Spontaneous light-polarization symmetry breaking for an anisotropic ring-cavity dye laser

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A semiclassical model of a unidirectional anisotropic ring cavity dye laser has been investigated analytically and numerically. In spite of the axial symmetry of the inversion angular distribution and the isotropy of the cavity, polarized initial fluctuations cause spontaneous light-polarization symmetry breaking in the form of alternative circularly polarized waves. Polarization symmetry is restored by increasing the orientational relaxation and pumping rates or the anisotropy of the cavity. [S1050-2947(99)08003-8]

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I. INTRODUCTION

It is well known [1–3] that if the parameter of a nonlinear optical system goes over a critical value, fluctuations cause a switching of linearly polarized waves into left or right elliptically polarized ones (an effect of spontaneous light-polarization symmetry breaking (SLPSB) [1]). This effect is frequently called ''optical threestability'' or ''polarization switching'' [1]. In experiments with isotropic conditions, switching into right and left elliptically polarized waves are equiprobable due to the property of space inverse symmetry with respect to electromagnetic interactions (P invariance [1]). Theoretical and experimental works on polarization instability have been carried out for atomic gases [2,3]. In theoretical investigations a model of two-level atoms (with a degenerate Zeeman ground-state level and a quickly relaxing upper level) placed into a Fabry-Perot resonator and excited by an electromagnetic field with a frequency close to the atomic resonance frequency (Λ system) has been utilized [2]. This model corresponds to transitions in Na vapors with a buffer gas for suppression of hole burning and hyperfine structure [1]. It has been shown theoretically and experimentally that under adiabatic decreases of the input light intensity, the back transition, namely from a elliptically polarized to a linearly polarized wave, takes place by a jump to lower intensity in comparison with direct transition (polarization hysteresis) [2]. Further theoretical and experimental investigations [3] have demonstrated the instability of the elliptically polarized waves, and the appearance of auto-oscillations, including chaos. If the light intensity is close to the saturation value, the output polarization returns to linear [1,2].

The effect of spontaneous light-polarization symmetry breaking can appear not only as a linearly polarized wave–elliptically polarized wave transition, but in the form of symmetry breaking in lasers without selection of polarization states as well [4–17]. In general, the problem consists of an investigation of the interaction between two nearly degenerate modes which have the same longitudinal and transverse spatial patterns and different polarization states, frequencies, and amplitudes [4–16]. (''a laser with a nearly isotropic resonator,'' ''Zeeman laser,'' and ''two-mode laser'' (see Ref. [5]). The following polarization states have been found [5]: (i) left and right circularly polarized states (poles on the Poincaré sphere [18]), (ii) a linearly polarized state with a rotating polarization plane (equator [18]); (iii) an elliptically polarized state with a rotating azimuth (circles parallel to the equator [18]). The second and third items forms of the SLPSB, because the lasing is polarized at short-time interval and is unpolarized after averaging over some much larger time interval.

All models of ''Zeeman lasers'' are finite dimension systems, but it has been found [14,15,19,20] that anisotropic dye lasers are infinite-dimension systems. Additional forms of SLPSB have been found for a dye laser with a saturable absorber: (i) asymmetric steady-state operations on two linearly polarized modes, (ii) asymmetric periodic auto-oscillation and (iii) chaotic auto-oscillations [14,15]. Experimentally it can be observed in the form of polarized lasing changing from one experimental realization to another in spite of an isotropic cavity and the pumping radiation polarized along the cavity axis. Lasing is polarized on the average in time as well, but it will be unpolarized after averaging over an ensemble of experimental realizations.

It has been also found that light-polarization symmetry is restored with an increase in the rate of orientational relaxation processes (the Brownian rotation of excited molecules or excitation energy migration [14,19]).

For a description of an anisotropic dye laser with a saturable absorber, a model based on adiabatic elimination of the medium polarization has been used [14,15]. Such an approach is quite relevant for an interpretation of most experimental results on dye laser dynamics. However experiments carried out by Hilman *et al.* [21] and Guerra and co-workers [22] demonstrated many of high-order bichromatic operations and spatiotemporal instabilities in cw rhodomine 6G dye lasers that can be explained based only on semiclassical models [22,23]. Experiments carried out by Kozlov and Sergeyev [17] showed that SLPSB can be observed for a dye laser with an isotropic cavity without a saturable absorber. The experimental results are beyond theoretical predictions based on the classical model [14,15]. This means that adiabatic elimination of the medium variables is not a suitable approach for a theoretical description of such results [17,22,23]. It is likely that the effect of SLPSB takes new forms for a semiclassical model of a dye laser, different from the forms of SLPSB considered before. Thus the goal of this

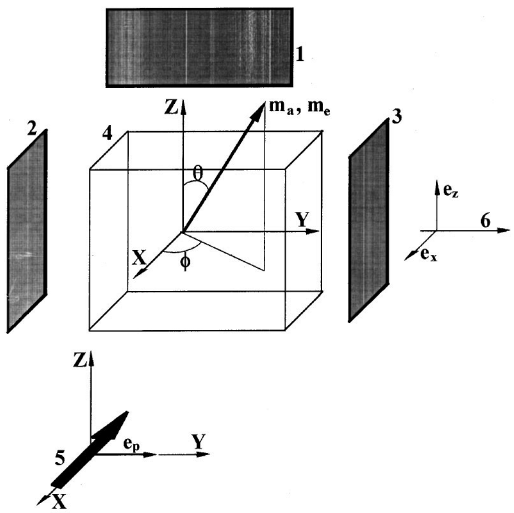

FIG. 1. Scheme of the ring cavity dye laser with linearly polarized pumping. 1, 2, and 3 show the cavity mirrors, 4 the dye cell, 5 the pumping radiation, and 6 the output emission.

paper consists of a theoretical investigation of the semiclassical model of a dye laser, in order to find new forms of spontaneous light-polarization symmetry breaking.

In Sec. I, we consider a semiclassical model of a ring cavity dye laser with pumping polarized along the cavity axis, and describe the main features of the orientational relaxation operators. We compare our model with some models derived for a description of the Zeeman laser dynamics. In Sec. II we describe forms of SLPSB for steady states. In Sec. III we derive expressions for eigenvalues to determine the local dynamics. Next we use numerical calculations to determine the global dynamics, and to classify time-dependent types of SLPSB for isotropic and anisotropic cavities. We investigate the light-polarization symmetry restoration caused by increasing the pumping and orientational relaxation rate and the cavity's anisotropy. In Sec. IV we summarize the results obtained in terms of the theory of nonlinear coupled oscillators.

II. MODEL OF AN ANISOTROPIC RING CAVITY DYE LASER

Let us consider a ring cavity dye laser in which the pumping has a transverse geometry [the pumping radiation propagates along the X axis in Fig. 1, and is polarized along the cavity axis (Y axis)]. Two polarization modes emitted by the laser [14,19] propagate along the Y axis, and are polarized along the X and Z axes. Let us denote the orientation of the dipole moments of the transitions with absorption and emission unit vectors $\mathbf{m}_a$ and $\mathbf{m}_e$. We assume $\mathbf{m}_a=\mathbf{m}_e=\mathbf{m}$, which corresponds to the excitation of the first absorption band for the rhodamine 6G dye lasers [24]. Then the orientation of the vector $\mathbf{m}$ with respect to the reference system XYZ may be written in the reference frame of the spherical angles $g=(\theta,\phi)$. Assuming that the ground state consists of one sublevel, we can derive the following system of equations from the system considered in Ref. [16]:

$$\frac{dE_i}{dt}=-k_i(1-i\delta)E_i+k_i\int P(g)(\mathbf{m}_e\mathbf{e}_i)dg,$$

$$\frac{dP(g)}{dt}=-(1+i\delta)P(g)+\langle D(g,S)\rangle_S\sum_{j=x,z}\rho_i(\mathbf{m}_e\mathbf{e}_j)E_j, \tag{1}$$

$$\frac{dD(g)}{dt}=\gamma\bigg(d_0(\mathbf{m_a}e_p)^2-\frac{1}{2}\sum_j\,(P(g)^*E_j+P(g)E_j^*)(\mathbf{m}_e\mathbf{e}_j)-(1-\hat{\mathcal{L}})D(g)\bigg),$$

$$k=(k_x+k_z)/2,\quad \rho_k=k/k_k,$$

Here we have decomposed the lasing electric field into the cross-polarized components E_i $(i=x,z)$. $P(g)$, $D(g)$ are the angular distributions of the induced dipole moment density and inversion, respectively. d_0 is the scaled parameter of the pumping. k_i, 1, and γ are the relaxation rates of E_i, P, and D, respectively. δ is the scaled detuning. $\hat{\mathcal{L}}$ is the orientational relaxation operator (Brownian rotation [19] or excitation energy migration [14]). For the excitation energy migration, $\hat{\mathcal{L}}x=-Sx+S\langle x\rangle_{s,g}$, [14], where $\langle\,\ldots\rangle_{S,g}=\int\int\cdots f(S)dS\,dg$. The value $S=R_0^6\Sigma_j(\vec{R}-\vec{R}_j)^{-6}$ determines the sum of the rates of excitation energy migration from one excited molecule to another unexcited molecule in the active medium (vector $\vec{R}_j$ determines the coordinates of the jth unexcited molecule). The value S has the distribution [14]

$$f(S)=\frac{N}{2S^{3/2}}\exp\bigg(\frac{-\pi N^2}{4S}\bigg), \tag{2}$$

where $N=\sqrt{2}c/c_0$ (c is the concentration of molecules in the active medium, $c_0=(4\pi R_0^3/3)^{-1}$, and R_0 is the critical distance [27]).

The Brownian rotation operator can be written in the form [15,19]:

$$\hat{\mathcal{L}}x=\mathcal{D}_0\bigg(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\bigg(\sin\theta\frac{\partial x}{\partial\theta}\bigg)+\frac{1}{\sin^2\theta}\frac{\partial^2 x}{\partial^2\phi}\bigg). \tag{3}$$

It has the property [28,29]

$$\hat{\mathcal{L}}D^l_{mn}=-l(l+1)\mathcal{D}_0D^l_{mn}, \tag{4}$$

where D^l_{mn} an Wigner's functions [28,29], and $\mathcal{D}_0$ is the dimensionless Brownian rotation factor.

Before analyzing system (1), let us determine the place of this model among the models of Zeeman lasers [5]. System (1) may be reduced to coupled equations describing the evolution of two circularly polarized components of a lasing electric field [5]. Orientational relaxation (OR) processes are quick, and pumping is isotropic with an intensity slightly above the first threshold one [24], and the conditions

$$k\ll\gamma,1 \tag{5}$$

hold true. Experimentally such an approach corresponds to a cw dye laser pumped by a xenon flashlamp [25]. Approximation (5) gives us an opportunity to use the procedure of adiabatic elimination of $P(g)$ and $D(g)$ [26]. It follows from Eq. (5) that $dP(g)/dt = dD(g)/dt = 0$ in Eqs. (1). Quick orientational relaxation or isotropic pumping means $(\mathbf{m}_a\mathbf{e}_p)^2 = \frac{1}{3}$. The close-to-threshold operation leads to $D(g) = D_0$, i.e., the inversion distribution is independent of angles. For an isotropic cavity $k_z = k_x = k$ and $\rho_i = 1$. Using the procedure of adiabatic elimination of variables $P(g)$ and $D(g)$, and the approximations mentioned above, from the basic equations (1) we derive the following system for circularly polarized components of a lasing electric field $E_\pm$:

$$\frac{dE_\pm}{dt} = a\cdot E_\pm - b(|E_+|^2 + |E_-|^2)E_\pm, \tag{6}$$

$$a = k(1-i\delta)\left(\frac{d_0}{9(1+\delta^2)} - 1\right), \quad b = \frac{kd_0(1-i\delta)}{27(1+\delta^2)^2},$$

If we exclude approximations of isotropic pumping and inversion $D(g) = D_0$, it is possible to rewrite system (6) as follows:

$$\frac{dE_+}{dt} = (a_1 - b_1\cdot(2|E_+|^2 + 3|E_-|^2))E_+ - b_1(E_-^2E_+^* + i|E_+|^2E_- - iE_-^*E_+^2),$$

$$\frac{dE_-}{dt} = (a_1 - b_1(2|E_-|^2 + 3|E_+|^2))E_- - b_1(E_+^2E_-^* - i|E_-|^2E_+ + iE_+^*E_-^2), \tag{7}$$

$$a_1 = k(1-i\delta)\left(\frac{d_0}{15(1+\delta^2)} - 1\right), \quad b_1 = \frac{kd_0(1-i\delta)}{105(1+\delta^2)^2}.$$

Both systems (6) and (7) are close to the system derived by means of third-order Lamb theory [5]:

$$\frac{dE_\pm}{dt} = E_\pm(\alpha_\pm - \beta_\pm|E_\pm|^2 - \theta_\pm|E_\mp|^2). \tag{8}$$

Thus model (1) may be reduced to the model derived based on third-order Lamb theory. It is well known that third-order Lamb theory does not allow for amplitude instability in quasi-isotropic lasers [5]. Moreover, limitations (5) and the close-to-threshold approximation fail for dye lasers with short resonators and laser pumping, respectively. Therefore, it is likely [5] that if we preserve the equations for $P(g)$ and $D(g)$ in Eq. (1), we can find amplitude instabilities for the quasi-isotropic case. System (1) is an infinite-dimensional system due to the angular dependences of $P(g)$ and $D(g)$, but the system considered in Ref. [5] is a finite-dimension one. Thus we believe model (1) is a new form of advanced Zeeman model—an infinite-dimensional Zeeman model.

III. STEADY-STATE SPONTANEOUS LIGHT-POLARIZATION SYMMETRY BREAKING FOR A RING CAVITY DYE LASER WITH A WEAK ANISOTROPY

Let us consider solutions of Eq. (1) in the form:

$$E_i(t) = |E_i(t)|\exp(i\phi_i(t)), \quad P(g,t) = |P(g,t)|\exp(i\psi(t)). \tag{9}$$

Using the approximation $k_z = k_x = k$ (isotropic cavity), and substituting Eq. (9) into Eqs. (1), we can derive the following system:

$$|\dot{E}_i| = -k|E_i| + k\int |P(g)|(\mathbf{m}_e\mathbf{e}_i)dg\,\cos(\psi - \phi_i),$$

$$|\dot{\phi}_i| = k\delta + \frac{k}{|E_i|}\int |P(g)|(\mathbf{m}_e\mathbf{e}_i)dg\,\sin(\psi - \phi_i),$$

$$|\dot{P}| = -|P| + D\sum_{i=z,x}|E_i|(\mathbf{m}_e\mathbf{e}_i)\cos(\phi_i - \psi), \tag{10}$$

$$\dot{\psi} = -\delta + \frac{D}{|P|}\sum_{i=z,x}|E_i|(\mathbf{m}_e\mathbf{e}_i)\sin(\phi_i - \psi),$$

$$\dot{D} = \gamma\left(d_0(\mathbf{m}_a\mathbf{e}_p)^2 - \sum_{j=x,z}|P||E_j|(\mathbf{m}_e\mathbf{e}_j)\times\cos(\psi - \phi_i) - (1-\hat{\mathcal{L}})D(g)\right).$$

It is easy to obtain steady-state solutions for system (10):

$$\phi = \phi_z - \phi_x = 0,\pi, \quad |E_x|^2 + |E_z|^2 = \tfrac{7}{3}(1+\delta^2)(1+4\mathcal{D}_{\mathrm{OR}})\mu,$$

$$\phi = \phi_z - \phi_x = \pm\pi/2, \tag{11}$$

$$|E_x|^2 = |E_z|^2 = \tfrac{7}{4}(1+\delta^2)(1+4\mathcal{D}_{\mathrm{OR}})\mu,$$

$$\mu = d_0/d_0^{(0)} - 1$$

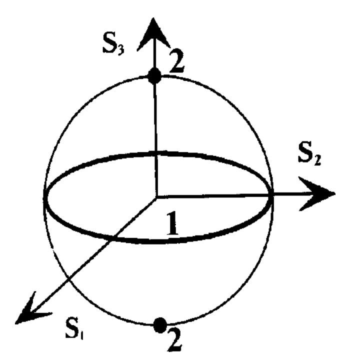

FIG. 2. Steady-state solutions for an anisotropic ring cavity dye laser in the reference frame of the Stokes parameters S_1, S_2, and S_3: $S_0 = |E_z|^2 + |E_x|^2$, $S_1 = |E_z|^2 - |E_x|^2$, $S_2 = 2|E_x||E_z|\cos\phi$, $S_3 = 2|E_x||E_z|\sin\phi$, and $S_0^2 = S_1^2 + S_2^2 + S_3^2$. 1 shows the linearly polarized waves, and 2 the circularly polarized waves. S_1, S_2, and S_3 are in arbitrary units.

where $d_0^{(0)}=15(1+\delta^2)/(1+4\mathcal{D}_{\rm OR})$ is the first threshold value for the pumping parameter. Here $\mathcal{D}_{\rm OR}=\mathcal{D}_0$ for Brownian rotation of excited molecules in the active medium, and $\mathcal{D}_{\rm OR}=N/3$ for excitation energy migration [15]. Expression (11) holds true for $\mathcal{D}_{\rm OR}\ll 1$.

It is convenient to describe polarization states in terms of Stokes parameters [30]:

$$\begin{aligned} S_0&=|E_z|^2+|E_x|^2,\\ S_1&=|E_z|^2-|E_x|^2,\\ S_2&=2|E_x||E_z|\cos\phi,\\ S_3&=2|E_x||E_z|\sin\phi,\\ S_0^2&=S_1^2+S_2^2+S_3^2. \end{aligned} \tag{12}$$

By means of the reference frame (S_1,S_2,S_3) (Poincaré sphere [18]), we rewrite steady-state solutions (11) in the forms

$$\begin{aligned} &S_1^2+S_2^2=\text{const},\quad S_3=0,\\ &S_1=S_2=0,\quad S_3=\pm\text{const}. \end{aligned} \tag{13}$$

All these solutions are shown in Fig. 2. The first set of solutions corresponds to all linearly polarized waves. The second one determines left and right circularly polarized waves. The same set of steady-state solutions was found in Ref. [5]. Any steady state can be reached from different sets of initial conditions. Because of luminescence in an active medium, fluctuations of initial conditions take place, and the polarization of the laser output is changed from realization to realization with mean values equal to zero [17]. To classify the types of SLPSB we introduce a variable, namely, a degree of polarization for lasing:

$$p=S_1/S_0. \tag{14}$$

It is quite clear that only linearly polarized solutions in Eq. (13) correspond to SLPSB because of $p\neq 0$. For circularly polarized lasing $p=0$, according to Eqs. (13) and (14), this means a symmetric solution. To find time-dependent forms of SLPSB we have to carry out a linear stability analysis of steady-state operations (13).

IV. TIME-DEPENDENT SPONTANEOUS LIGHT-POLARIZATION SYMMETRY BREAKING AND RESTORATION FOR ISOTROPIC AND ANISOTROPIC CAVITIES

The linear stability analysis carried out in the Appendix gives us an opportunity to obtain the following eigenvalues which determine the local dynamics near the poles and equator in Fig. 2:

$$\begin{aligned} &\lambda_1=0,\quad \lambda_2=-k+\sqrt{k(1+\delta^2)-(k\delta)^2},\\ &\lambda_3=-\gamma\pm i\omega_0,\quad \omega_0=\sqrt{2k(1+\delta^2)\mu\gamma-\gamma^2}. \end{aligned} \tag{15}$$

The second and the third eigenvalues correspond to the Shil'nikov dynamics [31]. Under the Shil'nikov condition [31,32]

$$|\mathrm{Re}(\lambda_3)/\lambda_2|<1, \tag{16}$$

the motion becomes chaotic. In expression (15) and relation (16) it follows that chaotic dynamics is possible when

$$\begin{aligned} &\gamma<-k+\sqrt{k(1+\delta^2)-k^2\delta^2},\\ &\mu>\mu_0=\frac{\gamma}{2k(1+\delta^2)}. \end{aligned} \tag{17}$$

Increasing the parameter μ, a Hoph bifurcation appears with the threshold parameter of the pumping, μ_1 and the threshold frequency ω_1:

$$\mu_1=\frac{1+\delta^2}{2k(3+\delta^2)},\quad \omega_1=\sqrt{\gamma(1+\delta^2)/(3+\delta^2)}. \tag{18}$$

Expressions (15) hold true for $\mu\ll 1$. For $\mu\gg 1$, we find

$$\begin{aligned} &\lambda_1=\lambda_3=0,\quad \lambda_2=-k+\sqrt{k(1+\delta^2)-(k\delta)^2},\\ &\lambda_4=-(1+k)\pm i\omega_2,\quad \omega_2=\delta\sqrt{(1-k^2)}, \end{aligned} \tag{19}$$

Because $|\mathrm{Re}(\lambda_4)/\lambda_2|\gg 1$, Shil'nikov dynamics is not possible. Thus we have steady-state solutions in the form determined only numerically from Eqs. (1). If $k\ll\gamma,1$ [Eqs. (6) and (7)], then $\mu_0,\mu_1\to\infty$. This means that the absence of Shil'nikov dynamics and auto-oscillations (Hopf bifurcations) for systems (6) and (7), i.e., the procedure of the adiabatic elimination of the variables $P(g)$ and $D(g)$, fails for the good cavity limit. It is clear that the procedure may be applicable for $k\sim\gamma,1$, if

$$\mu<\mu_0,\mu_1. \tag{20}$$

Thus expression (20) represents the quantitative condition of the applicability of the adiabatic elimination procedure.

To expand the local analysis and to determine the global behavior we solve equations (1) numerically. The results found for pumping polarized along the cavity axis in an isotropic cavity correspond to conditions quite relevant for dye lasers: $\mu=\frac{7}{3}$, $\delta=1$, $\gamma=0.01$, $\mathcal{D}_{\rm OR}=0$ [Fig. 3(a)–3(f)], $k_z=k_x=0.1$ [Figs. 3(a) and 3(b)], $k_z=k_x=0.3$ [Figs. 3(c) and 3(a)], and $k_z=k_x=0.5$ [Figs. 3(e) and 3(f)]. For $k=0.1$ we find, from expressions (17) and (18), $\mu_0=0.025$ and $\mu_1=2.5$. Because of $\mu_0<\mu<\mu_1$, for this case only Shil'nikov dynamics is possible. It has been found that for $\mu<\mu_0$, a numerical solution of Eqs. (1) takes the form of right or left circularly polarized waves, i.e., bistable behavior takes place. The interaction of the Shil'nikov dynamics and the neutral stability $\lambda_1=0$ for $\mu=\frac{7}{3}$ lead to a finite time localization near the north and south poles (Fig. 3). With increasing losses in the cavity from $k=0.1$ [Figs. 3(a) and 3(b)] up to $k=0.3$ [Figs. 3(c) and 3(d)], the time of localization increases, and a Hopf bifurcation appears (for $k=0.3$ we have $\mu>\mu_1=0.83$). Further increasing the losses up to $k_z=k_x=0.5$ leads to an increase in the frequency of the auto-oscillations [Figs. 3(e) and 3(f)]. The interaction of the neu-

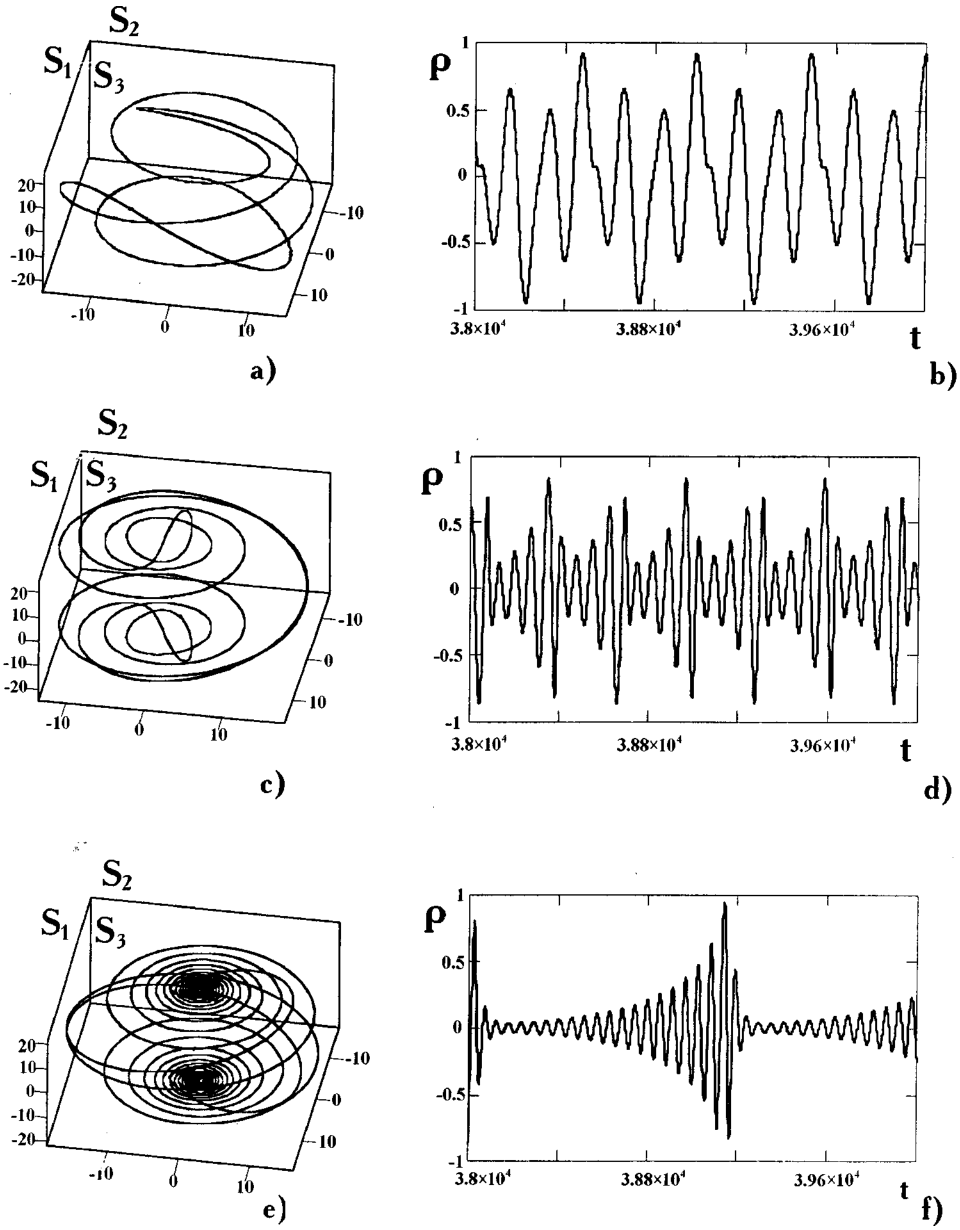

FIG. 3. Light-polarization operations for a ring cavity dye laser with an isotropic cavity. S_1, S_2, and S_3 are Stokes parameters, and $p=S_1/S_0$ is the degree of polarization. Laser parameters: $\mu=\frac{7}{3}$, $\gamma=0.01$, and $\delta=1$. $k_z=k_x=0.1$ [(a) and (b)], $k_z=k_x=0.3$ [(c) and (d)], and $k_z=k_x=0.5$ [(e) and (f)]: t is in units of the transverse relaxation time $t_\perp$, and p is a dimensionless value.

tral stability ($\lambda_1=0$) and instabilities in $\lambda_2>0$ and the Hopf bifurcation ($\lambda_3=\alpha+i\omega_1$, $\alpha>0$) leads to the following global dynamics: the trajectory is attracted to the limit cycle from the inner side, goes outside the limit cycle due to the neutral stability and, because of the instability in ($\lambda_2>0$), is attracted by the inner side of the limit cycle located close to the opposite pole [Figs. 3(c) and 3(e)]. In terms of light-polarization symmetry breaking, the dynamics operations shown in Fig. 3 may be called ''temporal spontaneous light-polarization symmetry breaking.'' This means that the laser output is polarized for a short-time interval [Figs. 3(b), 3(d), and 3(f)], and is unpolarized when the measured mean value, is averaged over some much larger time interval. It has been found numerically that all spiral attractors shown in Fig. 3 strongly depend on initial conditions, i.e., fluctuations of the lasing field (luminescence) cause a broadening of the trajectory. This kind of behavior is close to the Shil'nikov dynamics [31]. Increasing the pumping parameter from $\mu=\frac{7}{3}$ to $\mu=4$, the coupling between poles close to local attractors increases, and in the end a steady-state operation takes place: only two steady states $S_1=S_2=0$ and $S_3=\pm 81.4$ (south and north poles) are possible, and every state is accessible from different sets of initial conditions. This may be called polarization symmetry restoration, because the expression $p=0$ holds true.

Thus we have found that expressions (15) and (19) give us an opportunity to classify all possible polarization operations, and to determine conditions when new types of spontaneous polarization symmetry breaking appear. Numerical results with $k_z\neq k_x$ are beyond analytical investigations. If losses for the two polarization modes are slightly different ($k_x=0.3$, $k_x=0.31$), destabilization of the attractors shown in Fig. 3(c) takes place [Fig. 4(a)]. A quasiperiodic behavior [Figs. 3(c) and 5(a)] is transformed into a chaotic one [Figs.

4(a) and 5(b)]. For high pumping or cavity anisotropy we find that the laser output is linearly polarized close to the first threshold. For example, for the high cavity anisotropy ($k_z = 0.3$, $k_x = 0.5$) and for the pumping parameter $\mu = \frac{1}{3}$, the behavior is stabilized near states $S_1 = 3.719$ and $S_2 = S_3 = 0$ [the linearly polarized steady state (LPSS)]. Increasing μ up to $\mu = \frac{2}{3}$, the behavior is stabilized near $S_1 = 4.54$, $S_2 = 0.599$, and $S_3 = -3.168$ [the elliptically polarized steady state (EPSS)]. Here we have SLPSB in a form close to the classical one [1]: LPSS $\xrightarrow{\mu\uparrow}$ EPSS.

A numerical solution of Eqs. (1) for the case $\mathcal{D}_{\mathrm{OR}} > 0$ is difficult, but nevertheless the influence of orientational relaxation processes (ORP's) may be estimated. As follows from Eq. (11), parameter μ increases with increasing $\mathcal{D}_{\mathrm{OR}}$, i.e., the growth of the ORP rate has the same influence on the dynamics operations as the growth of the pumping power.

V. DISCUSSION

Results obtained in this paper allow us to classify the forms of spontaneous light-polarization symmetry breaking. For an anisotropic system it takes the form of a transition from a linearly polarized wave to a right or left elliptically polarized wave. For an isotropic system we have new forms. (i) For a dye laser with a saturable absorber [14,15] asymmetric steady state operations and asymmetric periodic and chaotic auto-oscillations are possible. Lasing changes its polarization from realization to realization, on the average remaining polarized in time, but it will be unpolarized after averaging on realizations. (ii) For an advanced Zeeman laser model [5], SLPSB appears as a linearly polarized state with a rotating polarization plane and an elliptically polarized state with a rotating azimuth. (iii) In this paper SLPSB for isotropic conditions appears in the form of spiral attractors: trajectories visit points located close to the poles and the equator on a Poincaré sphere.

In summary, we have found that the result of coupled oscillators theory [33] may account for light-polarization symmetry breaking and polarization symmetry restoration. According to the theory [33], weak coupling leads to a complex behavior, and increasing the coupling causes the stabilization of the behavior, i.e., coupled chaotic attractors go to a stable steady state (the Bar-Eli effect [33]). Coupling also increases with an increase in the difference between coupled oscillators [33]. As follows from the results of the paper, coupling is determined by the pumping parameter, the difference in losses for two polarization modes, and the rate of ORP. Complex chaotic attractors appear for weak coupling (Fig. 3), and stabilization takes place with an increase in the pumping and ORP rates, the difference in losses proceeding according to the general theory [33]. Thus spontaneous light-polarization symmetry breaking takes place for weak coupling, and polarization symmetry restoration (the Bar-Eli effect [33]) is possible for strong local attractor coupling.

APPENDIX: LINEAR STABILITY ANALYSIS

Substituting $|E_i| = |E_{i0}| + x_i$, $\phi_i = \phi_{i0} + y_i$, $|P| = |P_0| + u$, $\psi = \psi_0 + v$, and $D = D_0 + z$ into Eqs. (1), for normalized variables $\hat{x}_i = x/|E_{i0}|$, $\hat{u} = u/|P_0|$, and $\hat{z} = z/D_0$ we find

$$\begin{aligned}
\dot{\hat{x}}_i &= -k\hat{x}_i + k\hat{u}_i + k\delta(v - y_i),\\
\dot{y}_i &= k\delta\hat{x}_i - k\delta\hat{u}_i + k(v - y_i),\\
\dot{\hat{u}} &= -\hat{u} + \hat{z} + q_x\hat{x}_x + (1-q_x)\hat{x}_z\\
&\quad - \delta(q_x y_x + (1-q_x)y_z) + \delta v,\\
\dot{v} &= -\delta\hat{u} + \delta\hat{z} + \delta(q_x\hat{x}_x + (1-q_x)\hat{x}_z) + q_x y_x + (1-q_x)y_z - v,\\
\dot{\hat{z}} &= -\gamma\left((1-\hat{\mathcal{L}})\cdot\hat{z} + \sum_{i=x,z} m_i\cdot(\hat{x}_i + \hat{u} + \delta\cdot(v - y_i))\right),
\end{aligned} \tag{A1}$$

where

$$\begin{aligned}
\hat{u}_i &= \frac{\int |P_0|\cdot\hat{u}\cdot(\mathbf{m}_e\mathbf{e}_i)dg}{\int |P_0|\cdot(\mathbf{m}_e\mathbf{e}_i)dg},\\
\hat{z}_i &= \frac{\int |P_0|\cdot\hat{z}\cdot(\mathbf{m}_e\mathbf{e}_i)dg}{\int |P_0|\cdot(\mathbf{m}_e\mathbf{e}_i)dg},\\
q_x &= \frac{D_0}{|P_0|}|E_{x0}|\cos(\psi_0 - \phi_{x0})(\mathbf{m}_e\mathbf{e}_x),\\
m_i &= \frac{|P_0|}{D_0}|E_{i0}|\cos(\psi_0 - \phi_{i0})(\mathbf{m}_e\mathbf{e}_i).
\end{aligned} \tag{A2}$$

Introducing new variables

$$\begin{aligned}
&x_1 = (\hat{x}_z - \hat{x}_x)/2, \quad x_2 = (\hat{x}_z + \hat{x}_x)/2, \quad y_1 = (y_z - y_x)/2,\\
&y_2 = (y_z + y_x)/2,\\
&u_1 = (\hat{u}_z - \hat{u}_x)/2, \quad u_2 = (\hat{u}_z + \hat{u}_x)/2,\\
&z_1 = (\hat{z}_z - \hat{z}_x)/2, \quad z_2 = (\hat{z}_z + \hat{z}_x)/2,
\end{aligned} \tag{A3}$$

we rewrite equations (A1) in the form:

$$\begin{aligned}
\dot{x}_1 &= k(-x_1 + u_1 - \delta y_1),\\
\dot{y}_1 &= k(\delta\cdot x_1 - \delta u_1 - y_1),\\
\dot{u}_1 &= -\frac{m}{\delta}y_1,\\
\dot{x}_2 &= k(-x_2 + u_2 + \delta w_2),\\
\dot{w}_2 &= \delta(1-k)x_2 - (1+k)w_2 - \delta(1-k)u_2 + \delta z_2\\
&\quad - \delta(h_z + h_x - 1)x_1 - (h_z + h_x - 1)y_1,
\end{aligned} \tag{A4}$$

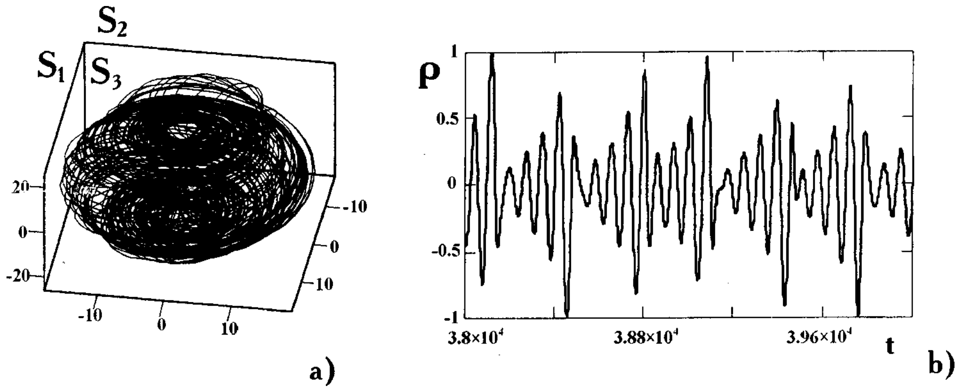

FIG. 4. Light-polarization operations for a ring cavity dye laser with an anisotropic cavity. S_1, S_2, and S_3 are Stokes parameters, and $p=S_1/S_0$ is the degree of polarization. Laser parameters: $\mu=\frac{7}{3}$, $\gamma=0.01$, and $\delta=1$. $k_z=0.3$. $k_x=0.31$.

$$\dot{u}_2=x_2+\delta\cdot w_2-u_2+z_2-(h_z+h_x-1)x_1+\delta(h_z+h_x-1)y_1.$$

Here $w_2=v-y_2$ and

$$m=(1+\delta^2)(h_x-h_z),$$
$$h_i=\frac{|E_{x0}|}{|E_{i0}|}\cos(\phi_{x0}-\psi_0)\cos(\phi_{i0}-\psi_0)\int D_0(\mathbf{m}_e\mathbf{e}_x)(\mathbf{m}_e\mathbf{e}_i)dg. \tag{A5}$$

To complete system (A4) it is necessary to write equation from which z_2 may be determined. As follows from Eqs. (A1) and relations (A3),

$$\dot{\hat{u}}=-\hat{u}+\hat{z}+(1-2q_x)x_1+x_2-\delta(1-2q_x)y_1+\delta w_2,$$
$$\dot{\hat{z}}=-\gamma\left((1-\hat{\mathcal{L}})\hat{z}+\sum_{i=x,z}m_i\hat{u}+(m_z-m_x)x_1+(m_x+m_z)x_2+\delta(m_x-m_z)y_1+(m_x+m_z)w_2\right). \tag{A6}$$

Substituting $\hat{u},\hat{z}\sim\exp(\lambda t)$ into Eq. (A6), and using relations (A2) and (A3) we find the following system as the projection of Eqs. (A4) and (A6) on the R^6 space $(u_1,x_1,y_1,x_2,w_2,u_2)$:

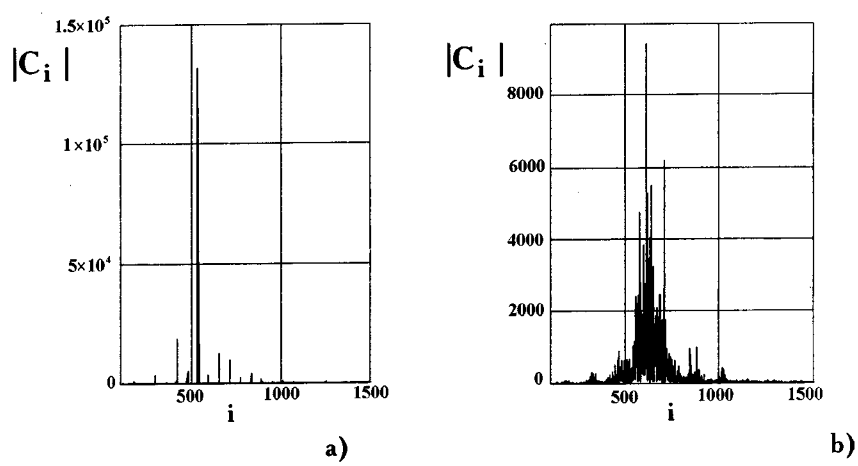

FIG. 5. Fourier transform for the intensity of the z-polarization mode. (a) Parameters correspond to Figs. 3(b) and 3(c). (b) Parameters correspond to Figs. 4(a) and 4(b). i is the number of the Fourier mode, and $|C_i|$ is the Fourier coefficient in arbitrary units.

$$\dot{\mathbf{U}}=\mathbf{L}(\lambda)\mathbf{U},\quad \mathbf{L}(\lambda)=\begin{pmatrix}\mathbf{A}_0 & \mathbf{B}_0\\ \mathbf{C}_0 & \mathbf{D}_0\end{pmatrix},\quad \mathbf{U}=\begin{pmatrix}x_1\\ y_1\\ u_1\\ x_2\\ w_2\\ u_2\end{pmatrix},$$

$$\mathbf{A}_0=\begin{pmatrix}-k & -k\delta & k\\ k\delta & -k & -k\delta\\ 0 & -m/\delta & 0\end{pmatrix},\quad \mathbf{B}_0=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$\mathbf{C}_0(\lambda)=\begin{pmatrix}0 & 0 & 0\\ -\delta(-1-\hat{f}_1+h_z+h_x) & -\delta^2\hat{f}_1+1-h_z-h_x & 0\\ \hat{f}_1-h_z-h_x+1 & \delta(-\hat{f}_1-1+h_z+h_x) & 0\end{pmatrix}, \tag{A7}$$

$$\mathbf{D}_0(\lambda)=\begin{pmatrix}-k & k\delta & k\\ \delta(1-k+\hat{f}_2) & -(1+k-\delta^2\hat{f}_2) & -\delta(1-k)\\ 1+\hat{f}_2 & \delta(1+\hat{f}_2) & -1\end{pmatrix}.$$

Here

$$\hat{f}_i=\frac{\int dg|P_0|f_i(\mathbf{m}_e\mathbf{e}_z)/g_1}{\int|P_0|(\mathbf{m}_e\mathbf{e}_z)dg}+\frac{\int dg|P_0|f_i(\mathbf{m}_e\mathbf{e}_x)/g_1}{\int|P_0|(\mathbf{m}_e\mathbf{e}_x)dg},\quad (i=1,2),$$

$$f_1(g)=-(1+\lambda)(m_z-m_x)-(m_z+m_x)(1-2q_x),\quad f_2(g)=-(2+\lambda)(m_z+m_x), \tag{A8}$$

$$g_1=(1+\lambda)(1-\hat{\mathcal{L}}+\lambda/\gamma)+\sum_{i=x,z}m_i\,.$$

Based on approximations $\mu\ll 1$ and $||\hat{\mathcal{L}}||\ll 1$, we formulate the eigenvalue problem for system (A7) in the forms

$$\lambda\big((k+\lambda)^2-k(1+\delta^2)+(k\delta)^2\big)=0,\quad \lambda^3+p(\lambda)\lambda^2+q(\lambda)\lambda+r(\lambda)=0. \tag{A9}$$

Here

$$p=p_0+\frac{(2+\lambda)\mu p_1}{(1+\lambda)(1+\lambda/\gamma)},\quad q=q_0+\frac{(2+\lambda)\mu q_1}{(1+\lambda)(1+\lambda/\gamma)},\quad r=\frac{(2+\lambda)\mu r_1}{(1+\lambda)(1+\lambda/\gamma)},$$

$$p_0=2(1+k),\quad p_1=\frac{\delta^2}{1+\delta^2},\quad q_0=(1+k)^2+\delta^2(1-k^2),\quad q_1=\frac{k}{1+\delta^2}, \tag{A10}$$

$$r_1=\frac{k\big(1+k+\delta^2(1-k)\big)}{1+\delta^2}.$$

Based on approximations $\mu\gg 1$, we formulate the eigenvalue problem for system (A7) in the forms

$$\lambda\big((k+\lambda)^2-k(1+\delta^2)+(k\delta)^2\big)=0,\quad \lambda(\lambda^2+p_0\lambda+q_0)=0. \tag{A11}$$

Using expressions (A9)–(A11) we find eigenvalues (15) and (19).

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