Reconstruction of a wave function from the *Q* function using a phase-retrieval method in quantum-state measurements of light

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A method of reconstructing a wave function from part of the *Q* function in a pure state of light is proposed. This method involves the use of a noniterative phase-retrieval method based on Gaussian filtering, which allows one to determine a convolution of the wave function with a known Gaussian function for the vacuum state from measured data of the *Q* function along only two parallel lines in phase space. By using a deconvolution process, the wave function is reconstructed from the convolution, provided that the resolution of the reconstructed wave function is limited by the extent of the Fourier transform of the Gaussian function. Numerical simulations demonstrate the applicability of this method to the reconstruction of the wave function from the *Q* function obtained from the photoelectric counting statistics in unbalanced homodyne detection. [S1050-2947(99)01706-0]

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I. INTRODUCTION

There has been a good deal of interest recently in quantum-state reconstruction. Quantum states can be fully characterized by the Wigner function defined in phase space. Vogel and Risken [1] theoretically pointed out that quadrature amplitude distributions measured in homodyne detection provide enough data to perform a complete reconstruction of the Wigner function. This method is called optical homodyne tomography. The pioneering experimental demonstration of this method was done by Smithey *et al.* [2,3]. The idea of quantum tomography has been applied to the quantum-state reconstruction of molecular vibrations [4]. Alternatively to the tomographic methods, more complicated (homodyne detection) schemes such as eight-port [5] and six-port [6] homodyne detection techniques were used for determining the quantum state of light in terms of the *Q* function.

Recently, a more simple measurement scheme has been proposed [7,8], in which various quasiprobability functions, including the Wigner and the *Q* functions, can be determined by unbalanced homodyne detection of the number statistics of the quantum state of interest, after introducing appropriate coherent displacements. This scheme is based upon an idea [9] that the complete Wigner function can be scanned by shifting the system or equivalently the frame of reference in the phase space. In this scheme, we can reconstruct a quasiprobability function in each point of the phase space independently, whereas in the tomographic and the multiport homodyning techniques the chosen grid of measured data essentially determines the quality of reconstruction. A method of this type has recently been used to reconstruct the motional state of a trapped atom [10].

In the measurement of quantum states, there is an interesting point that if the quantum state being measured is known *a priori* to be in a pure state, it may be unnecessary to obtain the Wigner function in a whole phase space in order to reconstruct the pure density matrix (wave function). In the tomographic method, Smithey *et al.* [3] pointed out that the wave function of a pure state may be determined from only two quadrature amplitude distributions measured in homodyne detection by using the iterative phase-retrieval algorithm (the Gerchberg-Saxton algorithm) [11] that was developed in the area of image reconstruction. This fact implies that there exists a mathematically analogous description between the quantum-state reconstruction and the image reconstruction using phase-retrieval methods. The determination of the phase of a complex function from its moduli (i.e., amplitude information) is referred to as a phase-retrieval problem. The study of the phase-retrieval problem has been actively done in the area of image reconstruction for about 20 years. Several methods [12–15] have been developed to solve this problem. Iterative phase-retrieval algorithms are widely used. The use of iterative phase-retrieval algorithms, however, is accompanied by convergence problems, and hence the algorithms sometimes stagnate in a local minimum solution different from a true one. In particular, Huiser *et al.* [16] pointed out that in the Gerchberg-Saxton algorithm there is a possibility of the solution converging to an incorrect nonanalytic solution for one-dimensional cases. Van Toorn and Ferwerda [17] also verified this fact in a computer simulation. On the other hand, an analytic (noniterative) phase-retrieval method by use of Gaussian filtering has been proposed recently [18]. This method is based on the mathematical properties of analytic functions, and ensures the uniqueness of the solution [19].

In this paper, a method for reconstructing a wave function from part of the *Q* function in a pure state by using the analytic phase-retrieval method is proposed. It is well known that the *Q* function can be expressed as a convolution of the Wigner function with the function that corresponds to the Wigner function of the vacuum state. As we shall see, such a convolution can be regarded as the square modulus of a convolution of the wave function in the pure state with a known Gaussian function for the vacuum state. Then, by applying the phase-retrieval method to the data of the square modulus along two parallel lines in phase space, the convolution of the wave function with the Gaussian function can be determined. The wave function can be reconstructed by the deconvolution of the known Gaussian function from the convolution, provided that the resolution of the reconstructed wave function is limited by the extent of the Fourier transform of the Gaussian function. In order to apply the present reconstruction method to the Q function in a pure state, the scheme of unbalanced homodyne detection [7,8] is the most efficient for the measurement of the Q function, because the Q function in each point of the phase space can be obtained independently by using that scheme. By using the present method with that detection scheme, the procedure of the measurement with the control of a local oscillator field can be reduced extremely, because the present method does not require the measurement of the whole Q function in phase space in contrast with the conventional ways of reconstruction.

This paper is organized as follows. In Sec. II, the reconstruction method of a wave function from its Q function is formulated. In Sec. III, the reconstruction method is tested by computer simulation of the measurement of odd coherent states with the unbalanced homodyne detection. Concluding remarks are given in Sec. IV.

II. FORMULATION OF RECONSTRUCTING A WAVE FUNCTION BY PHASE RETRIEVAL

It is well known that the Q function of a quantum state is defined by the diagonal matrix elements of a density operator $\hat{\rho}$ in a pure coherent state $|\alpha\rangle$:

$$Q(\alpha)=\frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle. \quad (1)$$

The Q function for a pure state $|\psi\rangle$ of a single-mode field is given by

$$Q(\alpha)=\frac{1}{\pi}|\langle\alpha|\psi\rangle|^2. \quad (2)$$

In the x representation, Eq. (2) can be written as [20]

$$Q(\alpha)=\frac{1}{\pi}\left|\int \psi(x)\exp\left[-\frac{1}{2}(x-\sqrt{2}\xi)^2-i\sqrt{2}\,\eta x\right]dx\right|^2, \quad (3)$$

where $\psi(x)=\langle x|\psi\rangle$, and we have set

$$\alpha=\xi+i\eta. \quad (4)$$

Combining the imaginary term with the quadratic term in the exponent of Eq. (3), we obtain

$$Q(\alpha)=\frac{1}{\pi}|\exp(-\eta^2-2i\eta\xi)F(\xi-i\eta)|^2, \quad (5)$$

where

$$F(\xi-i\eta)=\int \psi(x)\exp\left\{-\frac{1}{2}[x-\sqrt{2}(\xi-i\eta)]^2\right\}dx. \quad (6)$$

Equation (6) shows that the function $F(\xi-i\eta)$ corresponds to a convolution integral of the wave function with the Gaussian function for the complex variable $\xi-i\eta$. Thus we consider the properties of the Q function along two lines described by the equations $\eta=0$ and $\eta=c$, respectively, in the (ξ,η) plane, where c is assumed to be a constant. These Q functions can be written from Eq. (5) as

$$Q(\xi)=\frac{1}{\pi}|F(\xi)|^2 \quad (7)$$

and

$$Q(\xi+ic)=\frac{1}{\pi}\exp(-2c^2)|F(\xi-ic)|^2. \quad (8)$$

If the complex function $F(\xi)$ is reconstructed from data of the Q function along these lines in Eqs. (7) and (8), the wave function $\psi(x)$ can be obtained by eliminating the known Gaussian function (i.e., the wave function for the vacuum state) from the complex function $F(\xi)$. In order to reconstruct the complex function $F(\xi)$, we have to solve the problem that the phase of $F(\xi)$ is retrieved from the information of the moduli $|F(\xi)|$ and $|F(\xi-ic)|$. This problem can be solved by the analytic phase-retrieval method [18]. In this method the estimation of the phase is based on use of the logarithmic Hilbert transform [21] and a Fourier series expansion [19,22] as shown in the following way.

Let $F(\xi)$ be written as

$$F(\xi)=|F(\xi)|\exp[i\phi(\xi)], \quad (9)$$

where $|F(\xi)|$ and $\phi(\xi)$ are the modulus and the phase of $F(\xi)$, respectively. If we assume that the function $F(\xi)$ is given by the Fourier transform of a complex function with a finite extent, then the function $F(\alpha)$ becomes an entire function from a theorem formulated by Paley and Wiener [23]. This assumption is appropriate for the case in Eq. (6) because the inverse Fourier transform of Eq. (6) for $\eta=0$ (i.e., the product of the inverse Fourier transforms of the wave function and the Gaussian function) can be regarded as approximately band limited in practice. The entire function is analytic in the whole finite complex plane with the remarkable properties. One of them is the fact that the real and imaginary parts of $F(\xi)$ are related by the well-known Hilbert transforms or dispersion relations [24],

$$\mathrm{Re}\,F(\xi)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,F(\xi')}{\xi-\xi'}d\xi', \quad (10)$$

$$\mathrm{Im}\,F(\xi)=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,F(\xi')}{\xi-\xi'}d\xi', \quad (11)$$

where Re and Im indicate taking the real and imaginary parts, respectively, and P denotes that the Cauchy principal value is to be taken. These relationships can be obtained from the calculation of a contour integral in the complex lower half-plane. If either the real or imaginary part of $F(\xi)$ is obtained, the function $F(\xi)$ can be calculated from the relation of Eq. (10) or (11). In actual situations, however, only the modulus of $F(\xi)$ is directly obtained from the measurement of the Q function. Therefore, the relationship between the modulus and the phase of $F(\xi)$ is more desirable than that between the real and imaginary parts of $F(\xi)$. For this purpose, $F(\xi)$ is modified by taking its logarithm as follows:

$$\ln F(\xi)=\ln|F(\xi)|+i\phi(\xi). \qquad (12)$$

The Hilbert transform relationship between the real and imaginary parts of $\ln F(\xi)$ can be obtained from the calculation of a contour integral in the complex lower half plane by

$$\phi(\xi)=-\frac{\xi}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\ln|F(\xi')|}{\xi'(\xi-\xi')}d\xi'+\phi(0), \qquad (13)$$

where $\phi(0)$ is the constant phase at $\xi=0$. Equation (13) is called the logarithmic Hilbert transform for the function $\ln F(\xi)$, which was formulated by Burge *et al.* [21]. Since $\ln F(\alpha)$ has the same region of analyticity as $F(\alpha)$ except at the points where $F(\alpha)=0$, the relation of Eq. (13) can be established only in the case that $\ln F(\alpha)$ does not have any singularities in the complex lower half-plane. Unfortunately, the actual situation is not so simple, because many functions generally have zeros in the complex lower half-plane. Consequently, Eq. (13) cannot always be used to calculate the phase $\phi(\xi)$ from the modulus of $F(\xi)$, and the logarithmic Hilbert transform should be considered by taking into account the influence of zeros in the complex lower half-plane on the derivation process of the actual phase.

In consideration of this point, we now introduce the Hilbert function given by

$$F_h(\xi)=|F(\xi)|\exp[i\phi_h(\xi)], \qquad (14)$$

where $\phi_h(\xi)$ is the Hilbert phase:

$$\phi_h(\xi)=-\frac{\xi}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\ln|F(\xi')|}{\xi'(\xi-\xi')}d\xi'+\phi(0). \qquad (15)$$

In other words, the Hilbert function corresponds to a function whose all zeros in the complex lower half-plane are reflected onto the upper half-plane. It is well known that an entire function may be described everywhere by its zeros with the expression being known as a Hadamard product [24]:

$$F(\alpha)=\alpha^q B\prod_{j=1}^{\infty}\left(1-\frac{\alpha}{z_j}\right), \qquad (16)$$

where q is of the order of zero at the origin of the complex plane, B is a scaling constant, and z_j is the vector notation of the jth zero in the complex plane [i.e., $F(z_j)=0$]. Using the Hadamard product, we may represent the relation between the Hilbert function $F_h(\xi)$ with zeros only in the complex upper half-plane and the actual complex function $F(\xi)$ with zeros in both upper and lower planes as

$$F_h(\xi)=F(\xi)\prod_{j=1}^{M}\frac{\left(1-\frac{\xi}{z_j^*}\right)}{\left(1-\frac{\xi}{z_j}\right)}, \qquad (17)$$

where M is the number of zeros in the complex lower half-plane, and the asterisk denotes the complex conjugate.

Substitution of Eqs. (9) and (14) into Eq. (17) yields

$$|F(\xi)|\exp[i\phi_h(\xi)] =|F(\xi)|\exp[i\phi(\xi)] \times\exp\left\{-2i\sum_{j=1}^{M}[\arg(z_j-\xi)-\arg(z_j)]\right\}, \qquad (18)$$

where the modulus of the product term in Eq. (17) is unity and the symbol arg denotes the argument of the complex function $(z_j-\xi)$. The phase terms in Eq. (18) are given by

$$\phi_h(\xi)=\phi(\xi)-2\sum_{j=1}^{M}[\arg(z_j-\xi)-\arg(z_j)]. \qquad (19)$$

Since the Hilbert phase $\phi_h(\xi)$ is calculated by using Eq. (15) from the modulus of $F(\xi)$, the general logarithmic Hilbert transform involving the influence of zeros of $F(\xi)$ in the complex lower half-plane can be finally obtained from Eqs. (15) and (19) as

$$\phi(\xi)=-\frac{\xi}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\ln|F(\xi')|}{\xi'(\xi-\xi')}d\xi' +2\sum_{j=1}^{M}[\arg(z_j-\xi)-\arg(z_j)]+\phi(0). \qquad (20)$$

The first term on the right-hand side of Eq. (20) corresponding to the Hilbert phase implies the fundamental minimum condition of the phase. The second term in Eq. (20) supplements the information corresponding to the effect of the zeros of $F(\alpha)$ in the complex lower half-plane, which does not appear in the modulus $|F(\xi)|$ and is only contained in the phase $\phi(\xi)$. The rest term represents the constant phase, which does not appear in the positions of zeros of $F(\alpha)$ and the modulus $|F(\xi)|$. The ambiguity concerned with the constant phase is situated outside the phase retrieval from absolute magnitude distributions (i.e., moduli) and will be regarded here as an unimportant component. The phase $\phi(\xi)$ is evaluated from Eq. (20) except for a constant phase. Unfortunately, the zeros in the complex lower half-plane cannot be determined from only the modulus $|F(\xi)|$. However, the influence of zeros in the complex lower half-plane can be taken into account from two moduli $|F(\xi)|$ and $|F(\xi-ic)|$ in the following procedure.

Equation (20) is rewritten as

$$\phi(\xi)=\phi_h(\xi)+\phi_z(\xi), \qquad (21)$$

where $\phi_h(\xi)$ is the Hilbert phase and $\phi_z(\xi)$ is the phase with the influence of the zeros in the complex lower half-plane:

$$\phi_z(\xi)=2\sum_{j=1}^{N}[\arg(z_j-\xi)-\arg(z_j)]. \qquad (22)$$

Substitution of Eq. (21) into Eq. (9) gives

$$F(\xi)=F_h(\xi)\exp[i\phi_z(\xi)], \qquad (23)$$

where $F_h(\xi)$ is the Hilbert function given by Eq. (14). When the real variable ξ of $F(\xi)$ is expanded into the complex one, $\xi-ic$, Eq. (23) becomes

$$F(\xi-ic)=F_h(\xi-ic)\exp[i\phi_z(\xi-ic)]. \tag{24}$$

Thus the modulus of the function $F(\xi-ic)$ is given by

$$|F(\xi-ic)|=|F_h(\xi-ic)|\exp[-\mathrm{Im}\,\phi_z(\xi-ic)], \tag{25}$$

where Im denotes the imaginary part of a complex function. If the values of the moduli $|F(\xi-ic)|$ and $|F_h(\xi-ic)|$ are not zero, Eq. (25) can be rewritten as

$$\ln\frac{|F(\xi-ic)|}{|F_h(\xi-ic)|}=-\mathrm{Im}\,\phi_z(\xi-ic). \tag{26}$$

On the left-hand side of Eq. (26) the function $|F(\xi-ic)|$ can be derived from the Q function of Eq. (8) and the function $|F_h(\xi-ic)|$ is related to the Hilbert function $F_h(\xi)$ by the relationship

$$F_h(\xi-ic)=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}F_h(\xi')\exp(2\pi iu\xi')d\xi'\right]\times\exp(-2\pi cu)\exp(-2\pi i\xi u)du; \tag{27}$$

that is, Eq. (27) indicates that the function $F_h(\xi-ic)$ is the Fourier transform of the product of the inverse Fourier-transformed function of the Hilbert function $F_h(\xi)$ and the exponential function $\exp(-2\pi cu)$. The Hilbert function can be calculated from the Q function in Eq. (7) by using Eqs. (14) and (15).

Next we consider a method of computing the phase function $\phi_z(\xi)$ from Eq. (26). One approach to retrieving the phase is to represent $\phi_z(\xi)$ in terms of an appropriate basis function, e.g., a Fourier-series basis [19,22],

$$\phi_z(\xi)\cong\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi}{l}\xi+b_n\sin\frac{n\pi}{l}\xi\right), \tag{28}$$

where the observational region of the Q function is designated $-l<\xi<l$, and N is sufficiently large to enable the phase distribution to be reconstructed. Thus the unknown function $\phi_z(\xi)$ is represented by the unknown coefficients a_n and b_n $(n=1,\ldots,N)$. Substituting Eq. (28) into Eq. (26) and evaluating the imaginary part of $\phi_z(\xi-ic)$, we obtain

$$D(\xi)\cong\sum_{n=1}^{N}\left(-a_n\sin\frac{n\pi}{l}\xi+b_n\cos\frac{n\pi}{l}\xi\right)\sinh\left(\frac{n\pi}{l}c\right), \tag{29}$$

where $D(\xi)=\ln[|F(\xi-ic)|/|F_h(\xi-ic)|]$ is a known function. By calculating $D(\xi)$ at $2N$ values of ξ we obtain $2N$ simultaneous equations from which the unknown coefficients a_n and b_n $(n=1,\ldots,N)$ can be determined. The phase $\phi_z(\xi)$ with the influence of zeros in the complex lower half-plane is derived by substituting the results of the solution $(a_n,b_n,\ n=1,\ldots,N)$ into Eq. (28). Consequently, the phase $\phi(\xi)$ of the function $F(\xi)$ can be obtained by adding the phase $\phi_z(\xi)$ to the Hilbert phase $\phi_h(\xi)$. Note that, even if the relationship of Eq. (26) breaks down when the modulus $|F(\xi-ic)|$ and/or $|F_h(\xi-ic)|$ have zeros, the unknown coefficients a_n and b_n can be determined from the values of the modulus at the points except at the zeros. Finally, we reconstruct the wave function by eliminating the effect of the known Gaussian function from the function $F(\xi)$ consisting of the retrieved phase and the modulus that is derived from the Q function in Eq. (7). Because contamination of data is inevitable in practice, here we use a Wiener filter [25] for suppressing the amplification of noise due to the deconvolution. The inverse Fourier transform $f(u)$ of $F(\xi)$ is seen from Eq. (6) to be

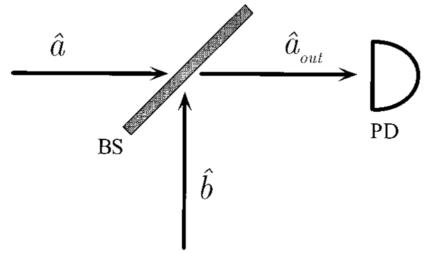

FIG. 1. Unbalanced homodyne detection scheme for the measurement of the Q function of light; BS denotes the beam splitter, PD is the photodetector, and the annihilation operators of the modes are indicated.

$$f(u)=\int F(\xi)\exp(2\pi iu\xi)d\xi,$$
$$=\Psi(u/\sqrt{2})g(u), \tag{30}$$

where the functions $\Psi(u)$ and $g(u)$ indicate the inverse Fourier transforms of the wave function $\psi(\xi)$ and the Gaussian function $\exp(-\xi^2)$, respectively. Using a Wiener filter, we can obtain

$$\Psi'(u/\sqrt{2})=\frac{f(u)g(u)}{g^2(u)+\epsilon}, \tag{31}$$

where ϵ is some small constant. Although ϵ should be a function u, experience with conventional deconvolution suggests that a constant term is usually sufficient. Then we can obtain an estimate of the wave function by Fourier transforming the result of Eq. (31).

III. COMPUTER SIMULATION

The present reconstruction method has been tested by computer simulation of the reconstruction of a wave function from part of the Q function in a pure state. There are some schemes for measuring quasiprobability distributions of a light field, such as the Q function. The unbalanced homodyne detection scheme [7,8] is the most suitable for the present reconstruction method, because this scheme allows one to measure the value of a quasiprobability distribution in each point of the phase space independently. Thus, by applying the present method to the data measured with the unbalanced homodyne detection scheme, the wave function of a light field in a pure state can be reconstructed from part of the Q function of the field without measuring the whole of the Q function. We consider the unbalanced homodyne detection scheme shown in Fig. 1, according to the treatment given by Wallentowitz and Vogel [7]. The detected field is a superposition of the signal and the local oscillator fields.

Such a combination is easily realized by means of a beam splitter. The superimposed light can be described by the beam splitter transformation

$$\hat{a}_{\rm out} = T\hat{a} + R\hat{b}; \tag{32}$$

$\hat{a}$, $\hat{b}$, and $\hat{a}_{\rm out}$ are photon annihilation operators of the signal field, the local oscillator field, and the superimposed field, respectively. T and R are the complex amplitude transmission and reflection coefficients of the beam splitter, respectively, which are assumed to have the relationships

$$|T|^2 + |R|^2 = 1, \tag{33}$$

$$\arg(T) - \arg(R) = \pm\frac{\pi}{2}. \tag{34}$$

If the local oscillator field is prepared in a coherent state $|\beta\rangle$, $\hat{b}|\beta\rangle = \beta|\beta\rangle$, the probability p_n of recording n counts with a photodetector of quantum efficiency ζ is written as

$$p_n(\alpha;\mu) = \left\langle :\frac{[\mu\hat{N}(\alpha)]^n}{n!}\exp[-\mu\hat{N}(\alpha)]:\right\rangle, \tag{35}$$

where the notation $::$ indicates normal ordering, $\alpha = -R\beta/T$, μ is the overall quantum efficiency

$$\mu = \zeta|T|^2, \tag{36}$$

and $\hat{N}(\alpha)$ is the displaced (signal-field) number operator

$$\hat{N}(\alpha) = \hat{D}(\alpha)\hat{a}^{\dagger}\hat{a}\hat{D}(\alpha)^{\dagger}, \tag{37}$$

in which $\hat{D}(\alpha)$ is the coherent displacement operator. With the homodyne counting distributions $p_n(\alpha;\mu)$, the s-parametrized quasiprobability distributions $P(\alpha;s)$ for the quantum state of the signal field can be represented [7,8] by

$$P(\alpha;s) = \frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}\left[-\frac{2-\mu(1-s)}{\mu(1-s)}\right]^n p_n(\alpha;\mu), \tag{38}$$

where s denotes the parameter of quasiprobability distributions with $s<1$, including the Wigner function ($s=0$) and the Q function ($s=-1$). This equation indicates that the quasiprobability $P(\alpha;s)$ is evaluated as a weighted sum over the counting distributions $p_n(\alpha;\mu)$. When the s values fulfill the condition $s \leqslant 1-1/\mu$, the weighting factors improve the convergence of the series in Eq. (38). Hence it is found that the Q function can be obtained from the full photoelectron statistics measured by means of a realistic photodetector with overall quantum efficiency $0.5 \leqslant \mu < 1$.

We demonstrate the present method for an odd coherent state,

$$|\alpha_-\rangle = A(|\alpha_0\rangle - |-\alpha_0\rangle), \tag{39}$$

where $|\alpha_0\rangle$ is a coherent state and A is a normalization constant $\{2[1-\exp(-2|\alpha_0|^2)]\}^{-1/2}$. The wave function of the odd coherent state is written in the x representation as

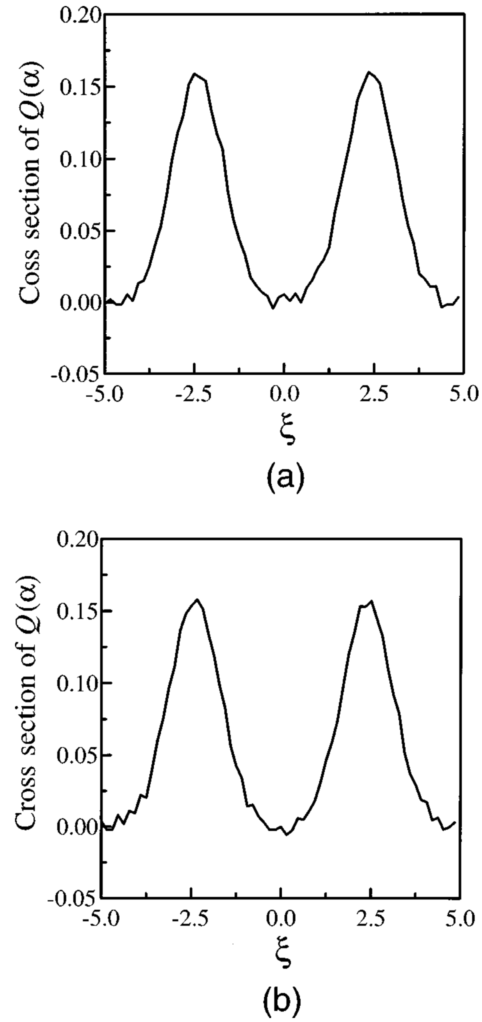

FIG. 2. Cross-sectional profiles of the Q function evaluated from the data of the unbalanced homodyne detection for an odd coherent state with $\alpha_0 = 2.4$ in the simulation with 8×10^3 events for each point of 64 sampling points. (a) and (b) are the cross-sectional profiles of the Q function along lines described by the equations $\eta=0$ and $\eta=10/64$, respectively, in the (ξ, η) plane.

$$\psi(x) = \frac{A}{\sqrt[4]{\pi}}\left\{\exp\left[-\frac{1}{2}(x-\sqrt{2}\xi_0)^2 + i\sqrt{2}\eta_0 x\right] - \exp\left[-\frac{1}{2}(x+\sqrt{2}\xi_0)^2 - i\sqrt{2}\eta_0 x\right]\right\}, \tag{40}$$

where $\alpha_0 = \xi_0 + i\eta_0$. By evaluating Eq. (35) with Eq. (39) we can obtain the probability $p_n(\alpha;\mu)$ for the odd coherent state in the unbalanced homodyne scheme of Fig. 1. In the present simulation, the photon-counting distributions detected with an efficiency of $\mu = 0.5$ were obtained by Monte Carlo calculations. Using Eq. (38), the Q function $P(\alpha;-1)$ was evaluated from the simulated photon-counting distributions $p_n(\alpha;0.5)$. Figures 2(a) and 2(b) show the cross-sectional profiles of the Q function $P(\alpha;-1)$ along lines described by the equations $\eta=0$ and $\eta=10/64$, respectively, in the (ξ,η) plane for an odd coherent state with $\alpha_0 = 2.4$, where the photon-counting statistics were simulated with a sample of 8×10^3 events for each of 64 sampling points in the extent $(-5 \leqslant \xi \leqslant 5)$ of the ξ coordinate. The absolute value of η in Fig. 2(b) was set to be the unit length of

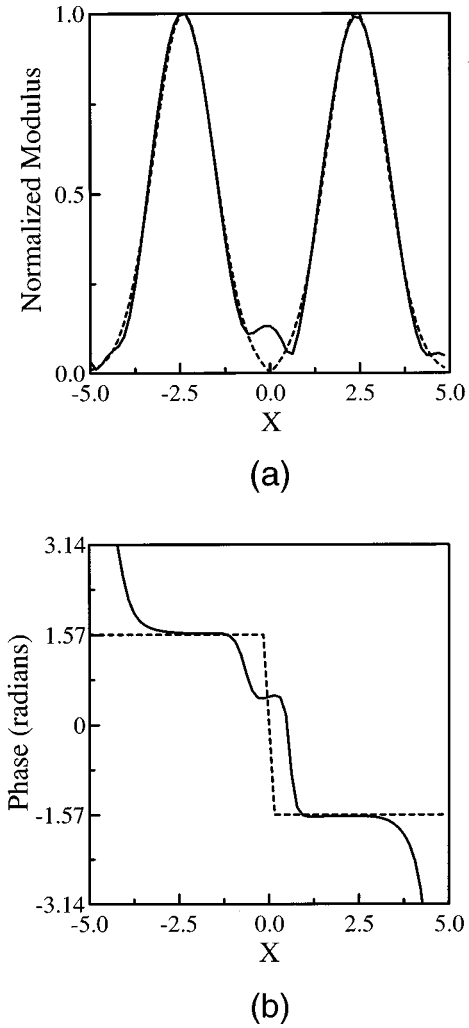

FIG. 3. Reconstruction of the wave function from the data of the Q function shown in Fig. 2: (a) normalized moduli and (b) phases of the reconstructed wave function (solid curves) and the true wave function (dotted curves).

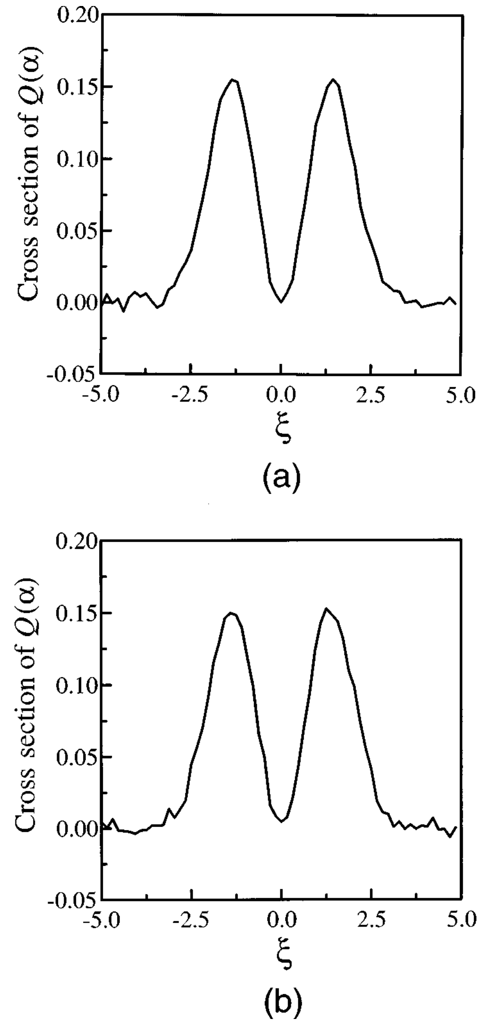

FIG. 4. Same as in Fig. 2 except that the state of light was an odd coherent state with $\alpha_0=1.3$.

the sampling points. Although there is not such a large difference between the absolute magnitude values of data in Figs. 2(a) and 2(b), the data in Figs. 2(a) and 2(b) have enough difference between these distributions for the extraction of phase information by the present phase-retrieval method. The solid curves in Figs. 3(a) and 3(b) show the modulus and the phase, respectively, of the reconstructed wave function from the data of the Q function in Figs. 2(a) and 2(b) by using the method described in Sec. II. Note that the principle value integral over the logarithm of the measured modulus in Eq. (20) [i.e., the convolution integral of the function $\ln|F(\xi)|/\xi$ with the function $1/\xi$] can easily be evaluated by taking a numerical inverse Fourier transform of the product of two Fourier transforms of the function $\ln|F(\xi)|/\xi$ and the function $1/\xi$. Then the accuracy in calculating the convolution integral can be increased [26] by using the analytic result for the Fourier transform of $1/\xi$ (i.e., the signum function with the coefficient $-\pi i$) and by calculating the Fourier transform of $\ln|F(\xi)|/\xi$ via the convolution of the signum function with the numerical Fourier transform of $\ln|F(\xi)|$. The reconstructed wave function in Fig. 3 corresponds to the function $\psi(\sqrt{2}x)$ that is linearly scaled as a result of the deconvolution of the known Gaussian function from the complex function consisting of the modulus [i.e., the square root of the Q function shown in Fig. 2(a)] and the retrieved phase. For comparison, the modulus and the phase of the true wave function scaled with $\sqrt{2}x$ in Eq. (40) are shown by the dotted curves in Figs. 3(a) and 3(b), respectively. Note that the constant phase of a wave function is the inevitable ambiguity in the reconstruction from the absolute magnitude data by using the phase-retrieval method.

Figures 4 and 5 show the example of reconstruction of the wave function for an odd coherent state with $\alpha_0=1.3$. Figures 4(a) and 4(b) show the cross-sectional profiles of the Q function along lines described by the equations $\eta=0$ and 10/64, respectively, in the (ξ,η) plane, where the photon-counting statistics were simulated with the same conditions as those in Fig. 2. The solid curves in Figs. 5(a) and 5(b) show the modulus and the phase, respectively, of the reconstructed wave function from the data of the Q function in Figs. 4(a) and 4(b). The dotted curves in Fig. 5 mean the true wave function that is scaled with $\sqrt{2}x$. It is found from Figs. 3 and 5 that the π phase difference between two Gaussian functions of the wave function with a real value for α_0 in Eq. (40) is almost faithfully retrieved. Note that the order of the sampling events (8×10^3) in the present simulation is com-

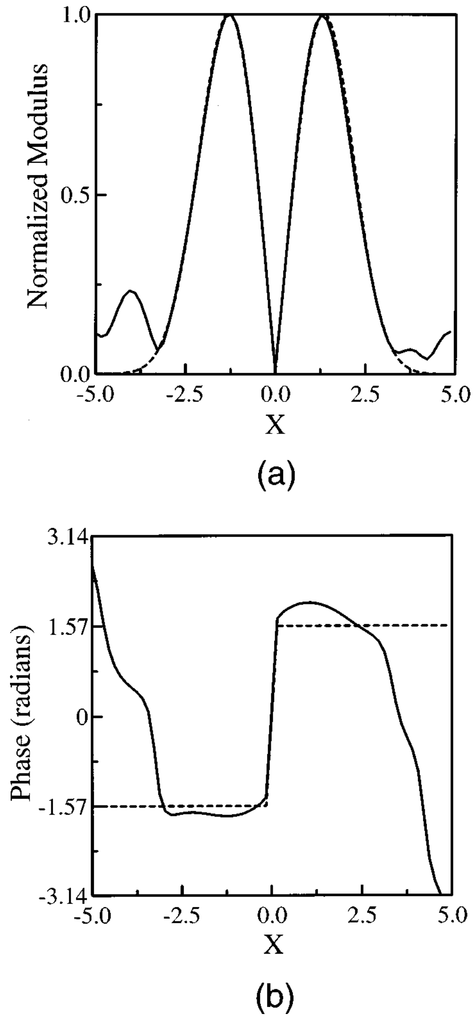

FIG. 5. Reconstruction of the wave function from the data of the Q function shown in Fig. 4: (a) normalized moduli and (b) phases of the reconstructed wave function (solid curves) and the true wave function (dotted curves).

parable to that of the simulation shown by Wallentowitz and Vogel [7], in which the samples of 10^3 and 5×10^3 events were used for simulating the reconstruction of quasiprobability functions from the photon counting in the same unbalanced homodyne detection scheme as utilized in this section. Therefore, the results of the reconstruction in Figs. 3 and 5 mean that the present reconstruction method is comparatively stable to noise.

It can be seen from Eq. (6) that the resolution of a reconstructed wave function is limited by the extent of the Fourier transform of the Gaussian function for the vacuum state. If the extent of the Fourier-transformed function for the vacuum state is defined by full width at $1/e$ values of its maximum, the minimum resolvable separation of two points at the x coordinate is $\pi/2$. This separation corresponds to the distance between two peaks of the wave function for the odd coherent state of $\alpha_0=\pi/4$. In practice, however, the resolution of a reconstructed wave function is also limited by the noise level of measured data. In the present simulation with a sample of 8×10^3 events for each point, the limit of the separation of two peaks was about 2.6 (i.e., $\alpha_0\cong1.3$).

IV. CONCLUSIONS

A reconstruction method of a wave function from part of the Q function in a pure state of light has been presented. In the first step of this method, the phase of the convolution of the wave function with the known Gaussian function for the vacuum state is retrieved from measured data of the Q function along two parallel lines in phase space by using the analytic phase-retrieval method based on use of the logarithmic Hilbert transform and a Fourier series expansion. In its second step, the wave function is reconstructed by a deconvolution of the known Gaussian function from the complex function consisting of the modulus (i.e., the square root of the measured Q function) and the retrieved phase along one line that passes through the center in the phase space. Especially it should be emphasized that the distributions of the Q function along only two lines contain enough information for the determination of wave functions in a pure state, provided that the resolution of the reconstructed wave functions is limited by the extent of the Fourier transform of the Gaussian function for the vacuum state. Computer-simulated examples of reconstructing the wave functions of odd coherent states demonstrated the applicability of the present method to the measurement of the Q function by using the unbalanced homodyne detection scheme. Since the present method employs the noniterative and analytic phase-retrieval algorithm, we can obviate the convergence problem that is usually encountered in iterative phase-retrieval algorithms such as the Gerchberg-Saxton algorithm [11], in which there is a possibility of the solution converging to a local minimum or a nonanalytic solution different from a true one. Besides, the present method does not have a twofold ambiguity [i.e., a problem that the wave function $\psi(x)$ and its complex conjugate $\psi^*(x)$ cannot be distinguished], which is the inevitable ambiguity in the Gerchberg-Saxton algorithm.

So far, some schemes have been developed for the reconstruction of quantum states of light. In the optical homodyne tomography [2,3], a four-port homodyne detection scheme is used to reconstruct the Wigner function from the measurement of the statistics of difference events in the two output channels of the detector for various values of the phase difference between local oscillator and signal field. Alternatively to the tomographic method, there are more complicated homodyne detection schemes used for determining the quantum state of light in terms of the Q function, such as six-port [6] and eight-port [5] detection schemes. On the other hand, the unbalanced homodyne detection scheme, which is utilized in the present method, is the simplest scheme presently known (i.e., a three-port scheme). By using the present method together with the unbalanced homodyne detection scheme, one can further simplify the procedure of the measurement with the control of a local oscillator field, because the present method allows one to reconstruct arbitrary wave functions from measured data along only two parallel lines without measuring the data in whole phase space in contrast with the conventional ways of reconstruction. In addition, it is another advantage in the present method that one can reconstruct wave functions even for overall quantum efficiency $0.5\leqslant\mu<1$, since the present method is based on the reconstruction from the Q function instead of the Wigner function. This makes it possible to use a realistic detector of quantum efficiency smaller than 1 for the measurement of quantum states in a pure state. In the measurement using the optical homodyne tomography, a high overall detection efficiency (nearly 100% efficiency) is required, because the measured distributions are used to reconstruct the Wigner functions, which always have higher frequency components than the *Q* functions. In the six-port and eight-port detection schemes, the recorded distributions for nonideal detectors are further smoothed, so that the reconstructed quasiprobability distributions become broader than the *Q* function. Consequently, the present method has the advantage of making the measurement of quantum states easier to do. In view of the resolution of the reconstruction, however, the applicability of the present method may be limited to the reconstruction of the wave functions of which the Fourier transforms have a narrower bandwidth than the extent of the Fourier transform of the Gaussian function for the vacuum state. Hence the improvement of the resolution in the present method is a remaining issue.

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