Ground-state energy of the spinor Bose-Einstein condensate

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It is known that for a weakly interacting Bose-Einstein condensate (BEC), the assumption of a two-body δ interaction described by a constant coupling strength gives rise to a divergent ground-state energy. A similar divergence occurs in the spinor condensate in which the spin-spin interaction is included in addition to the repulsive δ interaction. In this paper, we examine, in the standard Bogoliubov approximation, the ground-state energy of a homogeneous spinor BEC with hyperfine spin $f=1$. The renormalized coupling constants are calculated and expressed in terms of the bare ones using the standard second-order perturbation method. With these renormalized coupling constants, we show that the ultraviolet divergence of the ground-state energy can be exactly eliminated. [S1050-2947(99)01106-3]

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Recently, Stamper-Kurn *et al.* [1] have successfully confined a Bose-Einstein condensate (BEC) of ^{23}Na atoms in an optical dipole trap. In their experiment, a multicomponent BEC, which is characterized by the three hyperfine spin states $|f=1,m_f=\pm 1,0\rangle$, has been observed. This has opened an interesting possibility of exploring the multicomponent BEC with complicated internal spin dynamics in which not only the global U(1) symmetry but also the rotational SO(3) symmetry in spin space are involved [2,3].

An important feature of the spinor condensate is that, in addition to the repulsive binary hard-core collisions that give rise to the density-density interaction, atoms in the condensates can also couple to each other via the spin-exchange interaction. Assuming that the interaction for each spin exchange channel is again characterized by zero-range delta-potential scattering, one thus obtains the interacting term $\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}$ where $\hat{\mathbf{S}}$ is the spin-density operator. The competition between these two interactions thus leads to an intriguing scenario of the spin dynamics of the spinor BEC, which is characterized by a complex ground-state structure [2–4].

The question that arises now is how these two-body interactions alter the dynamical properties of the condensates. It is known that in a weakly interacting Bose-condensed system, the two-body interactions play a crucial role in determining the low-temperature properties of the systems, which will modify the ground state of the many-particle systems and cause a depletion of the condensate fraction even at zero temperature. Moreover, a divergence could possibly appear when we calculate the ground-state energy in the standard Bogoliubov approximation. This divergence is due to the naive assumption of a constant matrix element of binary interaction irrespective of the relative momenta of the interacting particles. A well-illustrated example is the ultraviolet divergence occurring in the calculated ground-state energy of the one-component BEC where the two-body interaction is described by the repulsive hard-core collisions with a momentum-independent coupling constant [8]. To eliminate such a divergence and gain more insight into the ground-state properties of the condensates, one has to calculate the *s*-wave scattering length at least to second order in the coupling constant. By expanding the coupling constant in powers of the *s*-wave scattering length the previously mentioned divergent ground-state energy can be rendered finite. This is expected since in a physically sensible theory the ground-state energy must assume a finite value when expressed in terms of physically measurable quantities. In other generalized Bose condensed systems such as the spinor BEC ultraviolet divergences of the same sort could also appear. It is, therefore, quite essential to verify if a similar procedure could completely remove these divergences, and in this paper we address this issue in detail for the $f=1$ spinor BEC in the presence of a constant magnetic field.

It should be noted that current BEC experiments are carried out in atomic traps, and the results for homogeneous systems are less likely to be applicable when inhomogeneous systems are under consideration. Nevertheless, the results obtained from a homogeneous system may serve to provide primary estimates for certain physical quantities of a trapped BEC (e.g., critical temperature [5] and damping rate of the collective excitation [6]). Moreover, in the WKB semiclassical approximation, the homogeneous results are directly used to determine the spectrum of elementary excitations, which is then exploited to calculate various thermodynamic quantities [7]. Apart from these, the extension of the present analytic approach to the inhomogeneous systems is technically difficult. As will become clear later, in order to remove the divergence and determine the ground-state energy, one needs precisely the elementary excitation spectra and the corresponding wave functions of the system. Unfortunately, seeking the closed forms of the elementary excitations for an inhomogeneous BEC remains as a great theoretical challenge even for the much simpler one-component BEC, to say nothing of the more complicated spinor BEC. On these grounds, we are therefore concerned mainly with the homogeneous system in this paper.

Consider an assembly of homogeneous dilute Bose gas with hyperfine spin $f=1$. The natural basis set to characterize such a system is the hyperfine spin states $|m_f=\pm 1,0\rangle$. However, in view of the special symmetrical forms of the spin-1 matrix representations one may adopt the basis set $\{|x\rangle,|y\rangle,|z\rangle\}$ whose elements are defined as the eigenstate of the αth component of the spin operator with eigenvalue 0, i.e., $S_\alpha|\alpha\rangle=0(\alpha=x,y,z)$ such that the matrix elements for the αth spin component is given by $\langle\gamma|S_\alpha|\beta\rangle=i\varepsilon_{\alpha\beta\gamma}$ where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor. This representation enables us to relate the spin-1 matrix elements to those of the space rotation, allowing the order parameter to behave as a vector under spin space rotation.

For the $f=1$ spinor BEC, the bosonic atomic field can be described by the three-component field operator $\mathbf{\Psi}$, with components $\psi_\alpha(\mathbf{r})(\alpha=x,y,z)$, and thus the particle number and spin densities can be written as $\hat{n}=\psi^\dagger_\alpha\psi_\alpha$, and $\hat{S}_\alpha=\psi^\dagger_\beta S_\alpha\psi_\beta=-i\epsilon_{\alpha\beta\gamma}\psi^\dagger_\beta\psi_\gamma$, respectively. Note that we have used the summation convention over the indices of component $\alpha,\beta,\ldots$ throughout this paper. Now, without loss of generality, the Hamiltonian density can be constructed in the presence of a constant magnetic field $\mathbf{B}$ pointing to the z direction [3]:

$$H=-\psi^\dagger_\alpha\frac{\nabla^2}{2m}\psi_\alpha+\frac{1}{2}g_n\hat{n}^2+\frac{1}{2}g_s\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}-\mathbf{\Omega}\cdot\hat{\mathbf{S}}\quad(\hbar=1),\tag{1}$$

where $\mathbf{\Omega}=\Omega\hat{z}=g_\mu\mathbf{B}$ (g_μ:gyromagnetic ratio) is the Larmor frequency in a vectorial notation. Expanding $\hat{n}$ and $\hat{S}_\alpha$ in terms of the field operators, Eq. (1) can be expressed as

$$H=-\psi^\dagger_\alpha\frac{\nabla^2}{2m}\psi_\alpha+\frac{1}{2}g_1\psi^\dagger_\beta\psi^\dagger_\alpha\psi_\alpha\psi_\beta+\frac{1}{2}g_2\psi^\dagger_\beta\psi^\dagger_\beta\psi_\alpha\psi_\alpha+i\epsilon_{\alpha\beta\gamma}\Omega_\gamma\psi^\dagger_\alpha\psi_\beta,\tag{2}$$

where the two new coupling constants are given by $g_1=g_n+g_s, g_2=-g_s$. According to the recent spectroscopic experiment by Abraham *et al.* [9], it is conceivable in general that g_2 is comparable to g_1 in magnitude and can be either positive or negative. It is known that the positive g_2 implies the ferromagnetic coupling while the negative one implies the antiferromagnetic coupling for the spin-exchange interaction.

Since the system is homogeneous, the field operator can be expanded in terms of creation and annihilation operators characterized by momentum $\mathbf{k}$,

$$\psi_\alpha(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}a_{\alpha,\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}},\tag{3}$$

where V denotes the volume of the system. Accordingly, the Hamiltonian in momentum space now reads as

$$H=H_0+H_{\mathrm{mag}}+H_{\mathrm{int}},\tag{4}$$

where

$$H_0=\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}a^\dagger_{\alpha,\mathbf{k}}a_{\alpha,\mathbf{k}},\tag{5}$$

$$H_{\mathrm{mag}}=\sum_{\mathbf{k}}i\varepsilon_{\alpha\beta\gamma}\Omega_\gamma a^\dagger_{\alpha,\mathbf{k}}a_{\beta,\mathbf{k}},\tag{6}$$

$$H_{\mathrm{int}}=\frac{g_1}{2V}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4}a^\dagger_{\beta,\mathbf{k}_4}a^\dagger_{\alpha,\mathbf{k}_3}a_{\alpha,\mathbf{k}_2}a_{\beta,\mathbf{k}_1}+\frac{g_2}{2V}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4}a^\dagger_{\beta,\mathbf{k}_4}a^\dagger_{\beta,\mathbf{k}_3}a_{\alpha,\mathbf{k}_2}a_{\alpha,\mathbf{k}_1}.\tag{7}$$

In the ground state, most particles occupy the $\mathbf{k}=0$ states. As a result, the scattering between two nonzero-momentum states can be ignored and the interacting part of the Hamiltonian can be replaced by

$$\begin{aligned}H_{\mathrm{int}}\simeq{}&\frac{g_1}{2V}\Big[a^\dagger_{\beta,0}a^\dagger_{\alpha,0}a_{\alpha,0}a_{\beta,0}+\sum_{\mathbf{k}\neq0}(a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\alpha,-\mathbf{k}}a_{\alpha,0}a_{\beta,0}\\&+a^\dagger_{\beta,0}a^\dagger_{\alpha,0}a_{\alpha,\mathbf{k}}a_{\beta,-\mathbf{k}}+2a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\alpha,0}a_{\alpha,\mathbf{k}}a_{\beta,0}\\&+2a^\dagger_{\beta,0}a^\dagger_{\alpha,\mathbf{k}}a_{\alpha,\mathbf{k}}a_{\beta,0})\Big]+\frac{g_2}{2V}\Big[a^\dagger_{\beta,0}a^\dagger_{\beta,0}a_{\alpha,0}a_{\alpha,0}\\&+\sum_{\mathbf{k}\neq0}(a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\beta,-\mathbf{k}}a_{\alpha,0}a_{\alpha,0}+a^\dagger_{\beta,0}a^\dagger_{\beta,0}a_{\alpha,\mathbf{k}}a_{\alpha,-\mathbf{k}}\\&+4a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\beta,0}a_{\alpha,\mathbf{k}}a_{\alpha,0})\Big].\end{aligned}\tag{8}$$

Before calculating the s-wave scattering lengths to second order, a couple of remarks are in order. First of all, for the sake of simplicity we shall disregard the magnetic interaction H_{mag} for a moment. Second, the s-wave scattering lengths are formally determined from the so-called T(ransition) matrix, which can be computed perturbatively by using the diagrammatic techniques. Moreover, it is known that the energy correction due to the two-body interactions can be directly related to the matrix elements of the T matrix [11]. Hence, one expects that the desired s-wave scattering lengths can be obtained from the calculations of energy correction. In fact, it is not hard to show that, to second order, our results agree with those obtained by the T-matrix approach. However, as the standard second-order perturbation methods are only required in our paper, the calculations can be greatly simplified. With these remarks in mind we are motivated to compute the energy correction due to H_{int}.

We first introduce a class of two-particle states defined by

$$|0,0;\varphi\rangle=\frac{1}{\sqrt{2}}\varphi^*_\alpha\varphi^*_\beta a^\dagger_{\beta,0}a^\dagger_{\alpha,0}|\mathrm{vac}\rangle,\tag{9}$$

where φ_α are constant parameters and $|\mathrm{vac}\rangle$ is the Fock vacuum. Such states can be normalized by imposing the condition $\varphi^*_\alpha\varphi_\alpha=|\varphi|^2=1$. Quite clearly, the unperturbed energy vanishes in the presence of the state Eq. (9). It is easy to show that the first-order energy correction due to H_{int} is

$$E^{(1)}_{\mathrm{int}}=\langle0,0;\varphi|H_{\mathrm{int}}|0,0;\varphi\rangle=\frac{g_1}{V}|\varphi|^4+\frac{g_2}{V}|\varphi^2|^2.\tag{10}$$

Next, we consider the second-order correction for the energy. Now, in view of the explicit form of $H_{\rm int}$ given in Eq. (8), the only possible intermediate states are those states of two noncondensate particles carrying opposite momenta,

$$|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle \equiv a^\dagger_{\alpha,\mathbf{k}} a^\dagger_{\beta,-\mathbf{k}}|\mathrm{vac}\rangle, \tag{11}$$

with which the unperturbed energy is given by

$$\langle \mathbf{k},-\mathbf{k};\alpha,\beta|H_0|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle = \epsilon_{\mathbf{k}}+\epsilon_{-\mathbf{k}} = 2\epsilon_{\mathbf{k}}. \tag{12}$$

Thus the second-order energy correction due to $H_{\rm int}$ is

$$E^{(2)}_{\rm int} = -\frac{1}{2}\sum_{\mathbf{k}\neq 0}\frac{|\langle 0,0;\varphi|H_{\rm int}|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle|^2}{2\epsilon_{\mathbf{k}}-0}. \tag{13}$$

The factor 1/2 in Eq. (13) is inserted in order to avoid the double counting of the momentum states. Now, we have

$$\begin{aligned}\langle 0,0;\varphi|H_{\rm int}|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle\\ &= \frac{g_1}{V}\langle 0,0;\varphi|a^\dagger_{\beta,0}a^\dagger_{\alpha,0}a_{\alpha,\mathbf{k}}a_{\beta,-\mathbf{k}}|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle\\ &\quad + \frac{g_2}{V}\langle 0,0;\varphi|a^\dagger_{\beta,0}a^\dagger_{\beta,0}a_{\alpha,\mathbf{k}}a_{\alpha,-\mathbf{k}}|\mathbf{k},-\mathbf{k};\alpha,\beta\rangle\delta_{\alpha\beta}\\ &= \frac{\sqrt{2}}{V}[g_1\varphi_\alpha\varphi_\beta + g_2\varphi^2\delta_{\alpha\beta}],\end{aligned} \tag{14}$$

and hence

$$\begin{aligned}E^{(2)}_{\rm int} &= -\frac{1}{V^2}\sum_{\mathbf{k}\neq 0}\frac{|g_1\varphi_\alpha\varphi_\beta+g_2\varphi^2\delta_{\alpha\beta}|^2}{2\epsilon_{\mathbf{k}}}\\ &= -\frac{1}{V}[g_1^2|\varphi|^4+(2g_1g_2+3g_2^2)|\varphi^2|^2]\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}.\end{aligned} \tag{15}$$

Obviously, the integral in Eq. (15) diverges as $|\mathbf{k}|\to\infty$. Choosing φ_α in such a way that $|\varphi^2|=0$, yields

$$g_1 = E^{(1)}_{\rm int}V, \tag{16}$$

indicating that g_1 is proportional to the first-order energy correction due to the two-particle interaction $H_{\rm int}$. At this order, g_1 is related to the corresponding s-wave scattering length a_1 by

$$g_1 = \frac{4\pi a_1}{m}. \tag{17}$$

Hence, to second order, a_1 is related to g_1 by the following equation:

$$\frac{4\pi a_1}{m} \equiv \tilde{g}_1 = (E^{(1)}_{\rm int}+E^{(2)}_{\rm int})V = g_1 - g_1^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}. \tag{18}$$

Here $\tilde{g}_1$ will be referred to as the renormalized coupling constant of g_1. Writing the original coupling g_1 in terms of the renormalized coupling $\tilde{g}_1$, we have at the same order,

$$g_1 = \tilde{g}_1 + \tilde{g}_1{}^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}, \tag{19}$$

which is equivalent to the results demonstrated in the one-component case [8]. Next, we consider the renormalization of g_2. Unlike g_1, g_2 cannot be isolated directly in the present formalism. Instead, we shall consider the sum g_1+g_2 that is equivalent to g_n as $g_1=g_n+g_s, g_2=-g_s$. To this end, we may take $\varphi^2=1$ such that $g_1+g_2=E^{(1)}_{\rm int}V$ and hence,

$$\begin{aligned}\tilde{g}_1+\tilde{g}_2 &= (E^{(1)}_{\rm int}+E^{(2)}_{\rm int})V\\ &= g_1+g_2-(g_1^2+2g_1g_2+3g_2^2)\\ &\quad\times\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}.\end{aligned} \tag{20}$$

Subtracting Eq. (18) from Eq. (20) and using Eq. (18) again, we have at this order,

$$g_2 = \tilde{g}_2 + (2\tilde{g}_1\tilde{g}_2+3\tilde{g}_2^2)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}. \tag{21}$$

Alternatively, g_n and g_s are related to the corresponding renormalized couplings by

$$g_n = \tilde{g}_n + (\tilde{g}_n^2+2\tilde{g}_s^2)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}, \tag{22}$$

$$g_s = \tilde{g}_s + (2\tilde{g}_n\tilde{g}_s-\tilde{g}_s^2)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}.$$

It should be noted that the renormalized coupling constants $\tilde{g}_1$ and $\tilde{g}_2$ are consistent with the one-loop renormalization obtained by using the Feynman diagram techniques [10]. These results are actually unaltered in the presence of a constant magnetic field. The point is that the two-body interaction term $H_{\rm int}$, in fact, commutes with the magnetic term $H_{\rm mag}$. As a consequence, despite the fact that the magnetic interaction would, inevitably, introduce a Zeeman energy shift to each hyperfine spin state, the total Zeeman energy is conserved in the two-particle scattering processes. Based on this point, one can easily check that both Eqs. (19) and (21) remain correct.

We now proceed to calculate the ground-state energy with the foregoing results. In the standard Bogoliubov approximation the operators $a_{\alpha,0}$ and $a^\dagger_{\alpha,0}$ are replaced by the classical number $\Phi_\alpha\sqrt{V}$ and $\Phi^*_\alpha\sqrt{V}$, respectively, such that $|\mathbf{\Phi}|^2 = N_0/V = n_0$ represents the density of condensate particles. Making these replacements in Eqs. (6) and (8) yields

$$H_{\rm mag}\to iV\varepsilon_{\alpha\beta\gamma}\Omega_\gamma\Phi^*_\alpha\Phi_\beta + \sum_{\mathbf{k}\neq 0} i\varepsilon_{\alpha\beta\gamma}\Omega_\gamma a^\dagger_{\alpha,\mathbf{k}}a_{\beta,\mathbf{k}} \tag{23}$$

and

$$H_{\rm int}\to\frac{1}{2}g_1V|\mathbf{\Phi}|^4+\frac{1}{2}g_2V|\mathbf{\Phi}^2|^2+\frac{g_1}{2}\sum_{\mathbf{k}\neq0}(\Phi_\alpha\Phi_\beta a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\alpha,-\mathbf{k}}+\Phi^*_\alpha\Phi^*_\beta a_{\alpha,\mathbf{k}}a_{\beta,-\mathbf{k}}+2\Phi^*_\alpha\Phi_\beta a^\dagger_{\beta,\mathbf{k}}a_{\alpha,\mathbf{k}}+2|\Phi|^2a^\dagger_{\alpha,\mathbf{k}}a_{\alpha,\mathbf{k}})+\frac{g_2}{2}\sum_{\mathbf{k}\neq0}(\Phi^2a^\dagger_{\beta,\mathbf{k}}a^\dagger_{\beta,-\mathbf{k}}+\Phi^{*2}a_{\alpha,\mathbf{k}}a_{\alpha,-\mathbf{k}}+4\Phi^*_\beta\Phi_\alpha a^\dagger_{\beta,\mathbf{k}}a_{\alpha,\mathbf{k}}). \tag{24}$$

Using Eqs. (23) and (24) we obtain the effective Hamiltonian,

$$H_{\rm eff}=H_{\rm con}+H_{\rm non}, \tag{25}$$

where

$$H_{\rm con}=\int d^3r\left[i\epsilon_{\alpha\beta\gamma}\Omega_\gamma\Phi^*_\alpha\Phi_\beta+\frac{g_1}{2}|\mathbf{\Phi}|^4+\frac{g_2}{2}|\mathbf{\Phi}^2|^2\right] \tag{26}$$

and

$$H_{\rm non}=\sum_{\mathbf{k}\neq0}(a^\dagger_{\alpha,\mathbf{k}}\mathcal{L}_{\alpha\beta}a_{\beta,\mathbf{k}}+\tfrac{1}{2}\mathcal{M}^*_{\alpha\beta}a_{\alpha,\mathbf{k}}a_{\beta,-\mathbf{k}}+\tfrac{1}{2}\mathcal{M}_{\alpha\beta}a^\dagger_{\alpha,\mathbf{k}}a^\dagger_{\beta,-\mathbf{k}}) \tag{27}$$

are the Hamiltonians for the condensate and noncondensate part, respectively. Here the matrix elements are given by

$$\mathcal{L}_{\alpha\beta}=\epsilon_{\mathbf{k}}\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\Omega_\gamma+g_1|\mathbf{\Phi}|^2\delta_{\alpha\beta}+g_1\Phi^*_\beta\Phi_\alpha+2g_2\Phi^*_\alpha\Phi_\beta, \tag{28}$$

$$\mathcal{M}_{\alpha\beta}=g_1\Phi_\alpha\Phi_\beta+g_2\Phi^2\delta_{\alpha\beta}.$$

Note that $H_{\rm eff}$ is precisely the Hartree-Fock-Bogoliubov Hamiltonian in the standard Bogoliubov approximation [10] whose ground-state structure can be determined by minimizing the integrand in Eq. (26). As a result, two different ground-state structures are found [2,3]:

$$\mathbf{\Phi}=\sqrt{n_0}(1/\sqrt{2},i/\sqrt{2},0)\quad\text{for}\quad n_0g_2>-\Omega, \tag{29}$$

which is referred to as the ''ferromagnetic'' state and

$$\mathbf{\Phi}=\sqrt{n_0}(\cos\theta,i\sin\theta,0)\quad\text{for}\quad n_0g_2<-\Omega, \tag{30}$$

as the ''polar'' state. Here the cosine and the sine in Eq. (30) are given by

$$\cos\theta=\frac{1}{2}\left(\sqrt{1+\frac{\Omega}{|g_2|n_0}}+\sqrt{1-\frac{\Omega}{|g_2|n_0}}\right),$$

$$\sin\theta=\frac{1}{2}\left(\sqrt{1+\frac{\Omega}{|g_2|n_0}}-\sqrt{1-\frac{\Omega}{|g_2|n_0}}\right). \tag{31}$$

Since $H_{\rm eff}$ is quadratic in $a^\dagger_{\alpha,\mathbf{k}}$ and $a_{\alpha,\mathbf{k}}$, we can diagonalize this Hamiltonian by using the generalized Bogoliubov transformation,

$$a_{\alpha,\mathbf{k}}=\sum_i[u^{(i)}_{\alpha,\mathbf{k}}b^{(i)}_{\mathbf{k}}-v^{(i)}_{\alpha,-\mathbf{k}}b^{\dagger(i)}_{-\mathbf{k}}], \tag{32}$$

where i is the mode index and $|u^{(i)}_{\alpha,\mathbf{k}}|^2-|v^{(i)}_{\alpha,\mathbf{k}}|^2=1$. In terms of the quasiparticle creation and annihilation operators $b^{(i)}_{\mathbf{k}}$ and $b^{\dagger(i)}_{\mathbf{k}}$, the Hamiltonian takes the following form:

$$H_{\rm eff}=H_{\rm con}+\sum_i\sum_{\mathbf{k}\neq0}E^{(i)}_{\mathbf{k}}(b^{\dagger(i)}_{\mathbf{k}}b^{(i)}_{\mathbf{k}}-|v^{(i)}_{\alpha,\mathbf{k}}|^2). \tag{33}$$

We now define the ground state, which is annihilated by all $b^{(i)}_{\mathbf{k}}$, i.e., $b^{(i)}_{\mathbf{k}}|\Theta_{\rm GND}\rangle=0$, so that the ground-state energy is found to be

$$E_{\rm GND}=V\left[i\varepsilon_{\alpha\beta\gamma}\Omega_\gamma\Phi^*_\alpha\Phi_\beta+\frac{g_1}{2}|\mathbf{\Phi}|^4+\frac{g_2}{2}|\mathbf{\Phi}^2|^2-\int\frac{d^3k}{(2\pi)^3}\sum_iE^{(i)}_{\mathbf{k}}|v^{(i)}_{\alpha,\mathbf{k}}|^2\right], \tag{34}$$

where the last integral indicates the energy shift due to the quasiparticle excitations. To calculate the ground-state energy, one needs to know precisely the values of $E^{(i)}_{\mathbf{k}}$ and $v^{(i)}_{\alpha,\mathbf{k}}$ for the quasiparticle modes. This can be done by using the standard Hartree-Fock-Bogoliubov mean-field method, and we have calculated $E^{(i)}_{\mathbf{k}}$ and $v^{(i)}_{\alpha,\mathbf{k}}$ for the quasiparticle modes. In the following, we devote our attention to the case in which the ground state is ''polar,'' since for the ''ferromagnetic'' case the excitation spectrum consists of one phononlike mode and two free-particle modes [3] so that the expression for the ground-state energy resembles that of the one-component scalar BEC [8]. As a result, we find that the low-lying excitations can be described by two gapless modes $E^{(\pm)}_{\mathbf{k}}$ and one massive mode $E^{(0)}_{\mathbf{k}}$:

$$E^{(\pm)}_{\mathbf{k}}=\sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+2n_0g^{(\pm)})},\qquad E^{(0)}_{\mathbf{k}}=\sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+2n_0g^{(0)})+\Omega^2}, \tag{35}$$

for which the corresponding nonvanishing mode functions are given by [10]

$$\begin{pmatrix}v^{(\pm)}_{x,\mathbf{k}}\\ v^{(\pm)}_{y,\mathbf{k}}\end{pmatrix}=\begin{pmatrix}+A^{(\pm)}\\ -B^{(\pm)}\end{pmatrix}\beta^{(\pm)}_{\mathbf{k}},\qquad v^{(0)}_{z,\mathbf{k}}=-\left(1-\frac{\Omega^2}{n_0^2g_2^2}\right)\beta^{(0)}_{\mathbf{k}}, \tag{36}$$

where

$$g^{(\pm)}=\frac{1}{2}\left[g_1\pm\sqrt{g_1^2+4g_2(g_1+g_2)\left(1-\frac{\Omega^2}{n_0^2g_2^2}\right)}\right],$$

$$g^{(0)}=|g_2|, \tag{37}$$

$$\beta^{(i)}_{\mathbf{k}}=\frac{n_0g^{(i)}}{\sqrt{2E^{(i)}_{\mathbf{k}}(E^{(i)}_{\mathbf{k}}+\epsilon_{\mathbf{k}}+n_0g^{(i)})}}\quad(i=\pm,0), \tag{38}$$

$$A^{(\pm)}=\frac{g_1\Omega/|g_2|}{\sqrt{\eta^{(\pm)2}+(g_1\Omega/g_2)^2}},\quad B^{(\pm)}=\frac{-i\,\eta^{(\pm)}}{\sqrt{\eta^{(\pm)2}+(g_1\Omega/g_2)^2}}, \tag{39}$$

$$\eta^{(\pm)}=n_0(g_1-2g^{(\pm)})+n_0(g_1+2g_2)\sqrt{1-(\Omega/n_0|g_2|)^2}. \tag{40}$$

Moreover, with the condensate wave functions described in Eqs. (30) and (31), we obtain the following results:

$$|\mathbf{\Phi}|^2=n_0,\quad |\mathbf{\Phi}^2|^2=n_0^2\left(1-\frac{\Omega^2}{n_0^2g_2^2}\right),$$

$$i\varepsilon_{\alpha\beta\gamma}\Omega_\gamma\Phi_\alpha^*\Phi_\beta=-\frac{\Omega^2}{|g_2|}. \tag{41}$$

However, one sees that all the three integrals,

$$\int\frac{d^3k}{(2\pi)^3}E_{\mathbf{k}}^{(i)}|v_{\alpha,\mathbf{k}}^{(i)}|^2\quad (i=\pm,0),$$

are divergent when $|\mathbf{k}|\to\infty$. Note also that they are essentially of second-order in the coupling constants g_1 and g_2. To eliminate these ultraviolet divergences we substitute Eqs. (19) and (21) into Eq. (34). The resulting expression for the ground-state energy is

$$\begin{aligned}E_{\rm GND}=&V\left[i\varepsilon_{\alpha\beta\gamma}\omega_\gamma\Phi_\alpha^*\Phi_\beta+\frac{\tilde g_1}{2}|\mathbf{\Phi}|^4+\frac{\tilde g_2}{2}|\mathbf{\Phi}^2|^2\right]\\&+V\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\epsilon_{\mathbf{k}}}\left\{\left[\frac{\tilde g_1^2}{2}|\mathbf{\Phi}|^4+\frac{1}{2}(2\tilde g_1\tilde g_2+3\tilde g_2^2)\right.\right.\\&\left.\left.\times|\mathbf{\Phi}^2|^2\right]-\sum_i E_{\mathbf{k}}^{(i)}(\tilde g_1,\tilde g_2)|v_{\alpha,\mathbf{k}}^{(i)}(\tilde g_1,\tilde g_2)|^2\right\}.\end{aligned} \tag{42}$$

Note that the condensate wave function $\mathbf{\Phi}$ determined by minimizing the sum of terms in the first line of Eq. (42) has the same form as that in Eq. (41) except that the renormalized coupling constants are substituted instead. Furthermore, the last term in Eq. (42) can be expanded in powers of the renormalized coupling constants. Since Eq. (42) is valid only up to second order in the renormalized couplings, it suffices to substitute $g_1=\tilde g_1$, $g_2=\tilde g_2$ in the expressions for $E_{\mathbf{k}}^{(i)}$ and $v_{\alpha,\mathbf{k}}^{(i)}$, i.e., in Eqs. (35)–(40). On these grounds, we are now ready to calculate $E_{\rm GND}$ given by Eq. (42). First, we note that

$$\tilde g^{(+)2}+\tilde g^{(-)2}=\tilde g_1^2+2\tilde g_1(\tilde g_1+\tilde g_2)\left(1-\frac{\Omega^2}{n_0^2\tilde g_2^2}\right), \tag{43}$$

and hence,

$$\tilde g_1^2|\mathbf{\Phi}|^4+(2\tilde g_1\tilde g_2+3\tilde g_2^2)|\mathbf{\Phi}^2|^2=n_0^2\left[\tilde g^{(+)2}+\tilde g^{(-)2}+\left(1-\frac{\Omega^2}{n_0^2\tilde g_2^2}\right)\tilde g^{(0)^2}\right]. \tag{44}$$

The integral in Eq. (42) is then equal to

$$\begin{aligned}\frac{1}{2}n_0^2V\int\frac{d^3k}{(2\pi)^3}&\left[\sum_{i=\pm}\tilde g^{(i)^2}\left(\frac{1}{2\epsilon_{\mathbf{k}}}-\frac{1}{(E_{\mathbf{k}}^{(i)}+\epsilon_{\mathbf{k}}+n_0\tilde g^{(i)})}\right)\right.\\&\left.+\left(\tilde g_2^2-\frac{\Omega^2}{n_0^2}\right)\left(\frac{1}{2\epsilon_{\mathbf{k}}}-\frac{1}{(E_{\mathbf{k}}^{(0)}+\epsilon_{\mathbf{k}}+n_0|\tilde g_2|)}\right)\right].\end{aligned} \tag{45}$$

Note that the first two terms are the same as that of the one-component case [8],

$$\begin{aligned}\frac{1}{2}n_0^2V\int\frac{d^3k}{(2\pi)^3}\tilde g^{(\pm)^2}&\left(\frac{1}{2\epsilon_{\mathbf{k}}}-\frac{1}{(E_{\mathbf{k}}^{(\pm)}+\epsilon_{\mathbf{k}}+n_0\tilde g^{(\pm)})}\right)\\&=\frac{2\pi n_0^{5/2}}{m}V\left(\frac{128}{15\sqrt{\pi}}a^{(\pm)^{5/2}}\right),\end{aligned} \tag{46}$$

where $a^{(\pm)}=m\tilde g^{(\pm)}/4\pi$ are the corresponding s-wave scattering wavelengths. The last integral can be expressed as

$$\begin{aligned}\frac{1}{2}n_0^2V\left(\tilde g_2^2-\frac{\Omega^2}{n_0^2}\right)&\int\frac{d^3k}{(2\pi)^3}\frac{E_{\mathbf{k}}^{(0)}-\epsilon_{\mathbf{k}}+n_0|\tilde g_2|}{2\epsilon_{\mathbf{k}}(E_{\mathbf{k}}^{(0)}+\epsilon_{\mathbf{k}}+n_0|\tilde g_2|)}\\&=\frac{2\pi V}{m}n_0^{5/2}\left(\frac{128}{15\sqrt{\pi}}|a_2|^{5/2}\right)(1-t^2)F(t^2),\end{aligned} \tag{47}$$

where $t=\Omega/n_0|\tilde g_2|$, and the function of integral is defined as

$$F(t^2)=\frac{15\sqrt{2}}{32}\int_0^\infty dx\frac{1-x^2+\sqrt{t^2+2x^2+x^4}}{1+x^2+\sqrt{t^2+2x^2+x^4}}\quad\text{for}\quad 0\leqslant t^2\leqslant 1, \tag{48}$$

which is a monotonically increasing function that cannot be analytically evaluated in general.

Finally, using Eq. (41) we get

$$\begin{aligned}&\left[i\varepsilon_{\alpha\beta\gamma}\omega_\gamma\Phi_\alpha^*\Phi_\beta+\frac{\tilde g_1}{2}|\mathbf{\Phi}|^4+\frac{\tilde g_2}{2}|\mathbf{\Phi}^2|^2\right]V\\&\quad=\left[\frac{\tilde g_1n_0^2}{2}+\frac{\tilde g_2n_0^2}{2}\left(1-\frac{\Omega^2}{n_0^2\tilde g_2^2}\right)-\frac{\Omega^2}{|\tilde g_2|}\right]V\\&\quad=\frac{2\pi n_0^2V}{m}(a_n-t^2a_s),\end{aligned} \tag{49}$$

and therefore the ground-state energy is given by

$$E_{\mathrm{GND}}=\frac{2\pi n_0^2 V}{m}\left[(a_n-t^2a_s)+\frac{128}{15\sqrt{\pi}}n_0^{1/2}\right.$$
$$\left.\times[a^{(+)5/2}+a^{(-)5/2}+(1-t^2)F(t^2)a_s^{5/2}]\right], \quad (50)$$

where $a_n=m\tilde{g}_n/4\pi, a_s=m\tilde{g}_s/4\pi$. The terms proportional to $a^{(\pm)5/2}$ are caused by the two gapless modes and have the same form of the phononlike mode in the one-component BEC. The last term in Eq. (50) is due to the massive mode, which depends solely on the scattering length a_s for the spin exchange channel and is suppressed by the increasing magnetic field.

In conclusion, we have analytically calculated the ground-state energy of a homogeneous spinor BEC with hyperfine spin $f=1$ based on the Bogoliubov approximation. In this weakly interacting system, the two-body interactions are described by the hard-core collisions and the spin-exchange interaction, which are characterized by the coupling constants g_1 and g_2, respectively. Using the second-order perturbation methods, the two bare coupling constants g_1 and g_2 are expressed in terms of their renormalized ones $\tilde{g}_1$ and $\tilde{g}_2$, which are directly related to the physically measurable s-wave wavelengths for the corresponding scattering channels. It is found that the renormalization of g_1 has the same form as that of the one-component scalar BEC. However, the renormalization of g_2 is more complicated and has a dependence on the renormalized coupling constant $\tilde{g}_1$. With the renormalized coupling constants, we are able to show that the ultraviolet divergence occurring in the calculation of ground-state energy can be completely removed.

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