[10] T. T. Chou *et al.*, Phys. Rev. A **53**, 4257 (1997).
[11] V. Bagnato *et al.*, Phys. Rev. A **35**, 4354 (1987).
[12] K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, New York, 1987).