Some general bounds for one-dimensional scattering

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One-dimensional scattering problems are of wide physical interest and are encountered in many diverse applications. In this paper I establish some very general bounds for reflection and transmission coefficients for one-dimensional potential scattering. Equivalently, these results may be phrased as general bounds on the Bogolubov coefficients or statements about the transfer matrix. A similar analysis can be provided for the parametric change of frequency of a harmonic oscillator. A number of specific examples are discussed. In particular I provide a general proof that sharp step function potentials always scatter more effectively than the corresponding smoothed potentials. The analysis also serves to collect together and unify what would otherwise appear to be quite unrelated results. [S1050-2947(99)08101-9]

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I. INTRODUCTION

One-dimensional scattering problems occur in a wide variety of physical contexts. In acoustics one might be interested in the propagation of sound waves down a long pipe, while in electromagnetism one might be interested in the physics of waveguides. In quantum physics the canonical examples are barrier penetration and reflection, while in classical physics an equivalent problem is the analysis of parametric resonances. All of these physical problems can be analyzed in the same mathematical framework, though for definiteness I shall present the discussion in terms of the Schrödinger equation, commenting on alternative formulations as appropriate.

For one-dimensional scattering problems there is a large catalog of specific potentials for which exact analytic results are known. There are also well-developed numerical techniques for estimating the scattering properties. In this paper I wish to take a different tack: I shall develop a number of very general and rather simple *bounds* on the reflection and transmission probabilities (equivalently, these bounds can be presented in terms of the Bogoliubov coefficients or in terms of statements about the transfer matrix). These bounds, because they are so general, are powerful aids in the *qualitative* understanding of one-dimensional scattering. Furthermore, this analysis provides a unifying theme that serves to connect together seemingly quite disparate results obtained in individual special cases.

II. GENERAL ANALYSIS

A. Shabat-Zakharov systems

Consider the one-dimensional time-independent Schrödinger equation [1–15]

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\,\psi(x)=E\,\psi(x). \tag{1}$$

If the potential asymptotically approaches a constant

$$V(x\to\pm\infty)\to V_{\pm\infty}, \tag{2}$$

then in each of the two asymptotic regions there are two independent solutions to the Schrödinger equation

$$\psi(\pm i;\pm\infty;x)\approx\frac{\exp(\pm ik_{\pm\infty}x)}{\sqrt{k_{\pm\infty}}}. \tag{3}$$

Here the $\pm i$ distinguishes right-moving modes e^{+ikx} from left-moving modes e^{-ikx}, while the $\pm\infty$ specifies which of the asymptotic regions we are in. Furthermore,

$$k_{\pm\infty}=\frac{\sqrt{2m(E-V_{\pm\infty})}}{\hbar}. \tag{4}$$

To even begin to set up a scattering problem the minimum requirement is that the potential asymptotically approaches some constant, and this assumption will be made henceforth.

The so-called Jost solutions [16] are exact solutions $J_{\pm}(x)$ of the Schrödinger equation that satisfy

$$J_+(x\to+\infty)\to\frac{\exp(+ik_{+\infty}x)}{\sqrt{k_{+\infty}}}, \tag{5}$$

$$J_-(x\to-\infty)\to\frac{\exp(-ik_{-\infty}x)}{\sqrt{k_{-\infty}}}, \tag{6}$$

and

$$J_+(x\to-\infty)\to\alpha\frac{\exp(+ik_{-\infty}x)}{\sqrt{k_{-\infty}}}+\beta\frac{\exp(-ik_{-\infty}x)}{\sqrt{k_{-\infty}}}, \tag{7}$$

$$J_-(x\to+\infty)\to\alpha^*\frac{\exp(-ik_{+\infty}x)}{\sqrt{k_{+\infty}}}+\beta^*\frac{\exp(+ik_{+\infty}x)}{\sqrt{k_{+\infty}}}. \tag{8}$$

Here α and β are the (right-moving) Bogoliubov coefficients, which are related to the (right-moving) reflection and transmission amplitudes by

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$$r=\frac{\beta}{\alpha}, \quad t=\frac{1}{\alpha}. \tag{9}$$

These conventions correspond to an incoming flux of right-moving particles (incident from the left) being partially transmitted and partially scattered. The left-moving Bogoliubov coefficients are just the complex conjugates of the right-moving coefficients; however, it should be borne in mind that the phases of β and β^* are physically meaningless in that they can be arbitrarily changed simply by moving the origin of coordinates. The phases of α and α^*, on the other hand, do contain real physical information.

In this paper I will derive some very general bounds on $|\alpha|$ and $|\beta|$, which also lead to general bounds on the reflection and transmission probabilities

$$R=|r|^2, \quad T=|t|^2. \tag{10}$$

The key idea is to rewrite the second-order Schrödinger equation as a particular type of Shabat-Zakharov [17] system: a particular set of two coupled first-order differential equations for which bounds can be easily established. A similar representation of the Schrödinger equation is briefly discussed by Peirls [18] and related representations are well known, often being used without giving an explicit reference (see, e.g., Ref. [19]). However, an exhaustive search has not uncovered prior use of the particular representation of this paper, nor the idea of using the representation to place bounds on one-dimensional scattering.

I start by introducing an arbitrary auxiliary function $\varphi(x)$ that may be either real or complex, though I do demand that $\varphi'(x)\neq 0$, and then defining

$$\psi(x)=a(x)\frac{\exp(+i\varphi)}{\sqrt{\varphi'}}+b(x)\frac{\exp(-i\varphi)}{\sqrt{\varphi'}}. \tag{11}$$

This representation effectively seeks to use quantities resembling the ''phase integral'' wave functions as a basis for the true wave function [20]. This representation is of course highly redundant since one complex number $\psi(x)$ has been traded for two complex numbers $a(x)$ and $b(x)$ plus an essentially arbitrary auxiliary function $\varphi(x)$. In order for this representation to be most useful it is best to arrange things so that $a(x)$ and $b(x)$ asymptotically approach constants at spatial infinity, which we shall soon see implies that we should pick the auxiliary function to satisfy

$$\varphi'(x)\to k_{\pm\infty} \quad \text{as} \quad x\to\pm\infty. \tag{12}$$

To trim down the number of degrees of freedom it is useful to impose a ''gauge condition''

$$\frac{d}{dx}\left(\frac{a}{\sqrt{\varphi'}}\right)e^{+i\varphi}+\frac{d}{dx}\left(\frac{b}{\sqrt{\varphi'}}\right)e^{-i\varphi}=0. \tag{13}$$

Subject to this gauge condition,

$$\frac{d\psi}{dx}=i\sqrt{\varphi'}\{a(x)\exp(+i\varphi)-b(x)\exp(-i\varphi)\}. \tag{14}$$

I now rewrite the Schrödinger equation in terms of two coupled first-order differential equations for these position-dependent Bogolubov coefficients. To do this note that

$$\frac{d^2\psi}{dx^2}=\frac{d}{dx}\left(i\frac{\varphi'}{\sqrt{\varphi'}}\{ae^{+i\varphi}-be^{-i\varphi}\}\right) \tag{15}$$

$$\begin{aligned}&=\frac{(i\varphi')^2}{\sqrt{\varphi'}}\{ae^{+i\varphi}+be^{-i\varphi}\}\\&\quad+i\varphi'\left\{\frac{d}{dx}\left(\frac{a}{\sqrt{\varphi'}}\right)e^{+i\varphi}-\frac{d}{dx}\left(\frac{b}{\sqrt{\varphi'}}\right)e^{-i\varphi}\right\}\\&\quad+i\frac{\varphi''}{\sqrt{\varphi'}}\{ae^{+i\varphi}-be^{-i\varphi}\}\end{aligned} \tag{16}$$

$$\begin{aligned}&=-\frac{\varphi'^2}{\sqrt{\varphi'}}\{ae^{+i\varphi}+be^{-i\varphi}\}\\&\quad+\frac{2i\varphi'}{\sqrt{\varphi'}}\frac{da}{dx}e^{+i\varphi}-i\frac{\varphi''}{\sqrt{\varphi'}}be^{-i\varphi}\end{aligned} \tag{17}$$

$$\begin{aligned}&=-\frac{\varphi'^2}{\sqrt{\varphi'}}\{ae^{+i\varphi}+be^{-i\varphi}\}\\&\quad-\frac{2i\varphi'}{\sqrt{\varphi'}}\frac{db}{dx}e^{-i\varphi}+i\frac{\varphi''}{\sqrt{\varphi'}}ae^{+i\varphi}.\end{aligned} \tag{18}$$

(The last two relations use the gauge condition.) Now insert these formulas into the Schrödinger equation written in the form

$$\frac{d^2\psi}{dx^2}=-k(x)^2\,\psi\equiv-\frac{2m[E-V(x)]}{\hbar^2}\psi \tag{19}$$

to deduce

$$\begin{aligned}\frac{da}{dx}=+\frac{1}{2\varphi'}\{&\varphi''b\exp(-2i\varphi)\\&+i[k^2(x)-(\varphi')^2][a+b\,\exp(-2i\varphi)]\},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{db}{dx}=+\frac{1}{2\varphi'}\{&\varphi''a\exp(+2i\varphi)\\&-i[k^2(x)-(\varphi')^2][b+a\,\exp(+2i\varphi)]\}.\end{aligned} \tag{21}$$

It is easy to verify that this first-order system is compatible with the gauge condition (13) and that by iterating the system twice (subject to this gauge condition) one recovers exactly the original Schrödinger equation. These equations hold for arbitrary φ, real or complex, and when written in matrix form, exhibit a deep connection with the transfer matrix formalism [21].

B. Bounds

To obtain our bounds on the Bogoliubov coefficients we start by restricting attention to the case that $\varphi(x)$ is a *real* function of x. (Since φ is an essentially arbitrary auxiliary function this is not a particularly restrictive condition.) Under this assumption the probability current is

$$\mathcal{J}=\mathrm{Im}\left\{\psi^{*}\frac{d\psi}{dx}\right\}=\{|a|^2-|b|^2\}. \tag{22}$$

Now at $x\sim+\infty$ the wave function is purely right moving and normalized to 1 because we are considering one-dimensional Jost solutions [16]. Then for all x we have a conserved quantity

$$|a|^2-|b|^2=1. \tag{23}$$

It is this result that makes it useful to interpret $a(x)$ and $b(x)$ as *position-dependent Bogoliubov coefficients* relative to the auxiliary function $\varphi(x)$. Now use the fact that

$$\frac{d|a|}{dx}=\frac{1}{2|a|}\left(a^{*}\frac{da}{dx}+a\frac{da^{*}}{dx}\right) \tag{24}$$

and use Eq. (20) to obtain

$$\begin{aligned}\frac{d|a|}{dx}=\frac{1}{2|a|}\frac{1}{2\varphi'}\{&\varphi''[a^{*}b\exp(-2i\varphi)+ab^{*}\exp(+2i\varphi)]\\&+i[k^2(x)-(\varphi')^2][a^{*}b\exp(-2i\varphi)\\&-ab^{*}\exp(+2i\varphi)]\},\end{aligned} \tag{25}$$

that is,

$$\begin{aligned}\frac{d|a|}{dx}=\frac{1}{2|a|}\frac{1}{2\varphi'}&\mathrm{Re}(\{\varphi''+i[k^2(x)-(\varphi')^2]\}\\&\times[a^{*}b\exp(-2i\varphi)]).\end{aligned} \tag{26}$$

The right-hand side can now be bounded from above, by systematically using $\mathrm{Re}(AB)\leqslant|A|\,|B|$. This leads to

$$\frac{d|a|}{dx}\leqslant\frac{\sqrt{(\varphi'')^2+[k^2(x)-(\varphi')^2]^2}}{2|\varphi'|}|b|. \tag{27}$$

It is essential that φ be real to have $|\exp(-2i\varphi)|=1$, which is the other key ingredient above. Now define the non-negative quantity

$$\vartheta[\varphi(x),k(x)]\equiv\frac{\sqrt{(\varphi'')^2+[k^2(x)-(\varphi')^2]^2}}{2|\varphi'|} \tag{28}$$

and use the conservation law (23) to write

$$\frac{d|a|}{dx}\leqslant\vartheta\sqrt{|a|^2-1}. \tag{29}$$

Integrate this inequality

$$\{\cosh^{-1}|a|\}\Big|_{x_i}^{x_f}\leqslant\int_{x_i}^{x_f}\vartheta\,dx. \tag{30}$$

Taking limits as $x_i\to-\infty$ and $x_f\to+\infty$,

$$\cosh^{-1}|\alpha|\leqslant\int_{-\infty}^{+\infty}\vartheta\,dx, \tag{31}$$

that is,

$$|\alpha|\leqslant\cosh\left(\int_{-\infty}^{+\infty}\vartheta\,dx\right), \tag{32}$$

which automatically implies

$$|\beta|\leqslant\sinh\left(\int_{-\infty}^{+\infty}\vartheta\,dx\right). \tag{33}$$

Since this result holds for *all real* choices of the auxiliary function $\varphi(x)$ (subject only to $\varphi'\neq0$ and $\varphi'\to k_{\pm\infty}$ as $x\to\pm\infty$), it encodes an enormously wide class of bounds on the Bogoliubov coefficients. When translated to reflection and transmission coefficients the equivalent statements are

$$T\geqslant\mathrm{sech}^2\left(\int_{-\infty}^{+\infty}\vartheta\,dx\right) \tag{34}$$

and

$$R\leqslant\tanh^2\left(\int_{-\infty}^{+\infty}\vartheta\,dx\right). \tag{35}$$

I shall soon turn this general result into more specific theorems.

C. Transfer matrix representation

The system of equations (20) and (21) can also be written in matrix form. It is convenient to define

$$\rho\equiv\varphi''+i[k^2(x)-(\varphi')^2]. \tag{36}$$

Then

$$\frac{d}{dx}\begin{bmatrix}a\\b\end{bmatrix}=\frac{1}{2\varphi'}\begin{bmatrix}i\,\mathrm{Im}[\rho] & \rho\exp(-2i\varphi)\\ \rho^{*}\exp(+2i\varphi) & -i\,\mathrm{Im}[\rho]\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}. \tag{37}$$

This has the formal solution

$$\begin{bmatrix}a(x_f)\\b(x_f)\end{bmatrix}=E(x_f,x_i)\begin{bmatrix}a(x_i)\\b(x_i)\end{bmatrix}, \tag{38}$$

in terms of a generalized position-dependent "transfer matrix" [21]

$$E(x_f,x_i)=\mathcal{P}\exp\left(\int_{x_i}^{x_f}\frac{1}{2\varphi'}\begin{bmatrix} i\,\mathrm{Im}[\rho] & \rho\exp(-2i\varphi)\\ \rho^*\exp(+2i\varphi) & -i\,\mathrm{Im}[\rho]\end{bmatrix}dx\right), \tag{39}$$

where the symbol $\mathcal{P}$ denotes ''path ordering.'' In particular, if we take $x_i\to-\infty$ and $x_f\to+\infty$ we obtain a formal but exact expression for the Bogoliubov coefficients

$$\begin{bmatrix}\alpha & \beta^*\\ \beta & \alpha^*\end{bmatrix}=E(\infty,-\infty)=\mathcal{P}\exp\left(\int_{-\infty}^{\infty}\frac{1}{2\varphi'}\begin{bmatrix} i\,\mathrm{Im}[\rho] & \rho\exp(-2i\varphi)\\ \rho^*\exp(+2i\varphi) & -i\,\mathrm{Im}[\rho]\end{bmatrix}dx\right). \tag{40}$$

The matrix E is *not* unitary, though it does have determinant 1. It is in fact an element of the group SU(1,1). Taking

$$\sigma_z=\begin{bmatrix}+1 & 0\\ 0 & -1\end{bmatrix} \tag{41}$$

so that $(\sigma_z)^2=+I$, and defining $E^\dagger=(E^*)^T$, it is easy to see that

$$E^\dagger\sigma_z E=\sigma_z. \tag{42}$$

This is the analog of the invariance of the Minkowski metric for Lorentz transformations in SO(3,1). Similarly, if we define the ''complex structure'' J by

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{43}$$

then $J^2=-I$ and

$$E^\dagger=JEJ. \tag{44}$$

III. SPECIAL CASE 1

Suppose now that the potential satisfies $V_{+\infty}=V_{-\infty}$. Also, choose the phase function $\varphi(x)$ to be $\varphi=k_\infty x$. We also require $k_\infty\neq 0$, that is, $E>V_{\pm\infty}$. This is the special case discussed in a different context by Peierls [18]. Then the evolution equations simplify tremendously and

$$\vartheta\to\frac{|k^2-k_\infty^2|}{2k_\infty}=\frac{m|V(x)-V_\infty|}{\hbar^2 k_\infty}. \tag{45}$$

Using $(\hbar k_\infty)^2=2m(E-V_\infty)$, the bounds become

$$T\geq\mathrm{sech}^2\left(\frac{1}{\hbar}\sqrt{\frac{m}{2(E-V_\infty)}}\int_{-\infty}^{+\infty}|V-V_\infty|\,dx\right) \tag{46}$$

and

$$R\leq\tanh^2\left(\frac{1}{\hbar}\sqrt{\frac{m}{2(E-V_\infty)}}\int_{-\infty}^{+\infty}|V-V_\infty|\,dx\right). \tag{47}$$

These bounds are exact nonperturbative results; however, for high energies it may be convenient to use the slightly less restrictive (but analytically much more tractable) bounds

$$T\geq 1-\frac{m\left(\int_{-\infty}^{+\infty}|V-V_\infty|dx\right)^2}{2(E-V_\infty)\hbar^2} \tag{48}$$

and

$$R\leq\frac{m\left(\int_{-\infty}^{+\infty}|V-V_\infty|dx\right)^2}{2(E-V_\infty)\hbar^2}. \tag{49}$$

This version of the bounds also holds for all energies, but is not very restrictive for low energy.

The transfer matrices can be analyzed by checking that the evolution equations simplify to

$$\frac{da}{dx}=\frac{-im(V-V_\infty)}{\hbar^2 k_\infty}\{a+b\exp(-2ik_\infty x)\}, \tag{50}$$

$$\frac{db}{dx}=\frac{+im(V-V_\infty)}{\hbar^2 k_\infty}\{a\exp(+2ik_\infty x)+b\}. \tag{51}$$

This can be written in matrix form as

$$\frac{d}{dx}\begin{bmatrix}a\\ b\end{bmatrix}=\frac{-im(V-V_\infty)}{\hbar^2 k_\infty}\times\begin{bmatrix}1 & \exp(-2ik_\infty x)\\ -\exp(+2ik_\infty x) & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}. \tag{52}$$

This version of the Shabat-Zakharov system [17] has a formal solution in terms of the transfer matrix

$$E(x_f,x_i)=\mathcal{P}\exp\left(\frac{-im}{\hbar^2 k_\infty}\int_{x_f}^{x_i}[V(x)-V_\infty]\times\begin{bmatrix}1 & e^{-2ik_\infty x}\\ -e^{+2ik_\infty x} & -1\end{bmatrix}dx\right). \tag{53}$$

The formal but exact expression for the Bogoliubov coefficients is now

$$\begin{bmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{bmatrix} = E(\infty, -\infty)$$
$$= \mathcal{P} \exp\left(\frac{-im}{\hbar^2 k_\infty} \int_{-\infty}^{\infty} [V(x) - V_\infty] \times \begin{bmatrix} 1 & e^{-2ik_\infty x} \\ -e^{+2ik_\infty x} & -1 \end{bmatrix} dx \right). \quad (54)$$

Furthermore, the form of the system (50) and (51) suggests that it might be useful to define

$$a = \tilde{a} \exp\left[+ \frac{im}{\hbar^2 k_\infty} \int_{-\infty}^{x} [V(y) - V_\infty] dy \right], \quad (55)$$

$$b = \tilde{b} \exp\left[- \frac{im}{\hbar^2 k_\infty} \int_{-\infty}^{x} [V(y) - V_\infty] dy \right]. \quad (56)$$

Then

$$\frac{d\tilde{a}}{dx} = \frac{-im[V(x) - V_\infty]}{\hbar^2 k_\infty} \tilde{b} \exp(-2ik_\infty x), \quad (57)$$

$$\frac{d\tilde{b}}{dx} = \frac{+im[V(x) - V_\infty]}{\hbar^2 k_\infty} \tilde{a} \exp(+2ik_\infty x). \quad (58)$$

This representation simplifies some of the results, for instance,

$$\begin{bmatrix} \tilde{\alpha} & \tilde{\beta}^* \\ \tilde{\beta} & \tilde{\alpha}^* \end{bmatrix} = \tilde{E}(\infty, -\infty)$$
$$= \mathcal{P} \exp\left(\frac{-im}{\hbar^2 k_\infty} \int_{-\infty}^{\infty} [V(x) - V_\infty] \times \begin{bmatrix} 0 & e^{-2ik_\infty x} \\ -e^{+2ik_\infty x} & 0 \end{bmatrix} dx \right). \quad (59)$$

This can be used as the basis of an approximation scheme for $\tilde{\beta}$. Suppose that for all x we have $|\tilde{b}(x)| \ll 1$, so that $|\tilde{a}(x)| \approx 1$. Then

$$\frac{d\tilde{b}}{dx} \approx \frac{+im[V(x) - V_\infty]}{\hbar^2 k_\infty} \exp(+2ik_\infty x). \quad (60)$$

This may be immediately integrated to yield

$$\tilde{\beta} \approx \frac{+im}{\hbar^2 k_\infty} \int_{-\infty}^{+\infty} [V(x) - V_\infty] \exp(+2ik_\infty x)\, dx. \quad (61)$$

This is immediately recognizable as the (first) Born approximation. If we instead work in terms of the original definition β,

$$\beta \approx \frac{+im}{\hbar^2 k_\infty} \exp\left[+ \frac{im}{\hbar^2 k_\infty} \int_{-\infty}^{+\infty} [V(x) - V_\infty]\, dx \right] \times \int_{-\infty}^{+\infty} [V(x) - V_\infty] \exp(+2ik_\infty x) \times \exp\left[- \frac{im}{\hbar^2 k_\infty} \int_{-\infty}^{x} [V(y) - V_\infty]\, dy \right] dx. \quad (62)$$

This is one form of the distorted Born wave approximation.

In short, this type of analysis collects together a large number of results that otherwise appear quite unrelated. By taking further specific cases of these bounds and related results it is possible to reproduce many analytically known results, such as for δ-function potentials, double-δ-function potentials, square wells, and sech^2 potentials, as discussed later in this paper. (See Sec. VI.)

IV. SPECIAL CASE 2

Suppose now we take $k(x) = \varphi'(x)$. This means that we are choosing our auxiliary function so that we use the WKB approximation for the true wave function as a ‘‘basis’’ for calculating the Bogoliubov coefficients. This choice is perfectly capable of handling the case $V_{+\infty} \neq V_{-\infty}$, but because of the assumed reality of φ is limited to considering scattering *over* the potential barrier. (This is the special case implicit in a different context in Ref. [19].) The evolution equations again simplify tremendously to yield

$$\frac{da}{dx} = + \frac{1}{2\varphi'} \{\varphi'' b \exp(-2i\varphi)\}, \quad (63)$$

$$\frac{db}{dx} = + \frac{1}{2\varphi'} \{\varphi'' a \exp(+2i\varphi)\}. \quad (64)$$

This form of the evolution equations can be related to the qualitative discussion of scattering over a potential barrier presented by Migdal and Krainov [22,23]. For this choice of auxiliary function

$$\vartheta \to \frac{|\varphi''|}{2|\varphi'|} = \frac{|k'|}{2|k|} \quad (65)$$

and the bounds become

$$T \geq \text{sech}^2\left(\frac{1}{2} \int_{-\infty}^{+\infty} \frac{|k'|}{|k|} dx \right) \quad (66)$$

and

$$R \leq \tanh^2\left(\frac{1}{2} \int_{-\infty}^{+\infty} \frac{|k'|}{|k|} dx \right). \quad (67)$$

The relevant transfer matrix is now

$$E(x_f,x_i)=\mathcal{P}\exp\left(\frac{1}{2}\int_{x_f}^{x_i}\frac{\varphi''}{\varphi'}\begin{bmatrix}0 & e^{-2i\varphi}\\ e^{+2i\varphi} & 0\end{bmatrix}dx\right). \quad (68)$$

The Bogolubov coefficients are now

$$\begin{bmatrix}\alpha & \beta^*\\ \beta & \alpha^*\end{bmatrix}=E(\infty,-\infty)$$

$$=\mathcal{P}\exp\left(\int_{-\infty}^{\infty}\frac{\varphi''}{\varphi'}\begin{bmatrix}0 & e^{-2i\varphi}\\ e^{+2i\varphi} & 0\end{bmatrix}dx\right). \quad (69)$$

This type of analysis collects together and unifies several analytically known results for scattering over the barrier, such as for asymmetric square wells and Poschl-Teller potentials. (See Sec. VI.) After a few general comments, I shall turn to specializing this still rather general result to more specific cases.

A. Reflection above the barrier

The system (63) and (64) can also be used as the basis of an approximation scheme for β. Suppose that for all x we have $|b(x)|\ll 1$, so that $|a(x)|\approx 1$. Then

$$\frac{db}{dx}\approx\frac{\varphi''}{2\varphi'}\exp(+2i\varphi). \quad (70)$$

This may be immediately integrated to yield

$$\beta\approx\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\varphi''(x)}{\varphi'(x)}\exp(+2i\varphi)dx \quad (71)$$

or the equivalent

$$\beta\approx\frac{1}{2}\int_{-\infty}^{+\infty}\frac{k'(x)}{k(x)}\exp\left(+2i\int_{-\infty}^{x}k(y)dy\right)dx. \quad (72)$$

This result serves to clarify the otherwise quite mysterious discussion of ‘‘reflection above the barrier’’ given by Migdal and Krainov [22,23]. Even though the WKB wave functions are buried in the representation of the wave function underlying the analysis leading to this approximation, the validity of this result for $|\beta|$ does not require validity of the WKB approximation.

If the shifted potential $V-V_\infty$ is ‘‘small’’ we can recover the Born approximation in the usual manner. In that case $k'\equiv mV'/\hbar^2k\approx mV'/\hbar^2k_\infty$, while $\exp(2i\int k)\approx\exp(2ik_\infty x)$. A single integration by parts then yields

$$\beta\approx -i\frac{m}{\hbar^2k_\infty}\int_{-\infty}^{+\infty}[V(x)-V_\infty]\exp(+2ik_\infty x)dx. \quad (73)$$

B. Under the barrier?

What goes wrong when we try to extend this analysis into the classically forbidden region? Analytically continuing the system (63) and (64) is trivial. Replace

$$\varphi'(x)=k\rightarrow i\kappa=i\sqrt{2m(V-E)}/\hbar \quad (74)$$

and write

$$\varphi(x)=\varphi_{\rm tp}+i\int_{\rm tp}^{x}\kappa(y)dy \quad (75)$$

(where ‘‘tp’’ denotes the turning point) to obtain

$$\frac{da}{dx}=+\frac{\kappa'}{2\kappa}b\exp(-2i\varphi_{\rm tp})\exp\left(+2\int\kappa\right), \quad (76)$$

$$\frac{db}{dx}=+\frac{\kappa'}{2\kappa}a\exp(+2i\varphi_{\rm tp})\exp\left(-2\int\kappa\right). \quad (77)$$

Thus we are *violating* our previous condition that φ be real, though we still require $\varphi'\neq 0$. This is a perfectly good Shabat-Zakharov system that works in the forbidden region. However, one cannot now use this to derive bounds on the transmission coefficient. The difficulty resides in the fact that the formula for the probability current is modified and that in the forbidden region the probability current is

$$\mathcal{J}=\text{Im}\left\{\psi^*\frac{d\psi}{dx}\right\}=2\,\text{Im}\{ab^*\exp(+2i\varphi_{\rm tp})\}. \quad (78)$$

For a properly normalized flux in the allowed region ($|a|^2-|b|^2=1$), we have in the forbidden region

$$2\,\text{Im}\{ab^*\exp(+2i\varphi_{\rm tp})\}=1. \quad (79)$$

While this does imply $2|a||b|>1$, the inequality is unfortunately in the wrong direction to be useful for placing bounds on the transmission coefficient.

C. Special case 2a

Suppose now that $V(x)$ is continuous and monotonically increasing or decreasing, varying from $V_{-\infty}=V(-\infty)$ to $V_{+\infty}=V(+\infty)$. Suppose $E\geqslant\max\{V_{-\infty},V_{+\infty}\}$ so there is no classical turning point. Then

$$\int_{-\infty}^{+\infty}\frac{|k'|}{|k|}dx=\left|\ln\left(\frac{k_{+\infty}}{k_{-\infty}}\right)\right| \quad (80)$$

and the transmission and reflection probabilities satisfy

$$T\geqslant\frac{4k_{+\infty}k_{-\infty}}{(k_{+\infty}+k_{-\infty})^2} \quad (81)$$

and

$$R\leqslant\frac{(k_{+\infty}-k_{-\infty})^2}{(k_{+\infty}+k_{-\infty})^2}. \quad (82)$$

These bounds are immediately recognizable as the exact analytic results for a step-function potential [1,7,8] and the result asserts that for arbitrary smooth monotonic potentials the step function provides upper and lower bounds on the exact result. If we are interested in physical situations such as a time-dependent refractive index [25,26] or particle production due to the expansion of the universe [27], this technique shows that *sudden* changes in refractive index or size of the universe provide a strict upper bound on particle production.

D. Special case 2b

Suppose now that $V(x)$ has a single unique extremum (either a peak or a valley), and provided that $E \geqslant \max\{V_\infty, V_{\text{extremum}}, V_{+\infty}\}$ so that there is no classical turning point, $k(x)$ moves monotonically from $k_{-\infty}$ to k_{extremum} and then back to $k_{+\infty}$. Under these circumstances

$$\int_{-\infty}^{+\infty} \frac{|k'|}{k} dx = \left| \ln\left[\frac{k_{\text{extremum}}}{k_{-\infty}}\right]\right| + \left| \ln\left[\frac{k_{\text{extremum}}}{k_{+\infty}}\right]\right| \tag{83}$$

$$= \left| \ln\left[\frac{k_{\text{extremum}}^2}{k_{-\infty} k_{+\infty}}\right]\right|. \tag{84}$$

This implies

$$|\alpha| \leqslant \cosh\left| \ln\left[\frac{k_{\text{extremum}}}{\sqrt{k_{-\infty} k_{+\infty}}}\right]\right|, \tag{85}$$

which yields

$$|\beta| \leqslant \sinh\left| \ln\left[\frac{k_{\text{extremum}}}{\sqrt{k_{-\infty} k_{+\infty}}}\right]\right|. \tag{86}$$

To be more specific, if in addition $V(-\infty)=0=V(+\infty)$, so that $k_{-\infty}=k_{+\infty}$, then we have

$$|\alpha| \leqslant \frac{k_{\text{extremum}}^2 + k_\infty^2}{2k_{\text{extremum}} k_\infty} \tag{87}$$

and

$$|\beta| \leqslant \frac{|k_{\text{extremum}}^2 - k_\infty^2|}{2k_{\text{extremum}} k_\infty}. \tag{88}$$

Translated into statements about the transmission and reflection probabilities this becomes

$$T \geqslant \frac{(E-V_\infty)(E-V_{\text{extremum}})}{(E-V_\infty)(E-V_{\text{extremum}}) + \frac{1}{4}(V_{\text{extremum}}-V_\infty)^2} \tag{89}$$

and

$$R \leqslant \frac{\frac{1}{4}(V_{\text{extremum}}-V_\infty)^2}{(E-V_\infty)(E-V_{\text{extremum}}) + \frac{1}{4}(V_{\text{extremum}}-V_\infty)^2} \tag{90}$$

or, equivalently,

$$T \geqslant 1 - \frac{(V_{\text{extremum}}-V_\infty)^2}{(2E-V_{\text{extremum}}-V_\infty)^2} \tag{91}$$

and

$$R \leqslant \frac{(V_{\text{extremum}}-V_\infty)^2}{(2E-V_{\text{extremum}}-V_\infty)^2}. \tag{92}$$

For low energies, these results are weaker than the bounds derived under special case 1 [Eqs. (46) and (47)] and [Eqs. (48) and (49)], but have the advantage of requiring more selective information about the potential. For high energies

$$E \geqslant \frac{\hbar^2 (V_{\text{extremum}}-V_\infty)^2}{2m(\int_{-\infty}^{+\infty} |V(x)-V_\infty| dx)^2}, \tag{93}$$

the present result (when it is applicable) leads to tighter bounds on the transmission and reflection coefficients.

Numerous generalizations of these formulas are possible. For example, at the cost of a little extra notation, we already have enough information to provide a bound on an *asymmetric* barrier or *asymmetric* well. As long as it has only a single extremum (maximum or minimum), we apply the previous equations to derive

$$|\alpha| \leqslant \frac{k_{\text{extremum}}^2 + k_{+\infty} k_{-\infty}}{2k_{\text{extremum}} \sqrt{k_{+\infty} k_{-\infty}}} \tag{94}$$

and

$$|\beta| \leqslant \frac{|k_{\text{extremum}}^2 - k_{+\infty} k_{-\infty}|}{2k_{\text{extremum}} \sqrt{k_{+\infty} k_{-\infty}}}. \tag{95}$$

Translated into statements about the transmission and reflection probabilities, this becomes

$$T \geqslant \frac{4k_{+\infty} k_{-\infty} k_{\text{extremum}}^2}{\{k_{\text{extremum}}^2 + k_{+\infty} k_{-\infty}\}^2} \tag{96}$$

and

$$R \leqslant \frac{\{k_{\text{extremum}}^2 - k_{+\infty} k_{-\infty}\}^2}{\{k_{\text{extremum}}^2 + k_{+\infty} k_{-\infty}\}^2} \tag{97}$$

or, equivalently,

$$T \geqslant \frac{4(E-V_{\text{extremum}})\sqrt{(E-V_{+\infty})(E-V_{-\infty})}}{[(E-V_{\text{extremum}}) + \sqrt{(E-V_{+\infty})(E-V_{-\infty})}]^2} \tag{98}$$

and

$$R \leqslant \frac{[(E-V_{\text{extremum}}) - \sqrt{(E-V_{+\infty})(E-V_{-\infty})}]^2}{[(E-V_{\text{extremum}}) + \sqrt{(E-V_{+\infty})(E-V_{-\infty})}]^2}. \tag{99}$$

This can be compared, for example, with known analytic results for the asymmetric square well; see Eq. (144) in Sec. VI.

E. Special case 2c

Suppose now that $V(x)$ has a number of extrema (both peaks and valleys). I allow $V(+\infty) \neq V(-\infty)$, but demand that for all extrema $E \geqslant \max\{V_{-\infty}, V_{+\infty}, V_{\text{extremum}}^i\}$ so that there is no classical turning point.

For definiteness, suppose the ordering is $-\infty \to$peak $\to$valley$\cdots$valley$\to$peak$\to +\infty$. Then

$$\int_{-\infty}^{+\infty} \frac{|k'|}{k} dx = \left| \ln\left[\frac{k_{\rm peak}^1}{k_{-\infty}} \right] \right| + \left| \ln\left[\frac{k_{\rm valley}^1}{k_{\rm peak}^1} \right] \right| + \cdots + \left| \ln\left[\frac{k_{\rm peak}^n}{k_{\rm valley}^{n-1}} \right] \right| + \left| \ln\left[\frac{k_{+\infty}}{k_{\rm peak}^n} \right] \right|. \quad (100)$$

Defining

$$\Pi_p(k) \equiv \prod_{\rm peaks} k_{\rm peak}^i, \quad (101)$$

$$\Pi_v(k) \equiv \prod_{\rm valleys} k_{\rm valley}^i, \quad (102)$$

$$\Pi_e(k) \equiv \prod_{\rm extrema} k_{\rm extremum}^i, \quad (103)$$

we see

$$\int_{-\infty}^{+\infty} \frac{|k'|}{k} dx = \left| \ln\left[\frac{\Pi_p^2(k)}{k_{-\infty} k_{+\infty} \Pi_v^2(k)} \right] \right|. \quad (104)$$

This bounds the Bogolubov coefficients as

$$|\alpha| \le \cosh \left| \ln\left[\frac{\Pi_p(k)}{\sqrt{k_{-\infty} k_{+\infty}} \Pi_v(k)} \right] \right|, \quad (105)$$

that is,

$$|\alpha| \le \frac{k_{-\infty} k_{+\infty} \Pi_v^2(k) + \Pi_p^2(k)}{2\sqrt{k_{+\infty} k_{-\infty}} \Pi_e(k)} \quad (106)$$

and

$$|\beta| \le \frac{|k_{-\infty} k_{+\infty} \Pi_v^2(k) - \Pi_p^2(k)|}{2\sqrt{k_{+\infty} k_{-\infty}} \Pi_e(k)}. \quad (107)$$

Then the transmission and reflection probabilities satisfy

$$T \ge \frac{4 k_{+\infty} k_{-\infty} \Pi_e^2(k)}{\{\Pi_p^2(k) + k_{+\infty} k_{-\infty} \Pi_v^2(k)\}^2} \quad (108)$$

and

$$R \le \frac{\{\Pi_p^2(k) - k_{+\infty} k_{-\infty} \Pi_v^2(k)\}^2}{\{\Pi_p^2(k) + k_{+\infty} k_{-\infty} \Pi_v^2(k)\}^2}. \quad (109)$$

In these formulas, peaks and valleys can be interchanged in the obvious way and by letting the initial or final peak sink down to $V_{\pm\infty}$ as appropriate we obtain bounds for sequences such as $-\infty \to$valley$\to$peak$\cdots$valley$\to$peak$\to +\infty$ or $-\infty \to$peak$\to$valley$\cdots$peak$\to$valley$\to +\infty$. In the case of one or zero extrema these formulas reduce to the previously given results [Eqs. (96) and (97)]. Further modifications of these formulas are still possible. The cost is that more specific assumptions are needed to derive more specific results.

V. PARAMETRIC OSCILLATIONS

Though the discussion so far has been presented in terms of the spatial properties of the time-independent Schrödinger equation, the mathematical structure of parametrically excited oscillations is identical, needing only a few minor translations to be brought into the current form. For a parametrically excited oscillator we have

$$\frac{d^2\phi}{dt^2} = \omega(t)^2 \phi. \quad (110)$$

Just map $t \to x$, $\omega(t) \to k(x)$, and $\phi \to \psi$. In the general analysis of Eqs. (28)–(35) the quantity ϑ should be replaced by

$$\vartheta[\varphi(t), \omega(t)] \equiv \frac{\sqrt{(\varphi'')^2 + [\omega^2 - (\varphi')^2]^2}}{2|\varphi'|}. \quad (111)$$

The analysis then parallels that of the Schrödinger equation. Some key results are given below.

A. Special case 1

If $\omega(-\infty) = \omega_0 = \omega(+\infty) \neq 0$, then by choosing the auxiliary function to be $\varphi = \omega_0 t$ we can use Eqs. (46) and (47) to deduce

$$|\alpha| \le \cosh\left(\frac{1}{2\omega_0} \int_{-\infty}^{+\infty} |\omega^2(t) - \omega_0^2| dt \right) \quad (112)$$

and

$$|\beta| \le \sinh\left(\frac{1}{2\omega_0} \int_{-\infty}^{+\infty} |\omega^2(t) - \omega_0^2| dt \right). \quad (113)$$

B. Special case 2

If $\omega(-\infty)$ and $\omega(+\infty) \neq 0$ are both finite so that suitable asymptotic states exist and assuming $\omega^2(t) \ge 0$ so that the frequency is always positive, then applying Eqs. (66) and (67) to the case of parametric resonance yields

$$|\alpha| \le \cosh \left| \int_{-\infty}^{+\infty} \frac{|\omega'(t)|}{|\omega(t)|} dt \right| \quad (114)$$

and

$$|\beta| \le \sinh \left| \int_{-\infty}^{+\infty} \frac{|\omega'(t)|}{|\omega(t)|} dt \right|. \quad (115)$$

C. Special case 2a

Suppose now that $\omega^2(t)$ is positive semidefinite, continuous, and monotonic increasing or decreasing, varying from $\omega_{-\infty} = \omega(-\infty) \neq 0$ to $\omega_{+\infty} = \omega(+\infty) \neq 0$. The Bogoliubov coefficients satisfy

$$|\alpha| \le \frac{\omega_{-\infty} + \omega_{+\infty}}{2\sqrt{\omega_{-\infty}\omega_{+\infty}}} \quad (116)$$

and

$$|\beta| \leqslant \frac{|\omega_{-\infty} - \omega_{+\infty}|}{2\sqrt{\omega_{-\infty}\omega_{+\infty}}}. \tag{117}$$

D. Special case 2b

Under the restriction $\omega(-\infty) = \omega_0 = \omega(+\infty) \neq 0$, with the additional constraint that $\omega(t)$ has a single unique extremum (either a maximum or a minimum but not both) and provided $\omega^2_{\text{extremum}} > 0$ so that we do not encounter complex frequencies (no classical turning point), the Bogoliubov coefficients satisfy

$$|\alpha| \leqslant \frac{\omega_0^2 + \omega^2_{\text{extremum}}}{2\omega_0 \omega_{\text{extremum}}} \tag{118}$$

and

$$|\beta| \leqslant \frac{|\omega_0^2 - \omega^2_{\text{extremum}}|}{2\omega_0 \omega_{\text{extremum}}}. \tag{119}$$

Suppose now that $\omega^2(t)$ has a single unique extremum (either a peak or a valley), but that $\omega(+\infty) \neq \omega(-\infty)$ and further that $\omega^2(t) > 0$ so that there is no classical turning point. The Bogoliubov coefficients satisfy

$$|\alpha| \leqslant \frac{\omega_{-\infty}\omega_{+\infty} + \omega^2_{\text{extremum}}}{2\sqrt{\omega_{-\infty}\omega_{+\infty}}\,\omega_{\text{extremum}}} \tag{120}$$

and

$$|\beta| \leqslant \frac{|\omega_{-\infty}\omega_{+\infty} - \omega^2_{\text{extremum}}|}{2\sqrt{\omega_{-\infty}\omega_{+\infty}}\,\omega_{\text{extremum}}}. \tag{121}$$

E. Special case 2c

Suppose now that $\omega(t)$ has a number of extrema (both peaks and valleys). I allow $\omega(+\infty) \neq \omega(-\infty)$, but demand that for all extrema $\omega^i_{\text{extremum}} > 0$ so that there is no classical turning point.

For definiteness, suppose the ordering is $-\infty \to$peak$\to$valley$\cdots$valley$\to$peak$\to +\infty$. Define

$$\Pi_p(\omega) \equiv \prod_{\text{peaks}} \omega^i_{\text{peak}}, \tag{122}$$

$$\Pi_v(\omega) \equiv \prod_{\text{valleys}} \omega^i_{\text{valley}}, \tag{123}$$

$$\Pi_e(\omega) \equiv \prod_{\text{extrema}} \omega^i_{\text{extremum}}. \tag{124}$$

The Bogoliubov coefficients satisfy

$$|\alpha| \leqslant \frac{\omega_{-\infty}\omega_{+\infty}\Pi_v^2(\omega) + \Pi_p^2(\omega)}{\sqrt{\omega_{+\infty}\omega_{-\infty}}\,\Pi_e(\omega)} \tag{125}$$

and

$$|\beta| \leqslant \frac{|\omega_{-\infty}\omega_{+\infty}\Pi_v^2(k) - \Pi_p^2(k)|}{\sqrt{\omega_{+\infty}\omega_{-\infty}}\,\Pi_e(k)}. \tag{126}$$

In these formulas, peaks and valleys can be interchanged in the obvious way and by letting the initial or final peak sink down to $\omega_{\pm\infty}$ as appropriate we obtain bounds for sequences such as $-\infty \to$valley$\to$peak$\cdots$valley$\to$peak$\to +\infty$ or $-\infty \to$peak$\to$valley$\cdots$peak$\to$valley$\to +\infty$. In the case of one or zero extrema these formulas reduce to the previously given results.

Again, further specializations of these formulas are still possible. As always, there is a trade-off between the strength of the result and its generality.

VI. COMPARISON WITH KNOWN ANALYTIC RESULTS

For comparison purposes, in this section I collect several known analytic results and show how they relate to the general results presented in this paper.

A. δ-function potential

For a δ-function potential

$$V(x) = \alpha\,\delta(x), \tag{127}$$

the transmission coefficient is known to be [2,3]

$$T = \frac{1}{1 + (m\alpha^2/2E\hbar)}. \tag{128}$$

This satisfies the bound (46) and also Eq. (48) and for $E \to \infty$ asymptotically approaches the bound, thus showing that the bound cannot be improved in the high-energy regime *unless additional hypotheses are made*.

Though these bounds were *derived* assuming well-behaved functions, the statements (46) and (48) continue to make good sense even for δ-function potentials. Thus any smooth set of well-behaved functions tending to a δ-function limit may be used to establish Eqs. (46) and (48) even for potentials containing δ-function contributions.

B. Double-δ-function potential

For the double δ function

$$V(x) = \alpha\{\delta(x - L/2) + \delta(x + L/2)\}, \tag{129}$$

the transmission coefficient is [11]

$$T = \frac{1}{1 + [(2m\alpha/\hbar^2 k)\cos(kL) + 1/2\,(2m\alpha/\hbar^2 k)^2 \sin(kL)]^2}. \tag{130}$$

It is an easy exercise to check that this satisfies the bounds (46) and (48).

C. Square barrier

Tunneling *over* a square barrier is an elementary problem that however, is not always discussed in the textbooks. (Tunneling *under* a square barrier is much more popular.) The exact transmission coefficient is

$$T=\frac{E(E-V_e)}{E(E-V_e)+\frac{1}{4}V_e^2\sin^2[\sqrt{2m(E-V_e)}L/\hbar]}. \tag{131}$$

(See Refs. [1] or [9].) If we rewrite this as

$$T=\frac{1}{1+(mV_e^2L^2/2E\hbar^2)/\sin^2[\sqrt{2m(E-V_e)}L/\hbar]/2m(E-V)L^2/\hbar^2}, \tag{132}$$

then it is clear that the bound (46) is satisfied. It is also possible to verify that this satisfies the general lower bound (66) that I have presented above and in fact oscillates between this lower bound and the upper $T\leqslant 1$ unitarity limit. For certain values of the barrier width $[k_{\text{extremum}}L=(2n+1)\pi/2]$ the square well saturates this bound, thus showing that this bound cannot be improved *unless additional hypotheses are made*.

D. tanh potential

For a smoothed step function of the form

$$V(x)=\frac{V_{-\infty}+V_{+\infty}}{2}+\frac{V_{+\infty}-V_{-\infty}}{2}\tanh\left(\frac{x}{L}\right), \tag{133}$$

the reflection coefficient is known analytically to be [1]

$$R=\left(\frac{\sinh[2\pi(k_{-\infty}-k_{+\infty})L]}{\sinh[2\pi(k_{-\infty}+k_{+\infty})L]}\right)^2. \tag{134}$$

This certainly satisfies the general bounds (81) and (82) enunciated above and as $L\to 0$ approaches and saturates the bound.

E. sech potential

For a sech^2 potential of the form

$$V(x)=V_e\,\text{sech}^2(x/L), \tag{135}$$

the transmission coefficient is known analytically to be [1]

$$T=\frac{\sinh^2[\pi\sqrt{2mE}L/\hbar]}{\sinh^2[\pi\sqrt{2mE}L/\hbar]+\cos^2[1/2\,\pi\sqrt{1-8mV_eL^2/\hbar^2}]}, \tag{136}$$

provided $8mV_eL^2<\hbar^2$. This satisfies the general bounds, both (34) and (81), enunciated above. (However, proving this is tedious.) Start by noting that for this sech potential

$$T\geqslant\tanh^2[\pi\sqrt{2mE}L/\hbar] \tag{137}$$

and use the inequality $(x>0)$

$$\tanh^2 x>\frac{x^2}{1+x^2}>\text{sech}^2(1/x). \tag{138}$$

Then

$$T\geqslant\text{sech}^2[\hbar/(\pi\sqrt{2mE}L)] \tag{139}$$

$$=\text{sech}^2\left[\frac{4}{\pi}\sqrt{\frac{m}{2E}}\frac{2L|V_e|}{\hbar}\frac{\hbar^2}{8m|V_e|L^2}\right]. \tag{140}$$

Provided the extremum is a peak $V_{\text{peak}}>0$, we can use the bound $8mV_{\text{peak}}L^2<\hbar^2$ to deduce

$$T\geqslant\text{sech}^2\left[\sqrt{\frac{m}{2E}}\frac{2L|V_{\text{peak}}|}{\hbar}\right]. \tag{141}$$

This is the particularization of Eq. (34) to the present case. If $V_e<0$ we need a different analysis.

F. Asymmetric square-well potential

For the asymmetric square well

$$V(x)=\begin{cases}V_1, & x<a\\ V_2, & a<x<b\\ V_3, & b<x\end{cases} \tag{142}$$

we define $k_i\equiv\sqrt{2m(E-V_i)}/\hbar$. The transmission coefficient is [15]

$$T=\frac{4k_1k_2^2k_3}{(k_1+k_3)^2k_2^2+[k_1^2k_3^2+k_2^2(k_2^2-k_1^2-k_3^2)]\sin^2(k_2L)}. \tag{143}$$

Then

$$T \geqslant \frac{4k_1 k_2^2 k_3}{(k_2^2 + k_1 k_3)^2}. \tag{144}$$

Similarly to the case for the symmetric square well, the transmission probability for the asymmetric square well oscillates between the bound (96) and the unitarity limit $T=1$. For certain values of the width of the well $[k_2 L=(2n+1)\pi/2]$ the transmission coefficient saturates the bound, thus showing that this bound cannot be improved *unless additional hypotheses are made*. Because $V_{-\infty} \neq V_{+\infty}$ the bound (34) is not applicable, at least not without modification from its original form.

G. Poschl-Teller potential

For the Poschl-Teller potential

$$V(x) = V_0 \cosh^2\mu\,(\tanh\{[(x-\mu L)/L]\} + \tanh\mu)^2 \tag{145}$$

we have

$$V_{-\infty} = V_0 e^{-2\mu}, \quad V_{\text{extremum}} = 0; \quad V_{-\infty} = V_0 e^{-2\mu}. \tag{146}$$

The transmission coefficient is [24]

$$T = \frac{2\ \sinh(\pi k_{-\infty} L)\sinh(\pi k_{+\infty} L)}{\cosh[\pi(k_{-\infty}+k_{+\infty})L] + \cos[\pi\sqrt{1+(8mV_0L^2/\hbar^2)\cosh^2\mu}]}. \tag{147}$$

It is now a straightforward if tedious exercise to check this analytic result against all the bounds derived in this paper.

VII. DISCUSSION

The various special cases discussed above are merely specific examples of the general results (32)–(35) illustrating the power of the technique. There are many other variations on the bounds presented above that can be derived for specific choices of $\varphi(x)$ and specific restrictions on the scattering potential $V(x)$.

The most general form of the bounds are given in Eqs. (32)–(35). Because of the large amount of freedom in choosing the function φ these bounds encode even more specific cases beyond those discussed in this paper and have the potential for leading to interesting specific cases. The special cases I discussed in this paper were chosen for directness and simplicity.

For instance, special case 1, as presented in Eqs. (46)–(49), has the advantage that it applies to scattering both over the barrier and under the barrier. On the other hand, special case 2, as presented in Eqs. (66) and (67) and their specializations, applies only to scattering over the barrier, but has the advantage of being much more selective in how much information is needed concerning the scattering potential. In summary, the bounds presented in this paper are useful in establishing *qualitative* analytic properties of one-dimensional scattering and as such are complementary to both explicit numerical investigations and the guidance extracted from exact analytic solutions.

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[1] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, New York, 1977).
[2] G. Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1969).
[3] S. Gasiorowicz, *Quantum Physics* (Wiley, New York, 1996).
[4] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1965).
[5] J. Singh, *Quantum Mechanics: Fundamentals and Applications to Technology* (Wiley, New York, 1997).
[6] P. M. Mathews and K. Venkatesan, *A Textbook of Quantum Mechanics* (McGraw-Hill, New York, 1978).
[7] A. Z. Capri, *Non-Relativistic Quantum Mechanics* (Benjamin-Cummings, Menlo Park, CA, 1985). See in particular pp. 95–109.
[8] P. Stehle, *Quantum Mechanics* (Holden-Day, San Francisco, 1996). See in particular pp. 57–60.
[9] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955).
[10] C. Cohen-Tannoudji, B. Dui, and F. Laloë, *Quantum Mechanics* (Wiley, New York, 1977).
[11] A. Galindo and P. Pascual, *Quantum Mechanics I* (Springer-Verlag, Berlin, 1990).
[12] D. Park, *Introduction to the Quantum Theory* (McGraw-Hill, New York, 1974).
[13] A. T. Fromhold, *Quantum Mechanics for Applied Physics and Engineering* (Academic, New York, 1981).
[14] M. Scharff, *Elementary Quantum Mechanics* (Wiley, London, 1969).

[15] A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1958).
[16] K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer-Verlag, New York, 1989).
[17] W. Eckhaus and A. van Harten, *The Inverse Scattering Transformation and the Theory of Solitons* (North-Holland, Amsterdam, 1981).
[18] R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, Princeton, 1979). See in particular pp. 21 and 22.
[19] M. Bordag, J. Lindig, and V. M. Mostepaneko, Class. Quantum Grav. **15**, 581 (1998).
[20] N. Froman and P. O. Froman, *JWKB Approximation: Contributions to the Theory* (North-Holland, Amsterdam, 1965).
[21] Transfer matrix techniques are discussed, at varying levels of detail, by Merzbacher [4], Singh [5], and Mathews and Venkatesan [6].
[22] A. B. Migdal, *Qualitative Methods in Quantum Theory* (Benjamin, London, 1977).
[23] A. B. Migdal and V. P. Krainov, *Approximation Methods in Quantum Mechanics* (Benjamin, London, 1969).
[24] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), pp. 1651–1660.
[25] S. Liberati, M. Visser, F. Belgiorno, and D. W. Sciama, e-print quant-ph/9805023; e-print quant-ph/9805031.
[26] E. Yablonovitch, Phys. Rev. Lett. **62**, 1742 (1989).
[27] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).