Lorentz-covariant quantum mechanics and preferred frame

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In this paper the relativistic quantum mechanics (QM) is considered in the framework of the nonstandard synchronization scheme, which preserves the Poincaré covariance but (at least formally) distinguishes an inertial frame. This enables one to avoid the problem with a strong formulation of local causality related to the breaking of Bell's inequalities in QM. Our analysis has been focused mainly on the problem of the existence of a proper position operator for massive particles. We have proved that in our framework such an operator exists for particles with arbitrary spin. This operator is Hermitian and covariant, it has commuting components, and its eigenvectors (localized states) are covariant too. We have found an explicit form of the position operator and demonstrated that it coincides with the Newton-Wigner one in the preferred frame. We have also defined a covariant spin operator and constructed an invariant spin squared operator. Moreover, full algebra of observables consisting of position, four-momentum, and spin operators is manifestly Poincaré covariant in this framework. Our results support expectations of other authors [J. S. Bell, in *Quantum Gravity*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, Oxford, 1981), p. 611; P. H. Eberhard, Nuovo Cimento B **46**, 392 (1978)] that a consistent formulation of quantum mechanics demands the existence of a preferred frame. [S1050-2947(99)04706-X]

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I. INTRODUCTION

In this paper we propose a formulation of the Poincaré-covariant quantum mechanics for a free particle. Our investigations are motivated by two old and still open problems: the violation of locality in quantum mechanics (breaking of Bell's inequalities) and the nonexistence of a covariant position operator as well as covariant localized states. It was recognized long ago that some correlation experiments (cf. [3–5]) imply that, ''... what happens macroscopically in one space-time region must in some cases depend on variables that are controlled by experimenters in far-away, space-like-separated regions'' [6]. This fact can be in conflict with special relativity; even more frustrating is a conflict with a strong version of local causality. It may be interesting to recall in this place the statement by Bell [7]: ''... For me then this is the real problem with quantum theory: the apparently essential conflict between any sharp formulation and fundamental relativity.'' According to Bell [1] (see also Eberhard [2]), a consistent formulation of relativistic quantum mechanics may require a preferred frame at the fundamental level. Here we follow these suggestions and introduce the Poincaré-covariant formulation of quantum mechanics which has a preferred frame built in.

It is important to realize that special relativity is in fact based on two main assumptions: the Poincaré covariance and the relativity principle. Our aim is to reformulate this theory in a way which preserves the Poincaré covariance but abandons the relativity principle and consequently allows one to introduce a preferred frame. Such a formulation of the relativity theory was given by one of the authors (J.R.) in [8,9]. It was shown there that by using a nonstandard synchronization procedure for clocks (called the Chang-Tangherlini synchronization in [9]), it is possible to obtain such a form of the transformations of coordinates between inertial observers, that while they still are Lorentz transformations, the time coordinate is only rescaled by a positive factor and the space coordinates do not mix with it. A price for this is the existence of a preferred frame in the theory and the dependence of the Lorentz group transformations on an additional parameter — the four-velocity of a preferred frame. Usually it is claimed that the existence of a preferred frame violates the Poincaré covariance. This is really the case if we restrict ourselves to the Minkowski space-time. But in our approach the additional set of parameters (four-velocity) allows us to preserve the Poincaré covariance but not necessarily the relativity principle. We also think that the existence of a preferred frame is not a serious problem from the physical point of view. In the real (expanding) Universe such a frame really does exist — it is the so-called comoving frame related to the matter and the cosmic background radiation frame. Furthermore, in our framework, an average light velocity over closed paths is still constant and equal to c, so Michelson-Morley-like experiments do not distinguish such a possibility from the standard one [10,9].

The formulation of the Poincaré-covariant quantum mechanics presented here seems to have a number of advantages over the standard formulation. First of all, the conflict between the causality and the quantum theory disappears. Second, the localization problem is solved. Various aspects of localizability of particles have been studied from the early days of quantum mechanics, but, in the relativistic case (as opposed to the nonrelativistic one) the fully satisfactory position operator has not been found yet. Let us explain at this point what we mean by the satisfactory position operator. Such an operator should be Hermitian, have commuting components (for massive particles), fulfill the canonical

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commutation relations with the momentum operators, be covariant, and have covariant eigenstates (localized states). The operator constructed in the framework of the theory presented here has all these properties. We also hope that this formulation of quantum mechanics will be very convenient in the case of the Poincaré-covariant de Broglie–Bohm approach to quantum mechanics.[1] To make this paper self-contained we devote the second section to the description of the Chang-Tangherlini synchronization scheme and the corresponding realization of the Lorentz group. More details on this subject can be found in [9]. In Sec. III we describe briefly the covariant canonical formalism for a relativistic free particle on the classical level, following [9,16]. Section IV contains a description of the quantum theory in the Chang-Tangherlini synchronization and the definition of a position operator. In Sec. V we construct and classify unitary orbits of the Poincaré group using the introduced position operator. We define also a covariant spin operator and construct the invariant operator of the spin squared. In Sec. VI we first review briefly the properties of the Newton-Wigner operator (in the authors' opinion this is the best position operator which has been constructed up to now; more information about the history of the localization problem and the full bibliography can be found in [17,18]). Then we find the explicit form of our position operator in the functional realization and show that in the preferred frame our operator coincides with the Newton-Wigner one. This section contains also a discussion of the position operator under the special choice of the integral measure. In this case the form of the position operator is the same as in the nonrelativistic quantum mechanics.

II. CHANG-TANGHERLINI (CT) SYNCHRONIZATION

In this section we briefly describe the main features of the CT synchronization scheme which we shall use in the sequel. The derivation of these results can be found in [9]. The idea adopted there is that the definition of the time coordinate depends on the choice of the synchronization scheme for clocks and that this choice is a matter of convention [10,19–22]. Using this freedom one can choose a synchronization procedure resulting in the desired form of the Lorentz transformations. Performing such a program we have to distinguish, at least formally, one inertial frame — the so-called preferred frame. Thus, at least formally, the relativity principle may be broken. We discuss this problem in Sec. II A. Now, each inertial frame is determined by the four-velocity of this frame with respect to the distinguished one. We denote four-velocity of the preferred frame as seen by an inertial observer by u^μ. Hereafter, quantities in the Einstein-Poincaré (EP) synchronization are denoted by the subscript (or superscript) E. Quantities in the CT synchronization usually have no index; only the time coordinate in CT synchronization is denoted by x^0_{CT}. We use the natural units ($\hbar=c=1$).

According to [8,9] the transformation law between inertial frames is determined by the following requirements: (i) the transformation group is isomorphic to the Lorentz group; (ii) the average value of the light speed over closed paths is constant and equal to 1; (iii) transformations are linear with respect to the coordinates (affinity); (iv) under the rotations the coordinates transform in a standard way (isotropy):

$$x'^0_{\mathrm{CT}}(u') = x^0_{\mathrm{CT}}(u),$$

$$\mathbf{x}'(u') = R\mathbf{x}(u),$$

where R is a rotation matrix; (v) the instant time hyperplane $x^0_{\mathrm{CT}} = \mathrm{const}$ is an invariant notion. Notice that (i)–(iv) are the standard requirements, while (v) is nonstandard. Consequently, in this synchronization the transformation of coordinates between inertial frames has the following form (for contravariant coordinates):

$$x'(u') = D(\Lambda,u)x(u), \tag{1}$$

where Λ is an element of the Lorentz group, u^μ is the four-velocity of the preferred frame with respect to the actual one, and $D(\Lambda,u)$ is a Λ- and u-dependent 4×4 matrix. Equation (1) is accompanied by

$$u' = D(\Lambda,u)u. \tag{2}$$

Matrices $D(\Lambda,u)$ fulfill the following group composition rule:

$$D(\Lambda_2, D(\Lambda_1,u)u)D(\Lambda_1,u) = D(\Lambda_2\Lambda_1,u) \tag{3}$$

so that

$$D^{-1}(\Lambda,u) = D(\Lambda^{-1}, D(\Lambda,u)u), \quad D(I,u) = I. \tag{4}$$

Let $T(u)$ be the intertwining matrix connecting coordinates in the CT and EP synchronizations. It means that for every contravariant four-vector A^μ we have

$$A^\mu = T(u)^\mu{}_\nu A^\nu_E. \tag{5}$$

Therefore $D(\Lambda,u)$ is of the following form:

$$D(\Lambda,u) = T(u')\Lambda T^{-1}(u). \tag{6}$$

One can find the explicit form of $T(u)$ [9], namely,

$$T(u) = \begin{pmatrix} 1 & -\mathbf{u}^T u^0 \\ 0 & I \end{pmatrix}. \tag{7}$$

Consequently for all rotations $R \in \mathrm{SO}(3)$, $D(\Lambda,u)$ is given by

$$D(R,u) = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \tag{8}$$

while for the boosts it is given by

[1] An exhaustive review of the interpretational problems of quantum mechanics can be found in [11–15].

$$D(W,u)=\begin{pmatrix} \frac{1}{W^0} & 0 \\ -\mathbf{W} & I+\frac{\mathbf{W}\otimes\mathbf{W}^T}{1+\sqrt{1+(\mathbf{W})^2}}-u^0\mathbf{W}\otimes\mathbf{u}^T \end{pmatrix}, \tag{9}$$

where W^μ denotes the four-velocity of the frame $O_{u'}$ with respect to the frame O_u. The four-velocity W^μ can be expressed in terms of u and u',

$$W^0=\frac{u^0}{u'^0},\quad \mathbf{W}=\frac{(u^0+u'^0)(\mathbf{u}-\mathbf{u}')}{[1+u^0u'^0(1+\mathbf{u}\mathbf{u}')]}. \tag{10}$$

Instead of W^μ we can also use velocity $\mathbf{V}=\mathbf{W}/W^0$. The form of all the above formulas in terms of $\mathbf{V}$ can be found by using the relation

$$\frac{1}{W^0}=\sqrt{(1+u^0\mathbf{u}\mathbf{V})^2-(\mathbf{V})^2}. \tag{11}$$

The explicit relationship between coordinates in EP and CT is given by

$$x_E^0=x_{\text{CT}}^0+u^0\mathbf{u}\mathbf{x},\quad \mathbf{x}_E=\mathbf{x},$$

$$u_E^0=\frac{1}{u^0},\quad \mathbf{u}_E=\mathbf{u}. \tag{12}$$

We see that only the time coordinate changes. Note also that in the same point in space we have $\Delta x_E^0=\Delta x_{\text{CT}}^0$ so the time lapse is the same in both synchronizations.

One can easily see that the line element

$$ds^2=g_{\mu\nu}(u)dx^\mu\,dx^\nu \tag{13}$$

is invariant under the transformations (6) if

$$g(u)=[T(u)\eta T^T(u)]^{-1}, \tag{14}$$

where η is the Minkowski metric tensor $\eta=\text{diag}(+,-,-,-)$. The explicit form of the covariant metric tensor reads

$$[g_{\mu\nu}]=\begin{pmatrix} 1 & u^0\mathbf{u}^T \\ u^0\mathbf{u} & -I+(u^0)^2\mathbf{u}\otimes\mathbf{u}^T \end{pmatrix}, \tag{15}$$

while the contravariant one has the form

$$g^{-1}(u)=\begin{pmatrix} (u^0)^2 & u^0\mathbf{u}^T \\ u^0\mathbf{u} & -I \end{pmatrix}. \tag{16}$$

This implies that the space line element is the Euclidean one: $dl^2=d\mathbf{x}^2$. Notice that the triangular form of the boost matrix (9) implies that under the Lorentz transformations the time coordinate is rescaled by a positive factor $[x'^0_{\text{CT}}=(1/W^0)x^0_{\text{CT}}]$; the space coordinates do not mix with it. One can also easily check that the following, very useful, relations hold:

$$\underline{\mathbf{u}}=\underline{\mathbf{0}}\ \text{ and }\ \frac{1}{(u^0)^2}-(\mathbf{u})^2=1. \tag{17}$$

Hereafter the three-vector part of a covariant (contravariant) four-vector a_μ (a^μ) will be denoted by $\underline{\mathbf{a}}$ $(\mathbf{a})$.

A. Geometric description of the CT synchronization

A geometric description of the special relativity in the CT synchronization scheme can be expressed in the language of frame bundles. For a reader who is not familiar with the notion of a fiber bundle, we collected all necessary definitions in the Appendix. The review of applications of fiber bundles in the context of special relativity can be found in [23]. Let M be the Minkowski space-time, $L_+^\uparrow$ the ortochronous Lorentz group (the group of space-time transformations), and let $F(L_+^\uparrow)$ be the set of all frames in the space M obtained by action of $L_+^\uparrow$ on one particular (but arbitrary) frame. Thus $F(L_+^\uparrow)$ is isomorphic to the group $L_+^\uparrow$ and the element of $F(L_+^\uparrow)$ corresponding to an element $g\in L_+^\uparrow$ is denoted by $e(g)$.

Now let us consider the following structure:

$$M_w=[L_+^\uparrow,(F(L_+^\uparrow)\times M,M,pr_2),\pi_w,\psi_w], \tag{18}$$

where pr_2 is the canonical projection on the second factor of the Cartesian product. Therefore $[F(L_+^\uparrow)\times M,M,pr_2]$ is a frame bundle with the typical fiber $F(L_+^\uparrow)$. π_w is a projection on a fixed timelike four-vector w, while ψ_w is the action of the group $L_+^\uparrow$ on the bundle $[F(L_+^\uparrow)\times M,M,pr_2]$ fulfilling the following conditions:

$$(e'(g),x')=(e(kg),x), \tag{19}$$

$$e^\mu(kg,x)=D(k,g)^\mu{}_\nu e^\nu(g,x), \tag{20}$$

$$D^{T-1}(k,g)\pi_w D^{-1}(k,g)=\pi_w, \tag{21}$$

where $k\in L_+^\uparrow$, $x\in M$. It is clear that the Lorentz transformations are considered as passive transformations — the action of the Lorentz group changes the observer, not the physical state. This means that the action of the Lorentz group changes the frame $e(g)$. Condition (19) means that the action ψ_w is trivial on the manifold M; the group $L_+^\uparrow$ acts only on the fiber. Condition (20) says that ψ_w acts linearly on frames. Now, we associate the time direction to w, which means that the projector $\pi_w=w\otimes w/w^2$ is equal to π_{e^0},

$$\pi_{e^0}=\pi_w. \tag{22}$$

After this identification, π_w reads in the e^μ basis $[\pi_w=(\pi_w)_{\mu\nu}e^\mu\otimes e^\nu]$

$$[(\pi_{e^0})_{\mu\nu}]=[(\pi_w)_{\mu\nu}]=\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}. \tag{23}$$

This construction defines a time orientation of M along w. The matrix $D(k,g)$ can be expressed by $D(\Lambda,u)$ given in Eqs. (8) and (9) as follows. Let $\Lambda_1=k$, $\Lambda_2=g$; then

$$D(k,g)=D(\Lambda_1,\Lambda_2\tilde{u}) \tag{24}$$

and $\tilde{u}=(1,\mathbf{0})$. Condition (21) means that the direction of the four-vector w is invariant under the action of the group $L_+^\uparrow$.

Thus we have a collection of time-oriented space-times M_w, where w is an arbitrary timelike four-vector. The objects M_w and $M_{w'}$ corresponding to different w and w' are evidently connected by the action of another Lorentz group $L_+^{\uparrow(S)}$ (called a synchronization group in [9]). The whole family of time-oriented space-times M_w together with the transformations φ of the synchronization group, treated as morphisms, form a category

$$\mathcal{A}=(M_w,\varphi). \tag{25}$$

The action φ of the synchronization group $L_+^{\uparrow(S)}$ is defined in the most natural way,

$$\varphi(M_w)=M_{\Lambda^S ow}, \quad \Lambda^S\in L_+^{\uparrow(S)}. \tag{26}$$

From the physical point of view all choices of an object in the category $\mathcal{A}$ are equivalent provided the relativity principle holds. However, if we want to introduce a covariant canonical formalism for a relativistic free particle on the classical level or to define a proper position operator for such a particle on the quantum level, we have to give up the relativity principle. In other words, a consistent description is possible if and only if we use a fixed element of the category $\mathcal{A}$. In this case also the causal problems connected with the breaking of Bell's inequalities in QM disappear. Summarizing, although the formulation of special relativity in terms of the category $\mathcal{A}$ is equivalent to the standard one, whenever the notions of localizability or absolute causality are incorporated, the group of morphisms is broken, i.e., a time orientation is selected. Using some particular $\mathcal{A}$ (family of CT synchronizations), one can consistently define the position operator, which is impossible within the EP scheme. Moreover, this construction shows that some notions, such as localizability, are simultaneously compatible with quantum mechanics and Poincaré covariance if and only if a privileged frame is distinguished.

III. CANONICAL FORMALISM IN THE CT SYNCHRONIZATION

For a relativistic free particle we postulate the following action functional:

$$S_{12}=-m\int_{\lambda_1}^{\lambda_2}\sqrt{ds^2}, \tag{27}$$

where $ds^2=g_{\mu\nu}(u)(dx^\mu/d\lambda)(dx^\nu/d\lambda)d\lambda^2$ and λ is a trajectory parameter. We define the four-velocity in the standard way: $\omega^\mu=dx^\mu/d\lambda=\dot{x}^\mu$. Then the velocity has the form $\mathbf{v}=d\mathbf{x}/dx^0_{\mathrm{CT}}=\boldsymbol{\omega}/\omega^0$. Choosing the parameter λ as the length of the trajectory, $d\lambda=\sqrt{ds^2}$, we obtain the following condition:

$$\omega^2=g_{\mu\nu}(u)\omega^\mu(u)\omega^\nu(u)=1, \tag{28}$$

which implies

$$\dot{\omega}^\mu=\ddot{x}^\mu=0. \tag{29}$$

Using the formula (15) we can derive from Eq. (27) the Lagrangian

$$L=-m\sqrt{(1+u^0\mathbf{uv})^2-(\mathbf{v})^2}. \tag{30}$$

Now we can calculate the canonical momenta

$$\pi_i=\frac{\partial L}{\partial v^i}=\frac{m[v^i-u^iu^0(1+u^0\mathbf{uv})]}{\sqrt{(1+u^0\mathbf{uv})^2-(\mathbf{v})^2}}=-m\omega_i \tag{31}$$

and the Hamiltonian

$$H=\pi_k v^k-L=\frac{1}{u^0}\left[\mathbf{u}\underline{\boldsymbol{\pi}}+\sqrt{(\mathbf{u}\underline{\boldsymbol{\pi}})^2+(\underline{\boldsymbol{\pi}})^2+m^2}\right]=m\omega_0, \tag{32}$$

where $\underline{\boldsymbol{\pi}}=(\pi_1,\pi_2,\pi_3)$. So the covariant four-momentum can be defined by

$$k_\mu=m\omega_\mu. \tag{33}$$

It is easy to check that k_μ fulfills the following dispersion relation:

$$k^2=g^{\mu\nu}(u)k_\mu k_\nu=m^2. \tag{34}$$

Now we introduce the Poisson bracket

$$\{A,B\}=-\left(\delta^\mu{}_\nu-\frac{k^\mu u_\nu}{uk}\right)\left(\frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial k_\nu}-\frac{\partial B}{\partial x^\mu}\frac{\partial A}{\partial k_\nu}\right), \tag{35}$$

where x^μ, k_ν are treated as independent variables; in particular, k_0 is not *a priori* connected with k_i via the dispersion relation (34). The Poisson bracket (35) satisfies all of the following necessary conditions: (i) it is bilinear and antisymmetric, satisfies the Leibniz rule, and fulfills the Jacobi identity; (ii) it is manifestly Poincaré covariant in the CT synchronization; (iii) it is consistent with the constraint (34), i.e., $\{k^2,k_\nu\}=\{k^2,x^\mu\}=0$, therefore there is no reason to introduce a Dirac bracket; (iv) it is consistent with the Hamilton equations (37). In particular, Eq. (35) implies

$$\{x^\mu,x^\nu\}=0, \quad \{x^0_{\mathrm{CT}},k_\mu\}=0,$$
$$\{x^i,k_j\}=-\delta^i_j, \quad \{x^i,k_0\}=k^i/k^0,$$
$$\{k_\mu,k_\nu\}=0. \tag{36}$$

The Hamilton equations for a free particle have the desirable form

$$\frac{dx^i}{dt}=\frac{\partial H}{\partial \pi_i}=\frac{k^i}{k^0}=v^i, \quad \frac{dk_i}{dt}=-\frac{\partial H}{\partial x^i}=0, \tag{37}$$

where H is given by Eq. (32) and $t=x^0_{\rm CT}$. In general, the equation of motion for an observable $\Omega(x^\mu,k_\nu)$ in terms of the Poisson bracket (35) reads

$$\frac{d\Omega}{dt}=\frac{\partial\Omega}{\partial t}+\{\Omega,k_0\}. \tag{38}$$

In Eq. (38) as well as in Eq. (35), k_0 is treated as an independent variable. The solution of Eq. (38) can be reduced to the constraint surface (34).

IV. QUANTUM THEORY IN THE CT SYNCHRONIZATION

The results of the previous sections imply that there is an absolute causality in the CT synchronization. Therefore all the problems with causality, connected with the violation of Bell inequalities, disappear. Quantum theory remains nonlocal but it is causal. In our approach we are able, in analogy to the classical Poisson algebra described in Sec. III, to introduce a Poincaré-covariant algebra of momentum and position operators satisfying all fundamental physical requirements. This is done in Sec. IV B. The properties of the position operator are discussed in detail in Secs. VI B and VI C.

A. Preliminaries

In the CT synchronization the following point of view is the most natural one: to each inertial observer O_u we associate its own Hilbert space H_u (the space of states). The state vectors in H_u are denoted by u: $|u,\ldots\rangle$. In other words, we have a bundle of Hilbert spaces corresponding to the bundle of frames described in Sec. II A. In such an interpretation we have to distinguish carefully between active and passive transformations, because active transformations are represented by operators acting on one Hilbert space while passive ones are represented by operators acting between different Hilbert spaces. So, in particular, the Lorentz group transformations are considered as passive ones. Now, let $U(\Lambda)$ be an operator representing a Lorentz group element Λ. We postulate the following, standard, transformation law for a contravariant four-vector operator:

$$U(\Lambda)\hat{A}(u)^\mu U^{-1}(\Lambda)=[D^{-1}(\Lambda,u)]^\mu_{\ \nu}\hat{A}(u')^\nu, \tag{39}$$

where $D(\Lambda,u)$ is given by Eqs. (8) and (9) and $u'=D(\Lambda,u)u$; for a covariant four-vector $\hat{A}(u)_\mu$ we have to replace D^{-1} by D^T on the right-hand side of Eq. (39). Let Ω be a four-vector observable. In the space H_u the observable Ω is represented by an operator $\hat{\Omega}^\mu(u)$. Now let two inertial observers O_u and $O_{u'}$ measure independently the value of the observable Ω for a physical system being in the same physical state.[2] Let this state be described by an eigenvector $|\omega,u,\ldots\rangle$ of the Ω^μ in the space H_u. Then in the space $H_{u'}$ the same state is described by the vector

$$|\omega',u',\ldots\rangle=U(\Lambda)|\omega,u,\ldots\rangle, \tag{40}$$

where $u'=D(\Lambda,u)u$, $\omega'=D(\Lambda,u)\omega$. As a result of measurement of Ω the observer O_u will receive the value ω,

$$\hat{\Omega}^\mu(u)|\omega,u,\ldots\rangle=\omega^\mu|\omega,u,\ldots\rangle. \tag{41}$$

As a result of measurement the observer $O_{u'}$ should obtain the value $\omega'=D(\Lambda,u)\omega$. Therefore, in the space of states $H_{u'}$ the observable Ω is represented by an operator $\hat{\Omega}^\mu(u')$, because

$$\begin{aligned}\hat{\Omega}(u')|\omega',u',\ldots\rangle&=U(\Lambda)D(\Lambda,u)\hat{\Omega}(u)|\omega,u,\ldots\rangle\\&=U(\Lambda)D(\Lambda,u)\omega(u)|\omega,u,\ldots\rangle\\&=\omega'(u')|\omega',u',\ldots\rangle,\end{aligned} \tag{42}$$

where we have used Eqs. (39) and (40). To conclude this section we provide the interpretation of the operator $\hat{\Omega}'(u')=D(\Lambda,u)\hat{\Omega}(u)$. We have

$$\begin{aligned}\hat{\Omega}'^\mu(u')|\omega,u,\ldots\rangle&=D^\mu_{\ \nu}(\Lambda,u)\hat{\Omega}^\nu(u)|\omega,u,\ldots\rangle\\&=D^\mu_{\ \nu}(\Lambda,u)\omega^\nu(u)|\omega,u,\ldots\rangle\\&=\omega'^\mu(u')|\omega,u,\ldots\rangle.\end{aligned} \tag{43}$$

Thus $\hat{\Omega}'(u')$ is an operator which, when acting on a vector describing the state of a physical system in the space of the observer O_u gives, the same result as seen by the observer $O_{u'}$ performing measurement on this system being in the same physical state.

B. Algebra of momenta and positions

In each space H_u one can introduce the Hermitian four-momentum operators $\hat{p}_\lambda(u)$ (generators of translations). These operators are interpreted as observables in the corresponding reference frame. In the CT synchronization we can go further and introduce Hermitian position operators $\hat{x}^\mu(u)$ in each space H_u. According to the Poisson bracket on the classical level (35) we postulate the following commutators between $\hat{x}^\mu(u)$ and $\hat{p}_\lambda(u)$:

$$[\hat{x}^\mu(u),\hat{p}_\lambda(u)]=i\left(\frac{u_\lambda\hat{p}^\mu(u)}{u\hat{p}(u)}-\delta^\mu_\lambda\right), \tag{44}$$

$$[\hat{p}_\mu(u),\hat{p}_\nu(u)]=0, \tag{45}$$

$$[\hat{x}^\mu(u),\hat{x}^\nu(u)]=0. \tag{46}$$

In particular,

$$[\hat{x}^0_{\rm CT}(u),\hat{p}_\lambda(u)]=0, \tag{47}$$

$$[\hat{x}^i(u),\hat{p}_j(u)]=-i\,\delta^i_j, \tag{48}$$

$$[\hat{x}^i(u),\hat{p}_0(u)]=i\,\frac{\hat{p}^i(u)}{\hat{p}^0(u)}. \tag{49}$$

We see that $\hat{x}^0_{\rm CT}$ commutes with all the observables. This allows us to interpret $\hat{x}^0_{\rm CT}$ as a parameter just like in the

[2] Of course, we should imagine an ensemble of identical copies of a physical system in the same prepared state.

standard nonrelativistic quantum mechanics. We have to stress here once again that the commutation relations (44)–(49) *are covariant* in the CT synchronization. This can be checked directly; one simply has to use Eqs. (8), (9), and (39) and to transform Eqs. (44), (45), and (46) to another reference frame. One can also check that

$$[\hat{x}^{\mu}(u),\hat{p}^{2}]=0, \tag{50}$$

which means that localized states have definite masses.

V. UNITARY ORBITS OF THE POINCARÉ GROUP FOR $k^2>0$ AND SPIN OPERATORS

According to our interpretation we deal with a bundle of Hilbert spaces H_u rather than with a single space of states. Therefore the transformations of the Lorentz group induce an orbit in this bundle. In this section we construct and classify unitary orbits of the Poincaré group in the bundle of Hilbert spaces. As we will see, the unitary orbits are parametrized by mass and spin, similarly as for the standard unitary representations of the Poincaré group.

A. Unitary orbits

As in the standard case we assume that the eigenvectors $|k,u,\ldots\rangle$ of the four-momentum operators

$$\hat{p}_{\mu}(u)|k,u,\ldots\rangle=k_{\mu}|k,u,\ldots\rangle, \tag{51}$$

with $k^2=m^2$, form a basis of the Hilbert space H_u. We adopt the following Lorentz-covariant normalization:

$$\langle k',u,\ldots|k,u,\ldots\rangle=2k^0\delta^3(\underline{\mathbf{k}}'-\underline{\mathbf{k}}), \tag{52}$$

where $\underline{k}$ denotes the space part of a covariant four-vector k_μ and $k^0=g^{0\mu}k_\mu$ is positive. The energy k_0 is the solution of the dispersion relation $k^2=m^2$ and is given by

$$k_0=\frac{1}{u^0}[-\mathbf{u}\underline{\mathbf{k}}+\sqrt{(\mathbf{u}\underline{\mathbf{k}})^2+(\underline{\mathbf{k}})^2+m^2}], \tag{53}$$

so

$$k^0\equiv\omega(\underline{\mathbf{k}})=u^0\sqrt{(\mathbf{u}\underline{\mathbf{k}})^2+(\underline{\mathbf{k}})^2+m^2}. \tag{54}$$

In the construction of the unitary irreducible orbits we use the operator $e^{-i\underline{\mathbf{q}}\hat{\mathbf{x}}(u)}$. Action of this operator on the basis states can be determined by using its unitarity, normalization of the basis vectors (52), and the commutation relations (44). Its final form is[3]

$$e^{-i\underline{\mathbf{q}}\hat{\mathbf{x}}(u)}|k,u,\ldots\rangle=e^{iq_0\hat{x}^0_{\mathrm{CT}}(u)}\sqrt{\frac{uk}{u(k+q)}}|k+q,u,\ldots\rangle, \tag{55}$$

where on the right-hand side of Eq. (55) q_0 is determined by $\underline{k}$, $\underline{q}$, and u,

$$q_0=\frac{1}{u^0}[-\mathbf{u}\underline{\mathbf{q}}-\sqrt{(\mathbf{u}\underline{\mathbf{k}})^2+(\underline{\mathbf{k}})^2+m^2}+\sqrt{(\mathbf{u}\underline{\mathbf{q}}+\mathbf{u}\underline{\mathbf{k}})^2+(\underline{\mathbf{k}}+\underline{\mathbf{q}})^2+m^2}]. \tag{56}$$

The basis vectors of the space H_u can be generated from a vector representing a particle at rest with respect to the preferred frame. First we act with $U(L_u)$ on such a vector; the resulting state has four-momentum mu_μ and belongs to H_u. Next, by means of the formula (55), we generate a vector in H_u with the four-momentum k_μ. Precisely

$$|k,u,\ldots\rangle=\sqrt{\frac{uk}{m}}e^{-i(k_\mu-mu_\mu)\hat{x}^{\mu}(u)}U(L_u)|\underset{\sim}{k},\tilde{u},\ldots\rangle, \tag{57}$$

where

$$\tilde{u}=(1,\mathbf{0}),\quad \underset{\sim}{k}=(m,\underline{\mathbf{0}}),\quad u=D(L_u,\tilde{u})\tilde{u}. \tag{58}$$

The orbit induced by the action of the operator $U(\Lambda)$ in the bundle of Hilbert spaces is fixed by the following covariant conditions: (i) $k^2=m^2$; (ii) $\varepsilon(k^0)=\mathrm{inv}$, for physical representations $k^0>0,\varepsilon(k^0)=1$. As a consequence there exists a positive defined, Lorentz-invariant measure

$$d\mu(k,m)=d^4k\,\theta(k^0)\,\delta(k^2-m^2). \tag{59}$$

Now, applying the Wigner method and using Eq. (39) one can easily determine the action of the operator $U(\Lambda)$ on a basis vector. We find

$$U(\Lambda)|k,u,m;s,\sigma\rangle=\mathcal{D}^{s}_{\sigma\lambda}{}^{-1}(R_{\Lambda,u})|k',u',m;s,\lambda\rangle, \tag{60}$$

where

$$u'=D(\Lambda,u)u=D(L_{u'},\tilde{u})\tilde{u}, \tag{61}$$

$$k'=D^{T-1}(\Lambda,u)k, \tag{62}$$

$$R_{\Lambda,u}=D(R_{\Lambda,u},\tilde{u})=D^{-1}(L_{u'},\tilde{u})D(\Lambda,u)D(L_u,\tilde{u})\subset SO(3), \tag{63}$$

and $\mathcal{D}^{s}_{\sigma\lambda}(R_{\Lambda,u})$ is the standard spin s rotation matrix $s=0,\frac{1}{2},1,\ldots$; $\sigma,\lambda=-s,-s+1,\ldots,s-1,s$. $D(R_{\Lambda,u},\tilde{u})$ is the Wigner rotation belonging to the little group of a vector $\tilde{u}$. Let us stress that in our approach, contrary to the standard one, representations of the Poincaré group are induced from the little group of a vector $\tilde{u}$, not $\underset{\sim}{k}$. Finally, the normalization (52) takes the form

$$\langle k,u,m;s,\lambda|k',u,m;s',\lambda'\rangle=2k^0\delta^3(\underline{\mathbf{k}}'-\underline{\mathbf{k}})\,\delta_{s's}\delta_{\lambda'\lambda}. \tag{64}$$

[3]There is a freedom in choosing the phase factor on the right-hand side of Eq. (55); it is determined here by the requirement that no phase factor on the right-hand side of Eq. (60) appears.

B. Spin

Now we describe in some detail the transformation properties of a second rank covariant tensor operator. These results are then used in the discussion of the spin.

Let $\hat{M}(u)=[\hat{M}_{\mu\nu}(u)]$ be a tensor operator. The transformation law for this tensor can be deduced from Eq. (39) and can be written in the matrix notation as

$$U(\Lambda)\hat{M}(u)U^{-1}(\Lambda)=D^T(\Lambda,u)\hat{M}(u')D(\Lambda,u). \quad (65)$$

The lower-triangular form of the matrix $D(\Lambda,u)$ [see Eqs. (8) and (9)] implies that the space part of $\hat{M}$ transforms into itself, namely,

$$U(\Lambda)\hat{M}_{ij}(u)U^{-1}(\Lambda)=\Omega_{ki}(\Lambda,u)\hat{M}_{kl}(u')\Omega_{lj}(\Lambda,u), \quad (66)$$

where $\Omega(\Lambda,u)$ denotes the space part of the matrix $D(\Lambda,u)$. It follows from the triangular form of $D(\Lambda,u)$ that

$$g_{ij}(u')=\Omega^{-1}_{ki}(\Lambda,u)g_{kl}(u)\Omega^{-1}_{lj}(\Lambda,u), \quad (67)$$

where g_{ij} are the space components of the covariant metric tensor $g_{\mu\nu}$. Therefore, one can easily show that the bilinear form

$$\hat{M}^2=\gamma_{ik}(u)\gamma_{jl}(u)\hat{M}_{ij}(u)\hat{M}_{kl}(u), \quad (68)$$

where

$$\gamma_{ij}(u)=[g_{ij}]^{-1}=-(\delta_{ij}+u^iu^j), \quad (69)$$

is a Poincaré-invariant operator. Let us introduce the spin operators $\hat{S}_{ij}(u)$ transforming covariantly according to Eq. (66), i.e.,

$$U(\Lambda)\hat{S}_{ij}(u)U^{-1}(\Lambda)=\Omega_{ki}(\Lambda,u)\hat{S}_{kl}(u')\Omega_{lj}(\Lambda,u), \quad (70)$$

and defined by the action on the basis vectors

$$\hat{S}_{ij}(u)|k,u,m;s,\lambda\rangle=-\mathcal{S}^s_{ij}(u)_{\lambda\sigma}|k,u,m;s,\sigma\rangle. \quad (71)$$

The application of Eq. (60) to Eqs. (70) and (71) leads to the following consistency condition:

$$\mathcal{D}^s(R_{\Lambda,u})\mathcal{S}^s_{ij}(u)\mathcal{D}^s(R^{-1}_{\Lambda,u})=\Omega_{ki}(\Lambda,u)\mathcal{S}^s_{kl}(u')\Omega_{lj}(\Lambda,u). \quad (72)$$

Therefore, using the fact that $R_{L_u,\tilde{u}}=I$ and

$$D(L_u,\tilde{u})=\begin{pmatrix} u^0 & 0 \\ \mathbf{u} & I+\dfrac{u^0}{1+u^0}\mathbf{u}\otimes\mathbf{u}^T\end{pmatrix} \quad (73)$$

[so $\Omega_{ij}(L_u,\tilde{u})=\delta_{ij}+(u^0/1+u^0)u^iu^j$], one obtains

$$\mathcal{S}^s_{ij}(u)=\tilde{\mathcal{S}}^s_{ij}+\frac{(u^0)^2}{1+u^0}(u^j\delta_{li}-u^i\delta_{lj})u^k\tilde{\mathcal{S}}^s_{kl}, \quad (74)$$

where $\tilde{\mathcal{S}}^s_{ij}:=\mathcal{S}^s_{ij}(\tilde{u})$ are assumed to be Hermitian matrix generators of the unitary representation $\mathcal{D}^s(R)$ of $SO(3)$, i.e.,

$$\tilde{\mathcal{S}}^s_{ij}=-\tilde{\mathcal{S}}^s_{ji}=(\tilde{\mathcal{S}}^s_{ij})^\dagger \quad (75)$$

and

$$[\tilde{\mathcal{S}}^s_{ij},\tilde{\mathcal{S}}^s_{kl}]=-i(\delta_{il}\tilde{\mathcal{S}}^s_{jk}+\delta_{jk}\tilde{\mathcal{S}}^s_{il}-\delta_{ik}\tilde{\mathcal{S}}^s_{jl}-\delta_{jl}\tilde{\mathcal{S}}^s_{ik}). \quad (76)$$

Consequently, in an arbitrary frame

$$\mathcal{S}^s_{ij}(u)=-\mathcal{S}^s_{ji}(u)=\mathcal{S}^{s\,\dagger}_{ij}(u) \quad (77)$$

and

$$[\mathcal{S}^s_{ij}(u),\mathcal{S}^s_{kl}(u)]=i[g_{il}(u)\mathcal{S}^s_{jk}(u)+g_{jk}(u)\mathcal{S}^s_{il}(u)-g_{ik}(u)\mathcal{S}^s_{jl}(u)-g_{jl}(u)\mathcal{S}^s_{ik}(u)]. \quad (78)$$

Therefore, the spin operators $\hat{S}_{ij}(u)=-\hat{S}_{ji}(u)$ are Hermitian and satisfy the same algebra,

$$[\hat{S}_{ij}(u),\hat{S}_{kl}(u)]=i[g_{il}(u)\hat{S}_{jk}(u)+g_{jk}(u)\hat{S}_{il}(u)-g_{ik}(u)\hat{S}_{jl}(u)-g_{jl}(u)\hat{S}_{ik}(u)]. \quad (79)$$

Now, according to Eq. (68) one can define the invariant spin square operator

$$\hat{S}^2=\frac{1}{2}\gamma_{ik}(u)\gamma_{jl}(u)\hat{S}_{ij}(u)\hat{S}_{kl}(u)=\frac{1}{2}\hat{S}_{ij}(u)\hat{S}_{ij}(u)+u^iu^j\hat{S}_{ik}(u)\hat{S}_{jk}(u). \quad (80)$$

Consequently

$$\hat{S}^2|k,u,m;s,\lambda\rangle=s(s+1)|k,u,m;s,\lambda\rangle. \quad (81)$$

Finally, as follows from Eqs. (71), (51), and (55), $\hat{S}_{ij}(u)$ commute with $\hat{p}_\mu(u)$ and $\hat{x}^\mu(u)$, i.e.,

$$[\hat{S}_{ij}(u),\hat{p}_\mu(u)]=[\hat{S}_{ij}(u),\hat{x}^\mu(u)]=0. \quad (82)$$

Summarizing, the covariant spin operator introduced here has the properties showing its advantage in comparison with the standard one. In particular, the algebra generated by $\hat{p}_\mu(u)$, $\hat{x}^\mu(u)$, and $\hat{S}_{ij}(u)$—Eqs. (44), (45), (46), (79), and (82)—is evidently covariant under the Poincaré group action.

VI. POSITION OPERATOR AND LOCALIZED STATES

In Sec. VI A we recall briefly the Newton-Wigner position operator. Section VI B is devoted to localized states and the derivation of a functional form of the position operator introduced in Sec. IV. Section VI C is devoted to description of localized states and position operator in the Hilbert space with a fully invariant measure, resembling the nonrelativistic one.

A. The Newton-Wigner operator

In the nonrelativistic quantum mechanics the situation is clear, and we can define the position operator which fulfills all the conditions stated in the Introduction (covariance is of course understood with respect to the Galilei group). Its construction and properties are very well known and we do not intend to describe them in this section. In the relativistic case the situation is much more complicated. One of the earliest definitions of the position operator is due to Newton and Wigner [24]. In this approach the authors try first to find states of the particle localized at a given point $(t,\mathbf{a})$ and then to write down the corresponding position operators. Let $S_{\mathbf{a}}$, the set of states $\psi_{\mathbf{a},0}$ localized at $\mathbf{a}\in\mathbb{R}^3$ at $t=0$, be the subset of the Hilbert space $\mathcal{H}$ of the unitary irreducible representation of the universal covering group of the Poincaré group. The Newton-Wigner postulates are as follows: (i) the set $S_{\mathbf{a}}$ is a linear subspace of $\mathcal{H}$; (ii) $S_{\mathbf{a}}$ is invariant under rotations around point $\mathbf{a}$, reflections in $\mathbf{a}$, and time inversions; (iii) $S_{\mathbf{a}}$ is orthogonal to all its space translates, i.e., under the space translations each $\psi_{\mathbf{a},0}\in S_{\mathbf{a}}$ transforms to a state from $\mathcal{H}$ which is orthogonal to all states from $S_{\mathbf{a}}$; (iv) certain regularity conditions. As an example, let us discuss shortly the Newton-Wigner position operator for a spinless particle. In this case $\mathcal{H}$ is a linear space of solutions to the Klein-Gordon equation with a positive energy. Using the Fourier transform one can obtain the states localized at $\mathbf{a}\in\mathbb{R}^3$ at $t=0$ in the momentum representation, namely

$$S_{\mathbf{a}}=\left\{\psi_{\mathbf{a},0}(k)=\frac{1}{(2\pi)^{3/2}}k_0^{1/2}e^{-i\vec{\mathbf{k}}\cdot\mathbf{a}}\right\}. \tag{83}$$

The corresponding position operators are given by

$$\hat{q}^k=-i\left(\frac{\partial}{\partial k_i}+\frac{1}{2}\frac{k^i}{(\mathbf{k})^2+m^2}\right). \tag{84}$$

The main results obtained by Newton and Wigner may be summarized as follows. For a massive particle with an arbitrary spin there exists a Hermitian position operator with commuting components, transforming like a vector under the rotations and satisfying the canonical commutation relations with the momentum operator, i.e., $[\hat{q}^k,\hat{k}_i]=-i\delta_i^k$. However, neither the position operator nor the localized states are covariant. Moreover, massless particles with spin are not localizable. Of course a lot of trials have been undertaken to remove all the unsatisfactory features of the Newton-Wigner approach, but to the best of the authors' knowledge, none of them has been fully successful. For a review see [17,18].

B. Localized states and momentum representation of the position operator

In this section we briefly describe some properties of the position operator introduced in Sec. IV B. First we find localized states in the Schrödinger picture. Equations (50) and (82) imply that $\hat{x}^\mu(u)$ commutes with $\hat{p}^2$ and $\hat{S}_{ij}(u)$ so these three operators all have common eigenvectors and consequently localized states have definite mass and spin. Let $|\boldsymbol{\xi},\tau,u,m;s,\lambda\rangle$ denote a state localized at the time τ in the space point $\boldsymbol{\xi}$,

$$\hat{\mathbf{x}}(u)|\boldsymbol{\xi},\tau,u,m;s,\lambda\rangle=\boldsymbol{\xi}|\boldsymbol{\xi},\tau,u,m;s,\lambda\rangle. \tag{85}$$

With the help of the invariant measure (59), the state $|\boldsymbol{\xi},\tau,u,m;s,\lambda\rangle$ can be expressed in terms of the basis vectors $|k,u,m;s,\lambda\rangle$,

$$|\boldsymbol{\xi},\tau,u,m;s,\lambda\rangle=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\underline{\mathbf{k}}}{2\omega(\underline{\mathbf{k}})}\sqrt{uk}e^{i\underline{\mathbf{k}}\boldsymbol{\xi}}|k,u,m;s,\lambda\rangle. \tag{86}$$

Now, after an arbitrary time t this state evolves to

$$|\boldsymbol{\xi},\tau,u,m;s,\lambda;t\rangle=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega(\underline{\mathbf{k}})}\sqrt{uk}e^{ik_\mu\xi^\mu}|k,u,m;s,\lambda\rangle \tag{87}$$

with $\xi^0=\tau-t$. One can easily check that these states are normalized as follows:

$$\langle\boldsymbol{\xi}',\tau,u,m;s',\lambda';t|\boldsymbol{\xi},\tau,u,m;s,\lambda;t\rangle=\frac{1}{2u^0}\delta^3(\boldsymbol{\xi}-\boldsymbol{\xi}')\delta_{ss'}\delta_{\lambda\lambda'}. \tag{88}$$

It is worthwhile to notice here that the states given by Eq. (86) are covariant in the CT synchronization, i.e., a state localized at the time $t=\tau$ for the observer O_u is localized at the time $t'=\tau'=D_0^0(\Lambda,u)\tau$ for the observer $O_{u'}$ too. Let us discuss a realization of the position operator in the momentum representation. Wave functions in momentum representation are defined in the standard way,

$$\psi_\lambda^{m,s}(k,u)=\langle k,u,m;s,\lambda|\psi\rangle, \tag{89}$$

or equivalently,

$$|\psi\rangle=\sum_\lambda\int\frac{d^3\underline{\mathbf{k}}}{2\omega(\underline{\mathbf{k}})}\psi_\lambda^{m,s}|k,u,m;s,\lambda\rangle. \tag{90}$$

The scalar product is given by

$$\langle\varphi|\psi\rangle=\sum_\lambda\int\frac{d^3\underline{\mathbf{k}}}{2\omega(\underline{\mathbf{k}})}\varphi_\lambda^{*m,s}(k,u)\psi_\lambda^{m,s}(k,u). \tag{91}$$

Now we can identify the wave functions related to the localized states (86); namely, we have

$$\chi_\lambda^{m,s}(\boldsymbol{\xi},\tau,k,u;\sigma;t)=\frac{1}{(2\pi)^{3/2}}\sqrt{uk}e^{ik_\mu\xi^\mu}\delta_{\sigma\lambda}. \tag{92}$$

It follows that in this realization

$$\hat{x}^i=-i\frac{\partial}{\partial k_i}+\frac{1}{2}i\left(\frac{u^i}{uk}-\frac{k^i}{(uk)^2}\right). \tag{93}$$

Evidently, for $\xi^0=0$ (i.e., for $t=\tau$) the functions χ are eigenvectors of $\hat{x}^i$. It can be easily demonstrated that in the preferred frame $[u=(1,\mathbf{0})]$ the function (92) reduces to the Newton-Wigner localized state (83); also the operator (93) coincides with the Newton-Wigner position operator (84) for a spinless particle.

C. Invariant measure

In the previous sections we used the Lorentz-invariant measure (59)

$$d\mu(k,m)=d^4k\ \theta(k^0)\,\delta(k^2-m^2).$$

We point out that this measure is not invariant under the action of the operator $e^{-i\underline{q}\hat{\mathbf{x}}(u)}$ [compare Eq. (55)]. Nevertheless, it is possible to find a measure which is both Poincaré invariant and invariant under the action of this operator. One can easily check that such a measure can be written as

$$d\overline{\mu}(k,m)=uk\,d\mu(k,m)=uk\,d^4k\ \delta(k^2-m^2)\,\theta(k^0). \tag{94}$$

The measure (94) simplifies some of the formulas discussed here. It resembles also the nonrelativistic one. Integrating $d\overline{\mu}(k,m)$ with respect to the k^0 we find

$$\int\ d\overline{\mu}(k,m)f(k)=\frac{1}{2u^0}\int\ d^3\underline{\mathbf{k}}\,f(\mathbf{k}_0,\underline{k}), \tag{95}$$

where k_0 is given by Eq. (53). Now the normalization (64) is not invariant under the action of the operator $e^{-i\underline{q}\hat{\mathbf{x}}(u)}$. To make it invariant we introduce rescaled basis vectors

$$|k,u,m;s,\lambda\rangle_{\rm inv}:=\frac{1}{\sqrt{uk}}|k,u,m;s,\lambda\rangle. \tag{96}$$

The rescaled vectors are normalized as follows:

$${}_{\rm inv}\langle k,u,m;s,\lambda|k',u,m;s',\lambda'\rangle_{\rm inv}=2u^0\delta^3(\underline{k}-\underline{k}')\,\delta_{ss'}\delta_{\lambda\lambda'}\,. \tag{97}$$

This normalization is invariant under the action of the operator $e^{-i\underline{q}\hat{\mathbf{x}}(u)}$ and it is simultaneously Lorentz invariant. Moreover,

$$e^{-i\underline{q}\hat{\mathbf{x}}(u)}|k,u,m;s,\lambda\rangle_{\rm inv}=e^{iq_0t}|k+q,u,m;s,\lambda\rangle_{\rm inv}, \tag{98}$$

where $q_0=q_0(\underline{k},\underline{q},u)$ is given by Eq. (56). The action of the operator $U(\Lambda)$ on the rescaled basis vectors has again the form (60), i.e.,

$$U(\Lambda)|k,u,m;s,\sigma\rangle_{\rm inv}=\mathcal{D}^{s}_{\sigma\lambda}{}^{-1}(R_{\Lambda,u})|k',u',m;s,\lambda\rangle_{\rm inv}. \tag{99}$$

Now let us return to the position operator and localized states. The localized states can be expressed in the new basis as follows:

$$|\boldsymbol{\xi},\tau,u,m;s,\lambda;t\rangle=\frac{1}{(2\pi)^{3/2}}\frac{1}{2u^0}\int\ d^3\underline{\mathbf{k}}\,e^{ik_\mu\xi^\mu}|k,u,m;s,\lambda\rangle_{\rm inv}, \tag{100}$$

where $\xi^0=\tau-t$. The corresponding wave functions localized at the time $t=\tau$ take the form

$$\tilde{\chi}^{m,s}_{\lambda}(\boldsymbol{\xi},\tau,k,u;\sigma;t)=\frac{1}{(2\pi)^{3/2}}e^{ik_\mu\xi^\mu}\delta_{\sigma\lambda}, \tag{101}$$

and the corresponding position operator takes the extremely simple form

$$\hat{x}^i=-i\frac{\partial}{\partial k_i} \tag{102}$$

as in the nonrelativistic case.

VII. CONCLUSIONS

Following the suggestions of some authors (Bell [1], Eberhard [2]) that a consistent formulation of quantum mechanics demands the existence of a preferred frame, we proposed here the Poincaré-covariant formulation of quantum mechanics with a built-in preferred frame. Our construction is based on the use of a nonstandard realization of the Poincaré group introduced in [9]. In this formulation the boost matrix has the lower-triangular form so the time coordinate rescales only under Lorentz transformations. Such a realization corresponds to a nonstandard synchronization of clocks (the CT synchronization), different from the standard coordinate time definition. Classically such a scheme is operationally indistinguishable from the standard one. Our construction shows that some notions such as localizability are simultaneously compatible with quantum mechanics and Poincaré covariance only if a privileged frame is distinguished. In this formulation of QM, causal problems connected with the violation of Bell's inequalities disappear: quantum theory remains nonlocal but it is causal in a strong sense. In this context we constructed and classified unitary orbits of the Poincaré group in the appropriate bundle of Hilbert spaces. The unitary orbits are parametrized by mass and spin, similarly to the standard unitary representations of the Poincaré group, although they are induced differently from SO(3). We introduced a Poincaré-covariant algebra of momentum and position operators satisfying all fundamental physical requirements. We proved that in our framework the position operator exists for particles with arbitrary spin. It fulfills all the requirements: it is Hermitian and covariant, it has commuting components, and moreover its eigenvectors (localized states) are also covariant. We found the explicit functional form of the position operator and demonstrated that in the preferred frame our operator coincides with the Newton-Wigner one. We also defined covariant spin operators and constructed an invariant spin squared operator. Moreover, full algebra of observables consisting of position, four-momentum, and spin operators is manifestly Poincaré-covariant in this framework. We hope that this formulation may be useful in the construction of the Poincaré-covariant version of the de Broglie–Bohm quantum mechanics as well.

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APPENDIX

Here we recall briefly some mathematical definitions. We follow [23], where one can find more detailed explanation (including examples).

A category $\mathcal{A}$ consists of a class of sets $A,B,C,\ldots,$ called the objects of $\mathcal{A}$ and a class of mappings between this sets Mor$(A,B),\ldots,$ called the morphisms of $\mathcal{A}$. If $f:A\to B$ and $g:B\to C$ is a pair of morphisms, it is always possible to construct the morphism $g\circ f:A\to C$. This composition is necessarily associative. It is further postulated that Mor(A,A) is not empty and always contains the identity morphism $\mathrm{id}_A:A\to A$.

A bundle is a triple (E,M,π) consisting of two manifolds E and M and a surjective map $\pi:E\to M$. The manifold E is called a total space, M is called a base space, and π is called a projection.

The simplest example is the Cartesian product bundle $(F\times M,M,\pi)$ of two manifolds F and M with $\pi=pr_2$ (canonical projection on the second factor of the Cartesian product) defined by $\pi(x,y)=pr_2(x,y)=y$ for all $x\in F,\ y\in M$.

In general, the inverse images $\pi^{-1}(x)$ of points $x\in M$ need not be isomorphic. If they are, one speaks of a fiber bundle. More precisely, a fiber bundle is a locally trivial bundle, i.e., there exists a manifold F, called the typical fiber, such that for each $x\in M$ there exists a neighborhood U of x such that the subbundle $[\pi^{-1}(U),U,\pi|_{\pi^{-1}(U)}]$ is isomorphic to the product bundle $(F\times U,U,pr_2)$.

An example of a fiber bundle is the bundle of linear frames. Let M be an n-dimensional manifold which is at the same time a vector space and let $P(M)$ be the set of all vector frames at all points of M. Let (e_i) be a basis at $x\in M$ and (r_i) a second basis at x. Then $r_i=e_j a_i^j$ with $(a_i^j)=a\in GL(n,\mathbb{R})$ (the general linear group in n dimensions). Thus there is an isomorphism $r\mapsto a$ and $GL(n,\mathbb{R})$ is the typical fiber. So we have constructed the fiber bundle $[P(M),M,\pi]$ with the typical fiber $GL(n,\mathbb{R})$. The projection π is the natural one which maps a basis at a point x to the point x. The fiber bundle $[P(M),M,\pi]$ is called a bundle of linear frames (or a frame bundle).

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