Zero sound modes of dilute Fermi gases with arbitrary spin

S.-K. Yip* and Tin-Lun Ho
Physics Department, The Ohio State University, Columbus, Ohio 43210
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Motivated by the recent success of optical trapping of alkali-metal bosons, we have studied the zero sound modes of dilute Fermi gases with arbitrary spin-f, which are spin-S excitations $(0\leqslant S\leqslant 2f)$. The dispersion of the mode (S) depends on a single Landau parameter $F^{(S)}$, which is related to the scattering lengths of the system through a simple formula. Measurement of (even a subset of) these modes in finite magnetic fields will enable one to determine all the interaction parameters of the system. [S1050-2947(99)07206-6]

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Since the discovery of Bose-Einstein condensation (BEC) in dilute gases of alkali-metal atoms [1], there have been experimental efforts to cool alkali-metal fermions such as ^{6}Li and ^{40}K down to the degenerate limit. In current experiments, alkali-metal atoms are confined in magnetic traps, which confine only the spin states aligned with the local magnetic field. As a result, the spinor nature of the atom is suppressed. The recent success of optical trapping [2], however, changes the situation. In optical traps, different spin components are degenerate in the absence of magnetic fields. One therefore has the opportunity to study dilute Bose gases with integer hyperfine spins (or simply spin) $f>0$ and Fermi gases with spins $f>1/2$. In a recent paper [3], we have discussed the structure of Cooper pairs of alkali-metal fermions in optical traps. Since most alkali-metal fermions have spin $f>1/2$, their Cooper pairs can have *even* spin J ranging from 0 to $2f-1$. The internal structures of these large spin Cooper pairs will give rise of to a great variety of superfluid phenomena.

The purpose of this paper is to study a key *normal*-state property of dilute Fermi gases with general spin f in the degenerate limit—their collisionless or ''zero'' sound. We shall show that in addition to the ordinary density mode, the system has additional modes corresponding to coherent interconversions of different spin species. These modes are the generalizations of the spin waves of spin 1/2 Fermi liquids. As we shall see, the dispersions of the zero sound modes contain the information on *all* the interaction parameters of the system, i.e., the set of s-wave scattering lengths $\{a_J\}$ of two spin-f atoms in the total spin J channel. Thus, observation of these modes will not only provide evidence of the degenerate nature of the system, but also information about the scattering lengths a_J, and hence the existence of superfluid ground state as well as their transition temperatures.

As in our previous study [3], we shall focus on the homogenous case, i.e., without external potential. This is a necessary step before studying trapped fermions. Moreover, it is conceivable that optical traps of the form of cylindrical *boxes* (rather than harmonic wells) be constructed in the future. In that case, the discussions here will be directly applicable. As in our previous work [3], our symmetry classification of the spin structure (which is a crucial step in our solution) also applies to arbitrary potentials.

In addition to homogeneity, we shall also consider the weak magnetic field limit, i.e., when the Zeeman energy is much smaller than the kinetic energy of the system. These are the regimes where the spinor nature of the Fermi gas is manifested most clearly. As demonstrated by the recent experiments at MIT [4], this limit can be easily achieved by specifying the total spin of the system. Since the low energy dynamics of the system is spin conserving [5], the specified spin cannot relax. The system, therefore, sees an effective magnetic field with which its spin would be in equilibrium, a field that can be much smaller than the external field B^{ext}. In the following, we shall refer to this effective field simply as ''magnetic field'' B, with the understanding that it is a Lagrange multiplier that determines the total spin of the system [4].

A. Zero magnetic field

We begin with the linearized kinetic equation for the distribution function matrix $\delta\hat{n}_{\mathbf{p}}$ in the collisionless regime,

$$\frac{\partial \delta\hat{n}_{\mathbf{p}}(\mathbf{r},t)}{\partial t}+\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}\left(\delta\hat{n}_{\mathbf{p}}(\mathbf{r},t)-\frac{\partial n^{o}_{\mathbf{p}}}{\partial\epsilon_{\mathbf{p}}}\delta\hat{\epsilon}_{\mathbf{p}}(\mathbf{r},t)\right)=0. \quad (1)$$

Our notations in Eq. (1) are the same as in Ref. [6]. Here, $n^{o}_{\mathbf{p}}$ is the $T=0$ Fermi function, $\mathbf{v}_{\mathbf{p}}=\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}$, $\delta\hat{n}_{\mathbf{p}}$ is a $(2f+1)\times(2f+1)$ matrix in spin space, $[\delta\hat{n}_{\mathbf{p}}(\mathbf{r},t)]_{\alpha\beta}=\int d\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\psi^{+}_{\beta}(\mathbf{r}-\mathbf{x}/2,t)\psi_{\alpha}(\mathbf{r}+\mathbf{x}/2,t)\rangle$ where ψ_α is the field operator. The energy matrix $\delta\hat{\epsilon}$ describes the change in the Hamiltonian due to $\delta\hat{n}$,

$$[\delta\epsilon_{\mathbf{p}}]_{\alpha\beta}=\int d\tau' f_{\alpha\gamma,\beta\delta}(\mathbf{p},\mathbf{p}')[\delta n_{\mathbf{p}'}]_{\delta\gamma}, \quad (2)$$

where $d\tau'$ means $d\mathbf{p}'/(2\pi)^3$, and $f_{\alpha\gamma,\beta\delta}(\mathbf{p},\mathbf{p}')$ are the Landau parameters, which can be extracted from the Hamiltonian of the system derived by one of us [5]. It is shown in Ref. [5] that only the lowest hyperfine states (with spin f) will remain in the optical trap and that the interactions between these spin-f atoms are spin conserving, of the form

$$H_{\text{int}}=\frac{1}{2}\sum_{J=0,2,\ldots}^{2f-1} g_J\sum_{M=-J}^{J}\int d\mathbf{r}O^{+}_{JM}(\mathbf{r})O_{JM}(\mathbf{r}), \quad (3)$$

*Present address: Physics Division, Center for Theoretical Sciences, P.O. Box 2-131, Hsinchu, Taiwan 300, R.O.C.

where $O_{JM}(\mathbf{r})=\Sigma_{\alpha\beta}\langle JM|ff\alpha\beta\rangle\psi_\alpha(\mathbf{r})\psi_\beta(\mathbf{r})$, and $\langle JM|f_1f_2\alpha\beta\rangle$ are the Clebsch-Gordan coefficients for forming a spin-J object from a spin-f_1 and a spin-f_2 particle [7], $g_J=4\pi\hbar^2 a_J/M_F$, and M_F is the mass of the atom. Pauli principle implies that only even J's exist in Eq. (3). Evaluating $\langle H_{\rm int}\rangle$ in Hartree-Fock approximation, and using the fact that $\delta\langle\psi_\alpha^+(\mathbf{r})\psi_\beta(\mathbf{r})\rangle=\int d\tau[\delta n_{\mathbf{p}}(\mathbf{r})]_{\beta\alpha}$, it is straightforward to show that

$$f_{\alpha\gamma,\beta\delta}=2\sum_{J=0,2,\ldots}^{2f-1} g_J\sum_{M=-J}^{J}\langle JM|ff\gamma\alpha\rangle\langle JM|ff\delta\beta\rangle, \quad (4)$$

which is momentum independent as a result of s-wave interaction. Note that if $g_J<0$, the system will have a superfluid instability towards spin-J Cooper pairs at a sufficiently low temperature $T_c^{(J)}$ [3]. Our discussions for negative g_J's therefore apply to temperatures above $T_c^{(J)}$ but low enough that the Fermi gas is degenerate. Before proceeding, we simplify Eq. (1) by writing $\delta\hat{n}_{\mathbf{p}}=(-\partial n^o_{\mathbf{p}}/\partial\epsilon_{\mathbf{p}})\hat{\nu}_{\hat{\mathbf{p}}}$, which turns Eqs. (1) and (2) into

$$\partial_t\hat{\nu}_{\hat{\mathbf{p}}}+\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}(\hat{\nu}_{\hat{\mathbf{p}}}+\delta\hat{\epsilon}_{\hat{\mathbf{p}}})=0,\quad \delta\hat{\epsilon}_{\hat{\mathbf{p}}}=N_F f_{\alpha\gamma,\beta\delta}\langle\hat{\nu}_{\hat{\mathbf{p}}}\rangle, \quad (5)$$

where $N_F=mk_F/2\pi^2\hbar^2$ is the density of state of a single spin component at the Fermi surface, k_F is the Fermi wave vector, and $\langle(\ )\rangle\equiv\int(d\hat{\mathbf{p}}/4\pi)(\ )$ denotes the angular average over the Fermi surface. Note that the quasiparticle energy $\delta\hat{\epsilon}_{\hat{\mathbf{p}}}$ is isotropic in k space as a consequence of the s-wave interactions between the particles.

Next, we note that a rotation $\vec{\theta}$ in spin space will cause a change $a_\alpha\to D^{(f)}_{\alpha\beta}(\vec{\theta})a_\beta$, where $D^{(f)}_{\alpha\beta}$ is the rotation matrix in the spin-f space. This implies $\hat{\nu}\to\hat{\nu}'=\hat{D}^{(f)}\hat{\nu}\hat{D}^{(f)+}$. From Eqs. (2) and (4), one can see that $\delta\hat{\epsilon}_{\mathbf{p}}$ transforms the same way, $\delta\hat{\epsilon}_{\mathbf{p}}\to\hat{D}^{(f)}\delta\hat{\epsilon}_{\mathbf{p}}\hat{D}^{(f)+}$. Since $\hat{\nu}$ is made up of two spin-f objects, it can be decomposed into a sum of spin-S quantities $\hat{\nu}^{(S,M)}$, which transform as $[\hat{D}^{(f)}\hat{\nu}^{(S,M)}\hat{D}^{(f)+}]_{\alpha\beta}=[\hat{\nu}^{(S,M')}]_{\alpha\beta}D^{(S)}_{M'M}$, where $0\leq S\leq 2f$, $-S\leq M\leq S$. The solution of this equation is easily seen to be $[\delta\hat{n}^{(S,M)}]_{\alpha\beta}\propto\langle f\alpha|SfM\beta\rangle$. We then have the representation

$$\begin{pmatrix}[\nu_{\mathbf{p}}(\mathbf{r},t)]_{\alpha\beta}\\ [\delta\epsilon_{\mathbf{p}}(\mathbf{r},t)]_{\alpha\beta}\end{pmatrix}=\sum_{S,M}\langle f\alpha|SfM\beta\rangle\begin{pmatrix}\nu^{(S,M)}_{\hat{\mathbf{p}}}(\mathbf{r},t)\\ \delta\epsilon^{(S,M)}_{\hat{\mathbf{p}}}(\mathbf{r},t)\end{pmatrix}. \quad (6)$$

Substituting Eq. (6) into Eq. (5) and using the identity

$$\sum_{\gamma\delta M'}\langle JM'|ff\gamma\alpha\rangle\langle JM'|ff\delta\beta\rangle\langle f\delta|SfM\gamma\rangle$$
$$=(-)^{2f-J}(2J+1)W(ffff;JS)\langle f\alpha|SfM\beta\rangle, \quad (7)$$

where W is the Racah coefficient [8], Eq. (5) becomes diagonal in the (S,M) modes,

$$\partial_t\hat{\nu}^{(S,M)}_{\hat{\mathbf{p}}}+\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}(\hat{\nu}^{(S,M)}_{\hat{\mathbf{p}}}+\delta\hat{\epsilon}^{(S,M)}_{\hat{\mathbf{p}}})=0, \quad (8)$$

$$\delta\epsilon^{(S,M)}_{\hat{\mathbf{p}}}=F^{(S)}\langle\nu^{(S,M)}_{\hat{\mathbf{p}}}\rangle, \quad (9)$$

$$F^{(S)}=-\sum_{J=0,2,\ldots}^{2f-1}\frac{4k_Fa_J}{\pi}(2J+1)W(ffff;JS), \quad (10)$$

where we have used the fact that $N_Fg_J=2k_Fa_J/\pi$ and $(-1)^{2f-J}=-1$ in obtaining Eq. (10). Equations (8) and (9) imply that

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_{\mathbf{p}}\cdot\nabla\right)\nu^{(S,M)}_{\hat{\mathbf{p}}}+F^{(S)}\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}\langle\nu^{(S,M)}_{\hat{\mathbf{p}}}\rangle=0, \quad (11)$$

which is precisely the equation for the ordinary zero sound mode with only $\ell=0$ spin-symmetric Landau parameter $F^s_{\ell=0}$ nonzero [6] and is given by $F^{(S)}$.

The dispersion relations of modes described by Eq. (11) is well known [6]. They are

$$1=F^{(S)}\int_{-1}^{1}\frac{dx}{2}\frac{qv_Fx}{\omega-qv_Fx}. \quad (12)$$

The properties of the modes depend crucially on the sign of the parameters $F^{(S)}$. When $F^{(S)}>0$, one has a well-defined propagating mode. When $-1<F^{(S)}<0$, the zero sound mode is Landau damped. When $F^{(S)}<-1$, the system is unstable against spin-S distortions. Because of the dilute limit, $k_Fa\ll 1$ and hence $|F^{(S)}|<1$, stability against spin-S distortions is guaranteed.

It is instructive to consider some special cases.

(i) The density modes $\hat{\nu}^{(S=0)}$ for fermions with arbitrary spin f. Using the fact that $W(ffff;J0)=(-1)^{2f-J}/(2f+1)$, we have

$$F^{(S=0)}=\frac{4}{\pi(2f+1)}\sum_{J=0,2,\ldots}^{2f-1}(2J+1)k_Fa_J. \quad (13)$$

In particular, if there are no superfluid instabilities in all angular momentum J channel, then $F^{(S=0)}>0$ and the density mode will not be Landau damped.

(ii) Fermions with spin-1/2, 3/2, and 5/2. For $f=1/2$, Eq. (13) reduces to the well-known results $F^{(S=0)}=-F^{(S=1)}=N_Fg_0=2k_Fa_0/\pi\hbar$ [6]. For $f=3/2$, using the tabulated values of the $6j$ symbols [9] to calculate the Racah coefficients, we find $F^{(S=0)}=k_F(a_0+5a_2)/\pi$, $F^{(S=1)}=F^{(S=3)}=-k_F(a_0+a_2)/\pi$, and $F^{(S=2)}=k_F(a_0-3a_2)/\pi$. Thus the $S=1$ and 3 modes are always degenerate, and the degeneracy between the $S=0$ and $S=2$ modes are lifted only by the interaction in the $J=2$ channel. If there are no superfluid instabilities in any J channel, i.e., both $a_0,a_2>0$, then $S=1$ and $S=3$ modes are always Landau damped.

For large f's there are no obvious systematics except for the $S=0$ result noted above. Modes for different S's are typically not degenerate, barring accidental values of g_J's. In the case of $f=5/2$, such as ^{22}Na and ^{86}Rb, we obtain

$$F^{(0)}=(2a_0/3+10a_2/3+6a_4)k_F/\pi,$$

$$F^{(1)}=-(2a_0/3+46a_2/21-6a_4/7)k_F/\pi,$$

$$F^{(2)}=(2a_0/3+a_2/3-3a_4)k_F/\pi,$$

$$F^{(3)}=-(2a_0/3-29a_2/21+19a_4/7)k_F/\pi,$$

$$F^{(4)}=(2a_0/3-5a_2/3-a_4)k_F/\pi,$$

and

$$F^{(5)}=-(2a_0/3+25a_2/21+a_4/7)k_F/\pi.$$

(iii) Undamped zeroth sound modes in the dilute limit. Zero sound modes (S,M) with $F^{(S)}>0$ are propagating [6]. Because of the dilute condition $k_F a_J \ll 1$, we have $F^{(S)}\ll 1$. In this limit, Eq. (12) can be integrated to give [6],

$$\omega^{(S)}(q)=qv_F(1+2e^{-2}e^{-2/F^{(S)}}). \tag{14}$$

Since $F^{(S)}\ll 1$, the exponential term in Eq. (14) will have little contribution. The frequencies of zero sound for all S are essentially given by qv_F. As a result, it will be difficult to obtain information on the interaction parameters from zero sound frequencies in zero field. On the other hand, we shall see that even a small magnetic field will cause significant changes in the zero sound dispersions, which lead to many observable features and enable one to determine all the interaction parameters.

B. Weak magnetic fields

When $B\neq 0$, the kinetic equation, Eq. (1), will have an additional term $\mathcal{I}_{\mathbf{p}}=(i/\hbar)[\hat{\epsilon}_{\mathbf{p}},\hat{n}_{\mathbf{p}}]$ on the left-hand side [6]. At the same time, the *equilibrium* distribution function and quasiparticle energy (denoted as $\hat{n}^o_{\mathbf{p},B}$ and $\hat{\epsilon}^o_{\mathbf{p},B}$, respectively) are altered from the zero field values ($\hat{n}^o_{\mathbf{p}}$ and $\hat{\epsilon}^o_{\mathbf{p}}$). The difference $\delta\hat{\epsilon}^o_{\mathbf{p}}=\hat{\epsilon}^o_{\mathbf{p},B}-\hat{\epsilon}^o_{\mathbf{p}}$ is $[\delta\epsilon^o_{\mathbf{p}}]_{\alpha\beta}=-\mu BF^z_{\alpha\beta}+\int d\tau' f_{\alpha\gamma,\beta\delta}[\delta n^o_{\mathbf{p}'}]_{\delta\gamma}$, with $\delta\hat{n}^o_{\mathbf{p}}=\hat{n}^o_{\mathbf{p},B}-\hat{n}^o_{\mathbf{p}}=(\partial n^o/\partial\epsilon_{\mathbf{p}})\delta\hat{\epsilon}^o_{\mathbf{p}}$. These two relations imply

$$[\delta\epsilon^o_{\mathbf{p}}]_{\alpha\beta}=-\mu BF^z_{\alpha\beta}-N_F f_{\alpha\gamma,\beta\delta}[\delta\epsilon^o_{\mathbf{p}}]_{\delta\gamma}, \tag{15}$$

where μ is the magnetic moment of the atom, and $F^z_{\alpha\beta}$ is the matrix representation of the z component of the hyperfine spin $\mathbf{F}$ operator. The solution of Eq. (15) is $\delta\hat{\epsilon}^o_{\mathbf{p}}=c\hat{F}^z$. Using the fact that $(F^z)_{\alpha\beta}\propto\langle f\alpha|1f0\beta\rangle$, it is easy to show from Eqs. (6), (10), and (15) that

$$\delta\hat{\epsilon}^o_{\mathbf{p}}=-\mu B^{\text{eff}}\hat{F}^z,\quad B^{\text{eff}}=B/(1+F^{(1)}). \tag{16}$$

Linearizing about the equilibrium configuration $\hat{n}^o_{\mathbf{p},B}$ and $\hat{\epsilon}^o_{\mathbf{p},B}$, we have $\mathcal{I}_{\mathbf{p}}=(i/\hbar)([\delta\hat{\epsilon}^o_{\mathbf{p}},\delta\hat{n}_{\mathbf{p}}]+[\delta\hat{\epsilon}_{\mathbf{p}},\delta\hat{n}^o_{\mathbf{p}}])$. From the definition $\delta\hat{n}_{\mathbf{p}}=-(\partial n^o_{\mathbf{p}}/\partial\epsilon_{\mathbf{p}})\hat{\nu}_{\mathbf{p}}$, the relation $\delta\hat{n}^o_{\mathbf{p}}=(\partial n^o_{\mathbf{p}}/\partial\epsilon_{\mathbf{p}})\delta\hat{\epsilon}^o_{\mathbf{p}}$, and the property $(\alpha-\beta)[\nu_{\mathbf{p}}]_{\alpha\beta}=\Sigma_{SM}\langle f\alpha|SfM\beta\rangle M\nu^{(S,M)}_{\mathbf{p}}$ which follows from Eq. (6), we have

$$[\mathcal{I}_{\mathbf{p}}]_{\alpha\beta}=\frac{i}{\hbar}\left(\frac{\partial n^o}{\partial\epsilon_{\mathbf{p}}}\right)\sum_{SM}\langle f\alpha|SfM\beta\rangle(\Omega M)(\nu^{(S,M)}_{\hat{\mathbf{p}}}+\delta\epsilon^{(S,M)}_{\hat{\mathbf{p}}}), \tag{17}$$

where $\Omega\equiv\mu B^{\text{eff}}/\hbar$. With this additional term on the left-hand side of Eq. (1) and repeating the procedure as before, we find that Eq. (9) remains unchanged, whereas Eq. (8) becomes

$$\partial_t\hat{\nu}^{(S,M)}_{\hat{\mathbf{p}}}+[\mathbf{v}_{\mathbf{p}}\cdot\boldsymbol{\nabla}-i\Omega M](\hat{\nu}^{(S,M)}_{\hat{\mathbf{p}}}+\delta\hat{\epsilon}^{(S,M)}_{\hat{\mathbf{p}}})=0. \tag{18}$$

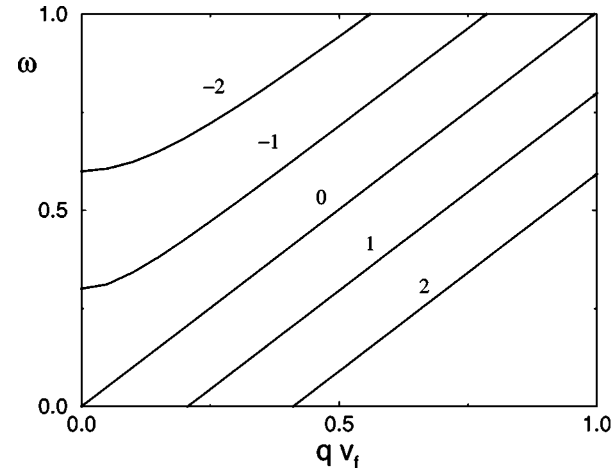

FIG. 1. The zero sound mode for $F^{(2)}=0.5>0$, f arbitrary, and $\Omega=+0.2$. From upper to lower, the curves correspond to $M=-2,-1,0,1,$ and 2, respectively. ω, qv_F, and Ω are plotted with arbitrary units. The vertical intercepts of curves $M=-2$ and -1 are $\omega^{(2,-2)}=2\Omega(1+F^{(2)})$ and $\omega^{(2,-1)}=\Omega(1+F^{(2)})$, respectively. The horizontal intercepts of the $M=1$ and $M=2$ curves are $q_1v_F=\Omega$ and $q_2v_F=2\Omega$, respectively. Zero sound modes for other S have different number of branches but behave similarly.

Thus, the zero sound modes can still be classified by the quantum numbers (S,M) in the weak field limit. The equation for the dispersion now becomes

$$1=F^{(S)}\int_{-1}^{1}\frac{dx}{2}\,\frac{qv_Fx-\Omega M}{\omega+\Omega M-qv_Fx}, \tag{19}$$

which, upon integration, gives

$$\frac{1}{F^{(S)}}=\frac{\omega}{2qv_F}\ln\frac{\omega+\Omega M+qv_F}{\omega+\Omega M-qv_F}-1,\quad \Omega=\frac{\mu B}{1+F^{(1)}}. \tag{20}$$

Since the collective modes are excitations above the ground state, we only need to study the $\omega>0$ solutions of Eq. (20). In the following, we shall discuss only the zero sound modes that are not Landau damped, which require $|\omega+\Omega M|>qv_F$ in Eq. (20). While many features of these propagating modes can be obtained analytically, we first display the numerical solutions of Eq. (20) for $S=3/2$ with $F^{(S)}>0$ and $F^{(S)}<0$ in Figs. 1 and 2, respectively. The notable features of these modes are:

(i) Zero sound modes near $q=0$. Near $q=0$, $qv_F/|\Omega M|\ll F^{(S)}$; it is easily seen from Eq. (20) that [10]

$$\omega^{(S,M)}(q)=-\Omega M(1+F^{(S)})\left[1+\frac{1}{3F^{(S)}}\left(\frac{qv_F}{\Omega M}\right)^2+\cdots\right]. \tag{21}$$

Since $\omega^{(S,M)}>0$, only $\Omega M<0$ modes can be excited at $q=0$. Note that *all $q=0$ modes in finite field are not Landau damped irrespective of the sign of $F^{(S)}$*. From Eq. (21), one can also see that all $\omega^{(S,-|M|)}$ modes increase (decrease) as q^2 for $F^{(S)}>0$ and <0 (see also Figs. 1 and 2).

(ii) The $F^{(S)}>0$ case. For $F^{(S)}>0$, zero sound modes with $\Omega M>0$ emerge from $\omega=0$ when $q>q_M\equiv\Omega M/v_F$ (see

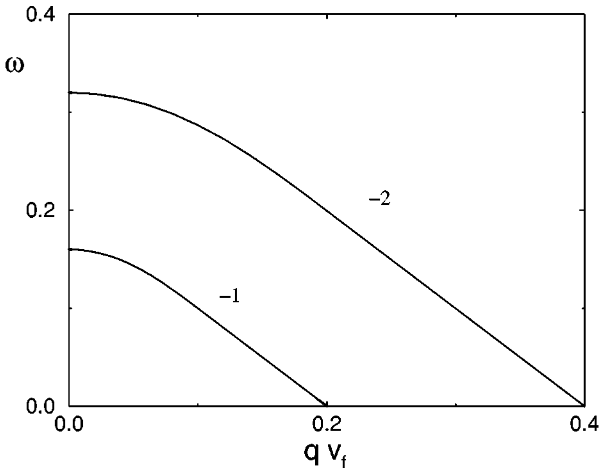

FIG. 2. The zero sound mode for the case $F^{(2)}=-0.2<0$, f arbitrary. From upper to lower, the curves correspond to $M=-2$ and -1. The value of Ω and the expressions for the vertical intercepts are identical to those in Fig. 1. The horizontal intercepts are $q_{-1}v_F=\Omega$ and $q_{-2}v_F=2\Omega$, respectively.

Fig. 1). Expanding Eq. (20) about $(\omega=0,q=q_M)$, one finds that the dispersion in this neighborhood is

$$\omega-(qv_F-\Omega M)=2\Omega M e^{-(1+1/F^{(S)})(2\Omega M/\omega)}. \quad (22)$$

Another simple feature one can derive from Eq. (20) is that as q increases so that $qv_F \gg |\Omega M|$, $\omega^{(S,M)}(q)\to\omega^{(S)}(q)-\Omega M$. The dispersions for all (S,M) modes become parallel to $\omega=qv_F$, with all $\Omega M<0(>0)$ modes shifted up (down) by an amount of $|M\Omega|$ (see Fig. 1). It is also straightforward to show that zero sound modes with $\Omega M\neq 0$ lie above the particle-hole continuum of that particular M state, i.e., $\omega>-\Omega M+qv_F$.

(iii) The $F^{(S)}<0$ case. We find that the $\Omega M<0$ modes decrease monotonically as q increases, and vanish at $q_{|M|}=|\Omega M|/v_F$ in a manner similar to Eq. (22). The entire mode M lies below the corresponding particle-hole continuum, i.e., $\omega<-\Omega M-qv_F$. Solving Eq. (20) graphically, one can also see that there are no solutions with $\omega>0$ when $\Omega M>0$, implying the absence of zero sound modes with $\Omega M>0$.

Determination of scattering lengths. For scattering lengths $|a_J|\sim 100a_B$ where a_B is the Bohr radius, a Fermi gas of density $\sim 10^{13}$ cm^{-1} will have $k_F a_J\sim 10^{-1}$, which implies $\{F^{(S)}\}\sim 10^{-1}$. As mentioned in Part (A), it will be hard to determine the scattering lengths a_J from the $B=0$ zero sound modes for these values of $\{F^{(S)}\}$ because of their small contributions. On the other hand, in the presence of magnetic field, different zero sound modes (S,M) are separated. Since the interaction contributions to the zero sound frequency at $q=0$ and to the critical wave-vector q_M are of the form $1+F^{(S)}$ instead of the essential singularity form in Eq. (14), their contributions should be measurable for $k_F a_J \sim 10^{-1}$ or even smaller. Note also that there are only $(2f+1)/2$ scattering lengths $[a_J,(J=0,2,\ldots,2f-1)]$ whereas the number of zeroth sound modes in finite field is $\Sigma_{S=0}^{2f}(2S+1)=(2f+1)^2$. Even though some of these modes may not be excited (as in the case of $F^{(S)}<0$), there are still more the conditions on a_J provided by the zero sound frequencies than the number of a_J themselves. Thus, it is possible to determine the entire set of scattering lengths $\{a_J\}$ from the zero sound dispersions.

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