Internal Josephson effect in trapped double condensates

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The two-component Bose-Einstein condensate exhibits oscillations in the individual particle number when the two species are coupled by a Josephson coupling. We present the results of numerical calculations based on coupled Gross-Pitaevskii equations, and compare the results with an approximate analytic theory. The small amplitude limit reproduces the conventional results, but the large signal regime offers some features characteristic of the trapped systems. [S1050-2947(99)09005-8]

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I. INTRODUCTION

The recent experimental achievement of Bose-Einstein condensation (BEC) in trapped clouds of alkali-metal atoms [1–3] has added one more system to our store of collectively condensed quantum systems. In contrast to the earlier known systems, superconductors and superfluids, the atomic system is characterized by a relatively low particle density and consequently a relatively weak interaction. Thus many experiments are well described by available mean-field methods, hence many of the ideas known from earlier condensates are applicable to the atomic BEC both experimentally and theoretically.

The conventional theoretical treatment of both superconductors and superfluids is based on the appearance of long-range phase correlations, which go together with a lack of precise particle conservation. This manifests itself physically as observability of the phase difference between two physically distinguishable condensates. The well known Josephson phenomenon is a direct consequence of this fact, and it is well established in superconducting junctions and leaks in superfluids [4–6].

When atomic BEC was discovered, it was immediately suggested that separated condensates would tunnel just like other separated ones [7–13]. Another possibility arose when experiments revealed that the same atoms could participate in two different condensates depending on their internal substate [14]. These are formed into partly separated regions due either to external forces (gravitation) or because of the mutual repulsion between the atoms forming the condensate. It is, however, in principle possible to create condensates sitting on top of each other, with different shapes deriving from the effective potentials they see at condensation. In both cases, an atom can be coherently transferred from one condensate to another in a coherent manner so that the phases of the two condensates become coupled. One possible coupling mechanism consists of coherent two-photon transitions between the levels involved [15]; in the ideal case, this would just correspond to a direct transfer from one condensate to the other. A recent paper [16] has suggested a possible experiment, where this effect can be observed. Its approach is based on the separation of the condensed components by an external potential, and a subsequent coupling of the two by external fields. However, even when the condensates are situated just on top of each other, the process of transfer is still possible, and due to the different shapes of the two components, the transfer of particles between them is still observable. Switching on the coherent interaction, when the two condensates are not in mutual equilibrium, will lead to transient transfer between the components, which in the weak coupling limit is equivalent with an internal Josephson effect. Due to the finite number of particles in each condensate, the perturbative region of linear oscillations is extremely small, and the full nonlinear behavior is interesting also. In fact, it appears possible to transfer a major fraction of the particles from one condensate to the other and back. This mechanism may well be utilized to prepare desired states for experiments by a suitably tailored coupling, which may be chosen to depend on time or space in a propitious way. The setup thus provides an excellent example of coherent control of the condensate order parameters.

The internal Josephson effect between two condensates based on the same physical objects has been suggested for a two-band superconductor [17] and for condensation in ^{3}He. We suggest that the BEC of trapped atoms offers an excellent opportunity to verify its existence experimentally.

The organization of the paper is as follows. Section II introduces the basic concepts and equations. In Sec. III we introduce an approximate analytic solution of the Josephson oscillations. This is expected to be a valid description for small oscillational amplitudes, but it can serve as a guide to the behavior even away from this region. In Sec. IV, we present the main results of this paper, the numerical computations of the Josephson oscillations as described by the coupled Gross-Pitaevskii equations. These results are compared with those of the analytic approximation, which turns out to be surprisingly good even beyond its expected region of validity. Finally, Sec. V comments on the calculations and their results.

II. FORMULATING THE PROBLEM

We consider a two-component Bose condensed gas in the external harmonic potentials

$$V_i(r,z)=\frac{1}{2}m(\Omega_{r_i}^2 r^2+\Omega_{z_i}^2 z^2),\quad i=1,2 \tag{1}$$

with the trap frequencies Ω_{α_i}, and assume a pair potential of the atom-atom interaction of the form

$$U_i=\frac{4\pi\hbar^2 a_i}{m}\delta(\mathbf{r}-\mathbf{r}')\equiv v_i\delta(\mathbf{r}-\mathbf{r}'),\quad i=1,2,3, \tag{2}$$

where a_1 and a_2 stand for the intraspecies scattering lengths, a_3 is the scattering length between the two different atoms, and m is the mass of the atoms, which is taken here to be the same for both species. The second quantized Hamiltonian can then be written in the form

$$\hat{H}=\hat{H}_1+\hat{H}_2+\hat{H}_3+\hat{H}_{\rm JC} \tag{3}$$

with

$$\hat{H}_i=\int d\mathbf{r}\left\{\hat{\Psi}_i^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2+V_i(r)\right)\hat{\Psi}_i(\mathbf{r}) +\frac{1}{2}v_i\hat{\Psi}_i^\dagger(\mathbf{r})\hat{\Psi}_i^\dagger(\mathbf{r})\hat{\Psi}_i(\mathbf{r})\hat{\Psi}_i(\mathbf{r})\right\},\quad i=1,2, \tag{4}$$

$$\hat{H}_3=v_3\int d\mathbf{r}\hat{\Psi}_1^\dagger(\mathbf{r})\hat{\Psi}_2^\dagger(\mathbf{r})\hat{\Psi}_1(\mathbf{r})\hat{\Psi}_2(\mathbf{r}), \tag{5}$$

$$\hat{H}_{JC}=-\Gamma\int d\mathbf{r}\{\hat{\Psi}_1^\dagger(\mathbf{r})\hat{\Psi}_2(\mathbf{r})+\hat{\Psi}_2^\dagger(\mathbf{r})\hat{\Psi}_1(\mathbf{r})\}, \tag{6}$$

where $\hat{\Psi}_i^\dagger(\mathbf{r})$ and $\hat{\Psi}_i(\mathbf{r})$ are the field operators of the bosons. $\hat{H}_1$ and $\hat{H}_2$ stand for the intraspecies Hamiltonians, whereas $\hat{H}_3$ is the coupling between the two species governed by the collisional interaction potential in Eq. (2). The Josephson coupling $\hat{H}_{\rm JC}$ destroys one atom of species 1 and creates one of species 2 and *vice versa*. The number of particles in each condensate is therefore not preserved, whereas the total number is.

Using the bosonic commutation relations, $[\hat{\Psi}_i(\mathbf{r}),\hat{\Psi}_j^\dagger(\mathbf{r}')]=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')$, we obtain the equations

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_1(\mathbf{r})=\left(-\frac{\hbar^2}{2m}\nabla^2+V_1(\mathbf{r})+v_1\hat{\Psi}_1^\dagger(\mathbf{r})\hat{\Psi}_1(\mathbf{r}) +v_3\hat{\Psi}_2^\dagger(\mathbf{r})\hat{\Psi}_2(\mathbf{r})\right)\hat{\Psi}_1(\mathbf{r})-\Gamma\hat{\Psi}_2(\mathbf{r}), \tag{7}$$

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_2(\mathbf{r})=\left(-\frac{\hbar^2}{2m}\nabla^2+V_2(\mathbf{r})+v_2\hat{\Psi}_2^\dagger(\mathbf{r})\hat{\Psi}_2(\mathbf{r}) +v_3\hat{\Psi}_1^\dagger(\mathbf{r})\hat{\Psi}_1(\mathbf{r})\right)\hat{\Psi}_2(\mathbf{r})-\Gamma\hat{\Psi}_1(\mathbf{r}). \tag{8}$$

Assuming zero temperature allows us to use the effective field approximation, $\hat{\Psi}_i(\mathbf{r})\rightarrow\langle\hat{\Psi}_i(\mathbf{r})\rangle\equiv\Psi_i(\mathbf{r})$, and to write Eqs. (7) and (8) in the form

$$i\hbar\frac{\partial}{\partial t}\Psi_1(\mathbf{r})=\left(-\frac{\hbar^2}{2m}\nabla^2+V_1(\mathbf{r})+v_1|\Psi_1(\mathbf{r})|^2 +v_3|\Psi_2(\mathbf{r})|^2\right)\Psi_1(\mathbf{r})-\Gamma\Psi_2(\mathbf{r}) \tag{9}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2(\mathbf{r})=\left(-\frac{\hbar^2}{2m}\nabla^2+V_2(\mathbf{r})+v_2|\Psi_2(\mathbf{r})|^2 +v_3|\Psi_1(\mathbf{r})|^2\right)\Psi_2(\mathbf{r})-\Gamma\Psi_1(\mathbf{r}). \tag{10}$$

When we start from a nonequilibrium situation, the system retains its time development and, without any additional physical mechanisms, it does not approach a steady-state solution. We therefore have to normalize the time-dependent solutions $\Psi_i(\mathbf{r})$ with respect to the total number of particles

$$\int d\mathbf{r}[|\Psi_1(\mathbf{r})|^2+|\Psi_2(\mathbf{r})|^2]=N_1+N_2. \tag{11}$$

III. ORIGIN OF THE DYNAMIC BEHAVIOR

The customary treatment of the Josephson effect as a weak coupling phenomenon [4] suggests that we can describe the oscillations in terms of phase alone. We hence use the ansatz

$$\Psi_i(\mathbf{r},t)=\psi_i(\mathbf{r})e^{-i\theta_i(t)}, \tag{12}$$

where we allow the phases θ_i to be complex functions of t with no spatial dependence and $\psi_i(\mathbf{r})$ is considered to be real. For spatially separate condensates coupled by a weak link, this assumption may be valid [11], but we have overlapping ones, coupled by the constant parameter Γ, which allows the rate of particle transfer to vary across the extension of the condensate. For Eq. (12) to be a valid ansatz, we have to assume a small Γ, a weak coupling. The particle transfer then remains small everywhere, and the weak coupling result is expected to emerge. For large oscillational amplitudes, the model solution can serve as a point of comparison only, but our detailed numerical computations show it to be surprisingly good even when a major part of the particles is transferred.

In order to obtain oscillating solutions, some quantity has to deviate from steady state. For simplicity we assume that the condensates contain equal particle numbers initially; $N_1=N_2=N$. They are allowed to settle to their Gross-Pitaevskii steady-state solutions without Josephson coupling, $\Gamma=0$, and the normalization condition

$$\int d\mathbf{r}\,\psi_i(\mathbf{r})^2=N_i \tag{13}$$

determines the initial chemical potentials from the equations

$$\mu_i\psi_i(\mathbf{r})=\left(-\frac{\hbar^2}{2m_i}\nabla^2+V_i(\mathbf{r})+v_i\psi_i(\mathbf{r})^2+v_3\psi_j(\mathbf{r})^2\right)\psi_i(\mathbf{r}),\quad i\neq j. \tag{14}$$

In the case of overlapping two-component condensates, we assume that the experimenter can switch on the Josephson coupling. Because $\mu_1\neq\mu_2$, oscillations start, but to compute these we need also initial conditions on the phases θ_i. Because the main result of the calculation is the oscillational frequency, we do not expect a strong dependence on the initial conditions on the phase, so we are free to choose them such that manageable analytic expressions appear. Throughout the calculations, the chemical potentials μ_i remain constant. This is correct within the validity range of the ansatz (12); these parameters just retain the memory of the intitial distribution of particles between the condensates as determined by Eqs. (14).

Inserting the ansatz Eq. (12) into Eqs. (9) and (10), multiplying by ψ_i, and integrating over $\mathbf{r}$, we get two first-order differential equations for the phases

$$\frac{d}{dt}\theta_1=\mu_1-\tilde{\Gamma}e^{i(\theta_1-\theta_2)}, \tag{15}$$

$$\frac{d}{dt}\theta_2=\mu_2-\tilde{\Gamma}e^{-i(\theta_1-\theta_2)}, \tag{16}$$

with

$$\tilde{\Gamma}=\Gamma\int d\mathbf{r}\psi_1(\mathbf{r})\psi_2(\mathbf{r}). \tag{17}$$

Introducing the notation $\theta_\pm=\theta_1\pm\theta_2$ and $\mu_\pm=\mu_1\pm\mu_2$ gives the more transparent equations

$$\frac{d}{dt}\theta_-=\mu_--2i\tilde{\Gamma}\sin\theta_-, \tag{18}$$

$$\frac{d}{dt}\theta_+=\mu_+-2\tilde{\Gamma}\cos\theta_-, \tag{19}$$

which may be integrated to give the solutions

$$\theta_-(t)=2\arctan\left\{\sqrt{1+4\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2}\times\tan\left[\frac{1}{2}\sqrt{1+4\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2}\mu_-t\right]+2i\frac{\tilde{\Gamma}}{\mu_-}\right\}-i\ln[\alpha] \tag{20}$$

$$=-\arctan\left(\frac{2\alpha\tan(\frac{1}{2}\alpha\mu_-t)}{1-4(\tilde{\Gamma}/\mu_-)^2-\alpha^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)\pm\pi-i\frac{1}{2}\ln\left[\frac{\alpha^2-4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\frac{1}{2}\alpha\mu_-t)}{\alpha^2+4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\frac{1}{2}\alpha\mu_-t)}\right], \tag{21}$$

$$\theta_+(t)=\mu_+t-2\tilde{\Gamma}\int^t dt'\cos(\theta_-(t')) \tag{22}$$

$$=\mu_+t+2\arctan\left(\frac{4\alpha\tilde{\Gamma}\mu_-\tan(\frac{1}{2}\alpha\mu_-t)}{4\tilde{\Gamma}^2-\mu_-^2-\alpha^2\mu_-^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)+i\frac{1}{2}\ln\left[\alpha^4-16\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2\cos^4(\frac{1}{2}\alpha\mu_-t)\right]-i\ln[\alpha] \tag{23}$$

with

$$\alpha=\sqrt{1+4\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2}. \tag{24}$$

Here we have used the somewhat unwieldy initial value $\theta_-(t=0)=2\arctan(2i\tilde{\Gamma}/\mu_-)$ in order to simplify the algebra. This initial condition only serves to simplify the analytic expressions. Physically it means that there is a specific relation between $\theta_-(t)$ and its derivative according to Eq. (18). As explained above, this choice is not expected to greatly affect the general consequences of the model. For general initial values, the algebra is still straightforward but considerably more tedious. Combining θ_- and θ_+ gives the final solutions for the phases

$$\theta_1(t)=\frac{1}{2}\mu_+t+\arctan\left(\frac{4\alpha\tilde{\Gamma}\mu_-\tan(\frac{1}{2}\alpha\mu_-t)}{4\tilde{\Gamma}^2-\mu_-^2-\alpha^2\mu_-^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)-\frac{1}{2}\arctan\left(\frac{2\alpha\tan(\frac{1}{2}\alpha\mu_-t)}{1-4(\tilde{\Gamma}/\mu_-)^2-\alpha^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)\pm\pi+i\frac{1}{2}\ln\left[\alpha^2+4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\tfrac{1}{2}\alpha\mu_-t)\right]-i\ln[\alpha], \tag{25}$$

$$\theta_2(t)=\frac{1}{2}\mu_+t+\arctan\left(\frac{4\alpha\tilde{\Gamma}\mu_-\tan(\frac{1}{2}\alpha\mu_-t)}{4\tilde{\Gamma}^2-\mu_-^2-\alpha^2\mu_-^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)+\frac{1}{2}\arctan\left(\frac{2\alpha\tan(\frac{1}{2}\alpha\mu_-t)}{1-4(\tilde{\Gamma}/\mu_-)^2-\alpha^2\tan^2(\frac{1}{2}\alpha\mu_-t)}\right)\pm\pi+i\frac{1}{2}\ln\left[\alpha^2-4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\tfrac{1}{2}\alpha\mu_-t)\right]-i\ln[\alpha]. \tag{26}$$

Only the imaginary parts give a contribution to the oscillations in the particle numbers which consequently gives the time-dependent behavior of the densities,

$$|\Psi_1(\mathbf{r},t)|^2=\psi_1(\mathbf{r})^2\left(\alpha^2+4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\tfrac{1}{2}\alpha\mu_-t)\right)\Big/\alpha^2 \tag{27}$$

$$|\Psi_2(\mathbf{r},t)|^2=\psi_2(\mathbf{r})^2\left(\alpha^2-4\frac{\tilde{\Gamma}}{\mu_-}\cos^2(\tfrac{1}{2}\alpha\mu_-t)\right)\Big/\alpha^2. \tag{28}$$

We immediately see that the solutions preserve the total particle number $2N$. The relative difference may then be written in the form

$$\eta=\frac{1}{2N}\int d\mathbf{r}[|\Psi_1(\mathbf{r},t)|^2-|\Psi_2(\mathbf{r},t)|^2]=\left(\frac{4\frac{\tilde{\Gamma}}{\mu_-}}{1+4\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2}\right)\times\frac{1}{2}[1+\cos(\alpha\mu_-t)], \tag{29}$$

which oscillates with the frequency

$$\omega_J = \sqrt{1+4\left(\frac{\tilde{\Gamma}}{\mu_-}\right)^2}\,\mu_- . \tag{30}$$

The ansatz in Eq. (12) assumes, in fact, that no symmetry breaking in the steady-state solutions takes place. This is also ensured by the assumption $\tilde{\Gamma} \ll \mu_-$, which is easily achieved if v_1 and v_2 are chosen different and v_3 large, which consequently gives a small contribution for the overlap integral in $\tilde{\Gamma}$. By keeping terms linear in $\tilde{\Gamma}/\mu_-$, we see that the particle number oscillations become independent of $\tilde{\Gamma}$ since α is of second order in $\tilde{\Gamma}/\mu_-$,

$$\eta \approx \left(4\frac{\tilde{\Gamma}}{\mu_-}\right)\frac{1}{2}[(1+\cos(\mu_- t)]. \tag{31}$$

The oscillations at the frequency $\mu_- = \mu_1 - \mu_2$ corresponds to the ordinary result for the Josephson effect, cf. Anderson [4]. For small oscillational amplitudes, the result (31) is expected to emerge from our numerical calculations as well. These also allow us to test the model when the oscillational amplitude becomes large, and we find it to describe the behavior beyond its assumed range of validity. The numerical results and the comparisons are presented in the next section.

IV. NUMERICAL RESULTS

Solving Eqs. (9) and (10) numerically allows us to use tailor-made couplings Γ, which may be both time and space dependent. The numerical calculations are performed in the usual way with discretized solutions and derivatives, which has been successfully used in earlier works concerning the nonlinear Schrödinger equation [18–20]. Discretizing the solutions is in this case favorable since we assume a cylindrical symmetry and consequently store the time-dependent solutions in matrices with the r and z dependence as rows and columns. The numerical calculation is started by solving the stationary Gross-Pitaevskii equation (14) with the method of steepest descent, and using that solution as an initial condition when applying the Josephson coupling in Eqs. (9) and (10). The time-dependent problem, which includes the Josephson coupling, is then solved by the method of finite differencing where the time step is split into two parts, giving an explicit scheme, see, e.g., [21]. The trap geometry has been chosen to be $\Omega_{r_i}/\Omega_{z_i} = \sqrt{8}$. In the equations above, the r and z coordinates are scaled by $\sqrt{\hbar/m\Omega_r}$ with $\Omega_{r_1} = \Omega_{r_2} \equiv \Omega_r$ and consequently the energies are scaled with $\hbar\Omega_r/2$ and time scaled with $2/\Omega_r$.

We apply a Josephson coupling to a system with $N_1 = N_2 = 21\,000$ atoms in each condensate, with the interaction strengths $v_1 = 0.04$, $v_2 = 0.042\,48$, and $v_3 = 0.041\,22$ (corresponding to $\Omega_r/2\pi = 10$ Hz for $v_1 = 0.04$ with rubidium atoms). The experiment is supposed to take place in such a way that the condensates settle to their respective ground states given by Eqs. (14) without the Josephson coupling; $\Gamma = 0$. When this is switched on at $t = 0$, the condensates are out of equilibrium and particles start to transfer between the components in the Josephson manner. The situation corresponds to that discussed in Sec. III, except that now the rate of transfer is allowed to vary spatially in accordance with the exact solution of the coupled Gross-Pitaevskii equations.

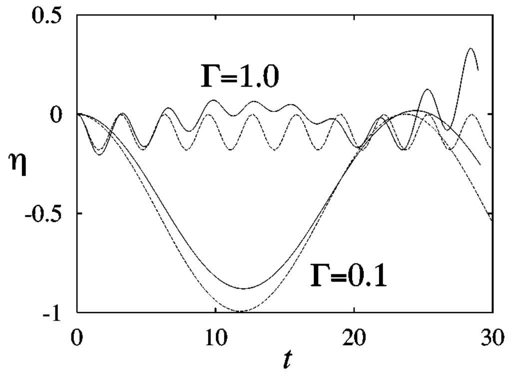

FIG. 1. The difference in the relative particle numbers $\eta = (N_1 - N_2)/(N_1 + N_2)$ show significant oscillations due to the Josephson coupling. Because of the strong coupling and the broken symmetry of the initial condition, the oscillation amplitude has a nontrivial behavior. The $\Gamma = 0.1$ case almost completely transfers all particles from one condensate to the other with a low frequency, whereas the $\Gamma = 1.0$ case oscillates with a smaller amplitude but a higher frequency. The dashed lines are the analytic results from Sec. III showing good agreement for the frequencies.

The calculations shown in Fig. 1 for two different coupling strengths, $\Gamma = 0.1$ and $\Gamma = 1.0$, clearly show the oscillational behavior in the particle number difference. The strong coupling, $\Gamma = 1.0$, causes nontrivial modulation of the maximum amplitudes, which the analytic results from Eq. (29) fail to describe in detail. However, the period $T_J = 3.16$, calculated with the overlap integral $\tilde{\Gamma}/\Gamma = 0.989$ and $\mu_- = -0.175$, still gives a surprisingly good description of the dynamic behavior even if the parameter $\tilde{\Gamma}/\mu_-$ is by no means small (for $\Gamma = 1.0$ we have $\tilde{\Gamma}/\mu_- = -5.64$). With the weaker coupling, $\Gamma = 0.1$, we have a stronger transfer of particles with $\eta_{\max} = 0.88$ and a longer period $T_J = 23.77$. The steady-state solutions from the Gross-Pitaevskii equation (14) are the symmetric configurations with condensate 2 forming a shell around the other one (see Ref. [20]). These solutions are in fact metastable in certain parameter regions. When we apply the Josephson coupling, the symmetric configuration is broken and the condensates strive to separate. This affects the oscillations since the overlap integral and the chemical potentials change. The symmetry breaking can be seen in Fig. 2, where the center of mass in the z direction is shown. In Fig. 3 we show the time dependence of the densities along the z axis, which exhibits a complicated behavior due to the broken symmetry and the strong coupling.

In order to avoid the strong coupling effects, we also evaluate the results in the perturbative regime. Looking at the case $|\tilde{\Gamma}/\mu_-| \ll 1$ we choose a system with $v_1 = 0.02$, $v_2 = 0.04$, $v_3 = 0.1$, and $N_1 = N_2 = 21\,000$ in the symmetric configuration. This gives us stable solutions with no risk of symmetry breaking because of a strong repulsion v_3. The overlap integral in Eq. (17) is then $\tilde{\Gamma}/\Gamma = 0.113$ and $\mu_- = -2.283$. The maximum amplitude from Eq. (29) is $\eta_{\max} = 0.20$ with the corresponding period $T_J = 2.739$ for $\Gamma = 1.0$ and $\eta_{\max} = 0.02$ with $T_J = 2.752$ for $\Gamma = 0.1$. The analytic results from Sec. III and the numerical calculations for $\Gamma = 1.0$ and $\Gamma = 0.1$ are shown in Fig. 4. The frequencies show

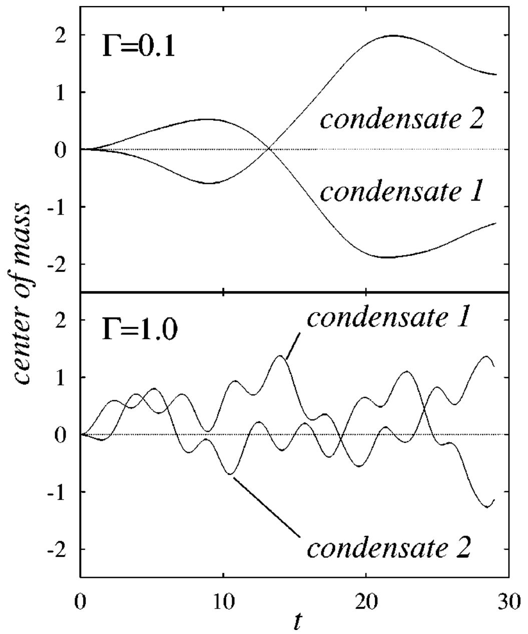

FIG. 2. The center of mass in the z direction shows that the initially symmetric configuration is unstable.

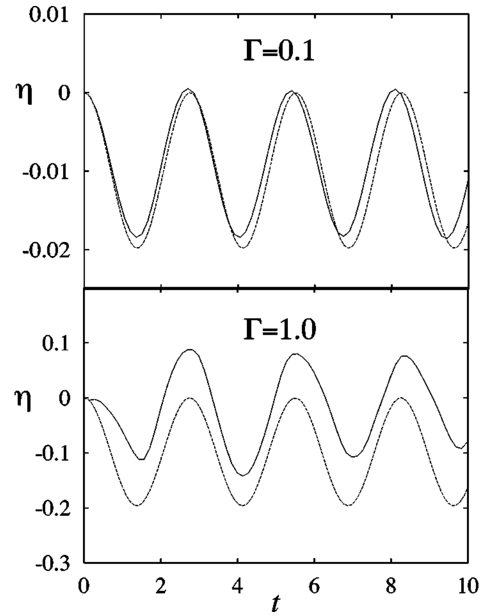

FIG. 4. With a stable initial configuration the analytic results (dashed lines) may be compared with the numerical calculations and show excellent agreement.

only a weak dependence on Γ, which agrees with the results obtained in Sec. III.

The analytic results from Sec. III work surprisingly well even when the assumption $|\tilde{\Gamma}/\mu_-|\ll 1$ is clearly violated. In Fig. 5 we show the maximum amplitude of the relative number of particles as a function of the parameter $|\tilde{\Gamma}/\mu_-|$. Here we see that the general behavior is well described by the amplitude in Eq. (29) even for $|\tilde{\Gamma}/\mu_-|>1$. The numerical calculations do not show a complete transfer of the particles in contrast to the analytic solution at $|\tilde{\Gamma}/\mu_-|=1/2$. This may be explained by the symmetry breaking taking place, where the condensates separate in the z direction, which causes dynamics not accounted for in the analytic model.

So far we have looked at static coupling strengths Γ. The numerical treatment allows us to use a coupling which is both time and space dependent. In Fig. 6 we show a cutoff situation where we take the same situation as in Figs. 1–3 with $\Gamma=0.1$ and cut off the Josephson coupling at $t=12.0$. With such a coupling the particles in condensate 1 are almost completely transferred to condensate 2 and remain there. In this case the symmetric configuration is broken as in Figs. 1–3, which can be seen as a splitting for $t>12$, where the small remaining part of condensate 1 is pushed to the side.

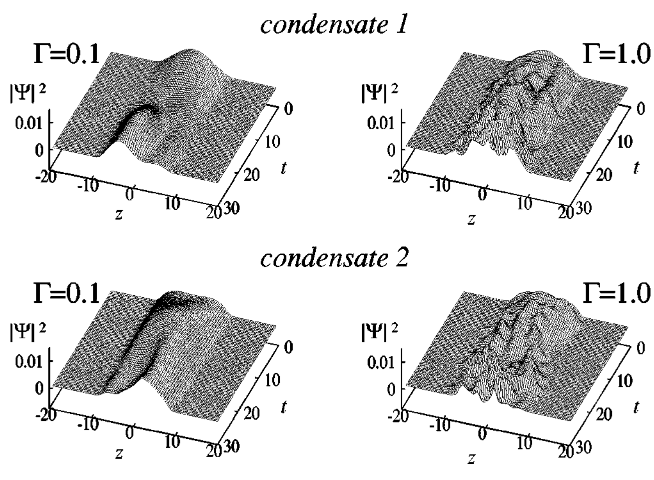

FIG. 3. The time-dependent oscillations in the densities show a nontrivial behavior due to the broken symmetry in both cases. The strong coupling causes wild oscillations in the densities, whereas the weak coupling oscillations are considerably slower than the trap frequency.

V. CONCLUSIONS

We have investigated the effect of introducing a coherent Josephson-type coupling between the components of a two-component condensate. To simplify the numerical treatment, we have chosen a cylindrical symmetry, which, however, is not experimentally unrealistic. For the same reason, the

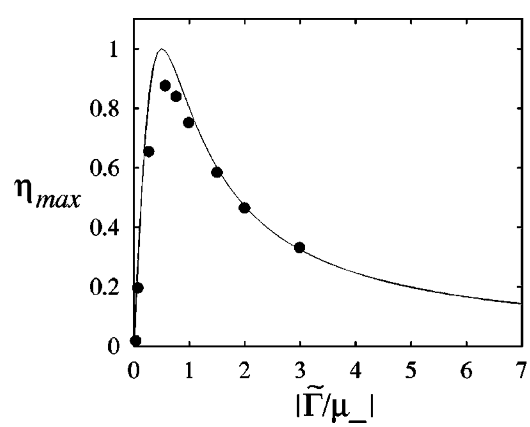

FIG. 5. A comparison of the analytic and numerical calculations for the oscillation amplitude in the relative particle number shows that the analytic solution (solid curve) works even if $|\tilde{\Gamma}/\mu_-|$ is not a small number. The filled circles show the numerically calculated results from Eqs. (9) and (10).

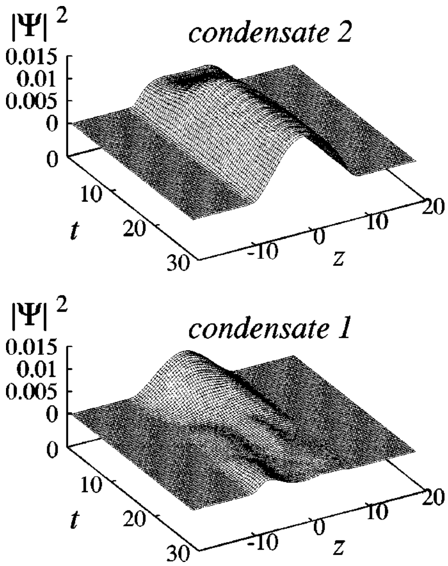

FIG. 6. By cutting off the Josephson coupling at $t=12.0$, it is possible to transfer almost completely all particles from condensate 1 to condensate 2.

ground state of the system, with zero Josephson coupling, is taken to be symmetric; we assume no spontaneous symmetry breaking due to external forces.

The effect of the Josephson coupling is first assumed to be just a time dependent modulation of the order parameter, Eq. (12), which neglects the possibility of having a spatially modulated exchange of particles between the condensates. This simplification has the advantage that it allows an analytic solution for the time dependence. The results presented in Sec. III show that the oscillations take place at the frequency

$$\omega_J=\frac{2\pi}{T_J}=\mu_-+2\tilde{\Gamma}\left(\frac{\tilde{\Gamma}}{\mu_-}\right)+\cdots, \tag{32}$$

which shows that for a weak coupling $\tilde{\Gamma}$, the zeroth-order result $\omega_J=\mu_1-\mu_2$ holds. This is shown in our numerical calculations in the perturbative limit. In Fig. 4 we show the results for $|\tilde{\Gamma}/\mu_-|=0.05$ and $|\tilde{\Gamma}/\mu_-|=0.005$. In both cases, the frequency is nearly correctly given by the analytic theory; for the weaker coupling, the amplitude is correct, too. In fact, the value of the oscillational amplitude is adequately given by the analytic model even if $(\tilde{\Gamma}/\mu_-)$ is not small, cf. Fig. 5.

However, in the configuration chosen, we know that the symmetric initial condition is not necessarily stable [20]. In the calculation shown in Fig. 1, the analytic model gives only an approximation to the correct frequency, but does not describe the amplitude variation correctly when the Josephson coupling becomes strong. This we ascribe to the instability of the spatial configuration caused by the symmetry breaking. The results reported in Figs. 2 and 3 support this interpretation. It is thus important to choose the parameter range propitiously when Josephson experiments are attempted. Due to the finite number of particles, one may not easily be able to achieve observable effects with infinitesimally small exchange of particles; in bulk systems this problem does not exist. On the other hand, the finite particle number suggests novel uses of the Josephson flipping of the population between the condensates. By suitably chosen tailoring of the Josephson coupling one can reach a desired initial state for various experiments. In Fig. 6 we show the simple example of nearly total transfer of all atoms to the one condensate. By using a suitable time dependence of the coupling Γ, one may achieve coherent control of the dynamics of the two-component condensate.

We have also pointed out that imposing a space dependence on the Josephson coupling, one may further direct the time evolution of the system. We have not explored this possibility in the present paper, but our numerical calculations could easily handle this case too. When the experimental tools and motivations exist, this possibility may offer new and unexpected directions for the BEC research.

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